Periodic magnetic curves on the 3-torus

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Preliminaries - magnetic curves

• The starting point in the study of magnetic curves was the classical treatment of static magnetic fields (time-independent) in $\mathbb{E}^3$.

• First results were obtained for magnetic fields on Riemannian manifolds.

We give now some general definitions.

• A closed 2-form F on a (complete) Riemannian manifold (M, g) is called a *magnetic field*.

• The *Lorentz force* of a magnetic background (M, g, F) is the skew symmetric $(1,1)-$type tensor field Φ on M satisfying

$$g(\Phi(X), Y) = F(X, Y), \quad \forall X, Y \in \chi(M). \tag{1}$$

• A *trajectory* generated by the magnetic field F is defined as a smooth curve γ on M fulfilling the *Lorentz equation* (or *Newton equation*):

$$\nabla_{\dot\gamma}\dot\gamma = \Phi(\dot\gamma), \tag{2}$$

∇ denotes the Levi Civita connection of g.

In a physical interpretation: the magnetic curve γ describes the trajectory of a charged particle under the action of F in the magnetic background (M, g, F).

• The skew symmetry of Lorentz force yields a basic property of magnetic curves, i.e. the following conservation law: *particles evolve with constant speed (and so with constant energy) along the magnetic trajectories*.

• Property for magnetic curves: $\frac{\mathrm{d}}{\mathrm{d}t} g(\gamma', \gamma') = 0$. In particular, a magnetic curve is called normal if it has unit energy, i.e. $||\gamma'|| = 1$.
In the sequel we study only unit speed curves.

Magnetic curves on surfaces

M -oriented surface,
$d\sigma$-the area element: $d\sigma(X, Y) = 1$ for any p.o. orth. $\{X, Y\}$ on M.

Any magnetic field on the surface M is determined from a smooth function f (the strength) by $F = fd\sigma$.
A parallel magnetic field F, i.e. a magnetic field with constant strength $f = q$, is called uniform magnetic field.

Proposition (Barros et al. 2005)

Let $F = qd\sigma$ be a uniform magnetic field with constant strength q, on a Riemannian surface (M, g). A curve γ, with constant velocity v, hence constant energy, is a magnetic curve of (M, g, F) if and only if it has constant curvature $\kappa = q/v$. □

Uniform magnetic curves on $\mathbb{E}^2$ and $\mathbb{S}^2(r)$

On the Euclidean plane: **circles with radius** $\frac{1}{|q|}$.

On the 2-sphere: **(small) circles with radius** $\frac{r\sqrt{e}}{\sqrt{e+r^2q^2}}$ $(< r)$

(e is the energy)

Uniform magnetic curves on $\mathbb{H}^2(-r)$, $r > 0$

The situation in hyperbolic plane is quite different:
Consider the upper half-plane model for the hyperbolic plane.
The description of the flowlines are due to Comtet, 1987.

(i) if $\frac{|q|}{\sqrt{r}} > 1$: **geodesic circles**, and therefore they are closed curves;

(ii) if $\frac{|q|}{\sqrt{r}} \leq 1$: **non-closed curves** of the upper half-plane; in particular they are tangent to the boundary and they are horocycles when $|q| = \sqrt{r}$.

Landau Hall problem on surfaces

Landau Hall problem on a surface of revolution:

- M. Barros, J.L. Cabrerizo, M. Fernández, A. Romero, *The Gauss-Landau-Hall problem on Riemannian surfaces*, J. Math. Phys. **46** (2005) 11, art. 112905.

The Landau-Hall problem on canal surfaces:

- M.I. Munteanu, *The Landau Hall problem on canal surfaces*, J. Math. Analysis Appl., **414** (2014) 2, 725–733.

Kähler magnetic fields

The problem was extended also for different ambient spaces.
For example, if the ambient is a complex space form, Kähler magnetic fields are studied in

- Adachi, T., *Kähler Magnetic flow for a manifold of constant holomorphic sectional curvature*, Tokyo J. Math. **18**, 473–483, (1995).

In particular, explicit trajectories for Kähler magnetic fields are found in the complex projective space $\mathbb{CP}^n$ in

- Adachi, T., *Kähler Magnetic fields on a complex projective space*, Proc. Japan Acad. **70** Ser. A, 12–13, (1994).

Contact magnetic fields

If the ambient is a contact metric manifold, the fundamental 2-form defines the so-called *contact magnetic field*. Interesting results are obtained when the manifold is Sasakian: for example, the angle between the velocity of a normal magnetic curve and the Reeb vector field is constant. Moreover, explicit description for normal flowlines of the contact magnetic field on a 3-dimensional Sasakian manifold is known:

- Cabrerizo, J.L., Fernández, M. and Gómez, J.S., *On the existence of almost contact structure and the contact magnetic field*, Acta Math. Hungar. **125**, 191–199, (2009).
- Cabrerizo, J.L., Fernández, M. and Gómez, J.S., *The contact magnetic flow in 3D Sasakian manifolds*, J. Phys. A: Math. Theor. **42**, art. 195201, (2009).

See also:

- Inoguchi, J. and Munteanu, M.I., *Periodic magnetic curves in Berger spheres*, Tohoku J. Math. **69** 1, 113–128, (2017).

Killing magnetic curves on Riemannian $3-$manifolds

In the case of a **3-dimensional Riemannian manifold** (M, g), 2-forms and vector fields may be identified via the Hodge star operator $\star$ and the volume form dv_g of the manifold.

Thus, magnetic fields mean **divergence free vector fields**.

In particular, Killing vector fields define an important class of magnetic fields, called Killing magnetic fields.

Killing vector field V:

$$g(\nabla_Y V, Z) + g(\nabla_Z V, Y) = 0$$

Killing vector fields $\longrightarrow$ divergence free vector fields.

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- S.L. Druță Romaniuc and M.I. Munteanu,
  Magnetic curves corresponding to Killing magnetic fields in $\mathbb{E}^3$,
  J. Math. Phys. **52**, 113506 (2011).

- M.I. Munteanu and A.I. Nistor,
  The classification of Killing magnetic curves in $\mathbb{S}^2 \times \mathbb{R}$,
  J. Geom. Phys. **62**, (2012) 170 – 182.

The classification of magnetic curves in $\mathbb{H}^2 \times \mathbb{R}$ was done in:

- A. I. Nistor, *Motion of charged particles in a Killing magnetic field in* $\mathbb{H}^2 \times \mathbb{R}$, Rendiconti Sem Matematico (Universita e Politecnico di Torino) special issue: Geom. Struc. on Riem. Man.-Bari, 73 (2015), 1-2, 161-170.

Review on the classical magnetic fields in $\mathbb{E}^3$

A basis of Killing vector fields on $\mathbb{E}^3$ is given by:

$$\{\partial_x,\ \partial_y,\ \partial_z,\ -y\partial_x + x\partial_y,\ -z\partial_y + y\partial_z,\ z\partial_x - x\partial_z\}$$

Example (1)

The Killing vector field $\xi_0 = \partial_z$ defines a magnetic field $F_0 = dx \wedge dy$ whose magnetic trajectories are helices with axis ∂_z,
$t \longmapsto (x_0 + a\cos t, y_0 + a\sin t, z_0 + bt)$, where $(x_0, y_0, z_0) \in \mathbb{R}^3$ and $a, b \in \mathbb{R}$.

- Barros, M., Cabrerizo, J. L., Fernández, M. and Romero, A., *Magnetic vortex filament flows*, J. Math. Phys. **48**, art. 082904, (2007).
- Barros, M., Romero, A., Cabrerizo, J.L. and Fernández, M., *The Gauss-Landau-Hall problem on Riemannian surfaces*, J. Math. Phys. **46**, art. 112905, (2005).
- Barros, M. and Romero, A., *Magnetic vortices*, EPL **77**, art. 34002, (2007).

Review on the classical magnetic fields in $\mathbb{E}^3$

Example (2)

The Killing vector field $V = -y\partial_x + x\partial_y$ (meaning the rotation around z-axis) defines the the magnetic field $F_V = -(xdx + ydy) \wedge dz$ in $\mathbb{E}^3$ and the corresponding magnetic curves are:

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- Druță-Romaniuc, S.L. and Munteanu, M.I., *Magnetic curves corresponding to Killing magnetic fields in* $\mathbb{E}^3$, J. Math. Phys. **52**, art. 113506, (2011).
- Munteanu, M. I., *Magnetic curves in the Euclidean space: one example, several approaches*, Publ. Inst. Math. (Beograd), **94** (108), 141–150, (2013).

Review on the classical magnetic fields in $\mathbb{E}^3$

Example

Consider the 2-form $F_0 = dx \wedge dy$ and the magnetic field $F_q = qF_0$, where $q \neq 0$ is the magnitude. Denote by Φ_q the Lorentz force, whose action on the vector space $\mathbb{R}^3$ is given by:

$$\Phi_q \partial_x = q\partial_y,\ \Phi_q \partial_y = -q\partial_x,\ \Phi_q \frac{\partial}{\partial z} = 0.$$

Let $\gamma(s) = (x(s), y(s), z(s))$ be such a curve. Solving the Lorentz equation $\gamma'' = \Phi_q(\gamma')$ we obtain

$$\gamma : \begin{cases} x(s) = \frac{u_0}{q}\sin(qs) + \frac{v_0}{q}\cos(qs) + x_0 - \frac{v_0}{q}, \\ y(s) = -\frac{u_0}{q}\cos(qs) + \frac{v_0}{q}\sin(qs) + y_0 + \frac{u_0}{q}, \\ z(s) = w_0 s + z_0, \end{cases} \tag{3}$$

Review on the classical magnetic fields in $\mathbb{E}^3$

The curvature κ and the torsion τ of γ: $\kappa^2 = q^2(1 - w_0^2), \quad \tau = qw_0$.
Even both the curvature κ and the torsion τ depend on the strength q, the ratio $\frac{\tau}{\kappa}$ does not.

(a) If $w_0 = 0$, then the curvature is $\kappa = q$ and the torsion is $\tau = 0$, so the magnetic trajectory is a **horizontal circle**.

(b) If $w_0 = \pm 1$, then $\kappa = 0$ and the magnetic curves are **vertical lines**.

(c) In all the other cases the magnetic curves are **circular helices**.

Remark

A magnetic curve γ given by (3) is **periodic** (in $\mathbb{E}^3$) if and only if $w_0 = 0$, that is when the initial speed is horizontal; i.e. it is a horizontal circle of curvature $\kappa = q$.

Contact metric structures on a 3-torus

• By a *contact manifold* we mean a C^∞ manifold M^{2n+1} together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. In particular, the $(2n+1)$-form $\eta \wedge (d\eta)^n$ is a volume form on M and hence the contact manifold M is orientable.

• The 1-form η is called the *contact form*.

• We denote by $\mathcal{D}$ the *contact distribution*, namely for any point $p \in M$ we consider the subspace $\mathcal{D}_p = \{v \in T_pM \;:\; \eta_p(v) = 0\}$.

• We can define a global vector field ξ, called the *Reeb vector field*, by the conditions $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$, for all X tangent to M.

Contact metric structures on a 3-torus

• A differentiable manifold M^{2n+1} is said to have an *almost contact metric structure* (φ, ξ, η, g) if it admits a field φ of endomorphisms of tangent spaces, a vector field ξ, a 1-form η s.t.

$$\eta(\xi) = 1,\ \varphi^2 = -\mathrm{I} + \eta \otimes \xi,\ \varphi\xi = 0,\ \eta \circ \varphi = 0,$$

and a compatible Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),\ \forall X, Y \in \mathfrak{X}(M^{2n+1}).$$

• *The fundamental 2-form* of the almost contact metric structure (φ, ξ, η, g) is defined as a 2-form Ω on $(M^{2n+1}, \varphi, \xi, \eta, g)$ by

$$\Omega(X, Y) = g(\varphi X, Y), \forall X, Y \in \mathfrak{X}(M^{2n+1}) \tag{3}$$

• If $\Omega = d\eta$, then $(M^{2n+1}, \varphi, \xi, \eta, g)$ is called a *contact metric manifold*.
• A contact metric structure (φ, ξ, η, g) is said to be *K-contact*, if ξ is a Killing vector field with respect to g.

First result

Consider the manifold $\mathbb{R}^3$ with standard coordinates (x, y, z) and the 1-form $\eta_0 = dz - ydx$.

- η_0 is a contact form on $\mathbb{R}^3$, since $\eta_0 \wedge d\eta_0 = dx \wedge dy \wedge dz \neq 0$.
- the characteristic vector field is $\xi_0 = \frac{\partial}{\partial z}$
- the contact distribution $\mathcal{D}$ is spanned by

$$X = \partial_x + y\frac{\partial}{\partial z} , \quad Y = \partial_y .$$

In the following, we take

- the Euclidean metric, which is not compatible with the contact structure,
- the magnetic field $F_q = qd\eta_0 = qF_0 = qdx \wedge dy$.

First result

Consider the flat 3-torus $\mathbb{T}^3 = \mathbb{R}^3 / \mathbb{Z}^3$.
An interesting class of curves are the closed ones which return to the initial position after a certain amount of time. Therefore, we ask:

when the magnetic curve γ, *given by* (3), *induces a closed curve* $[\gamma]$ *on* $\mathbb{T}^3$*?*

With this end in view, we should find (the smallest) $T > 0$ such that

$$\gamma(s+T) - \gamma(s) \in \mathbb{Z}^3, \ \forall s \in \mathbb{R}.$$

We compute

$$\begin{aligned} x(s+T) - x(s) &= \frac{2}{q} \sin \frac{qT}{2} \left[u_0 \cos(qs + \frac{qT}{2}) - v_0 \sin(qs + \frac{qT}{2}) \right], \\ y(s+T) - y(s) &= \frac{2}{q} \sin \frac{qT}{2} \left[u_0 \sin(qs + \frac{qT}{2}) + v_0 \cos(qs + \frac{qT}{2}) \right], \\ z(s+T) - z(s) &= w_0 T. \end{aligned} \tag{4}$$

First result

Theorem

The helix (3) induces a closed curve on the 3-torus $\mathbb{T}^3$ if and only if

(i) either $u_0 = v_0 = 0$, case in which $T = 1$ and $[\gamma]$ is a circle, obtained by the factorization of the z-axis,

(ii) or $w_0 = 0$, case in which the period $T = \frac{2\pi}{|q|}$ and $[\gamma]$ is a closed curve on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$, obtained from a circle in the 2-plane $\mathbb{R}(x, y)$,

(iii) or, in the general case when $w_0 \neq 0$, $u_0^2 + v_0^2 \neq 0$, the following periodicity condition is satisfied

$$\frac{q}{2\pi w_0} \in \mathbb{Q},$$

where $\mathbb{Q}$ denotes the set of rational numbers.

Second result

A more "exotic" example of a contact structure on $\mathbb{R}^3$ is given by the contact form

$$\eta = \cos(pz)\ dx + \sin(pz)\ dy, \tag{4}$$

where $p \in \mathbb{R}$, $p \neq 0$. See e.g. Example 3.2.6 in :

- Blair, D. E., *Riemannian geometry of contact and symplectic manifolds*, Progress in Math. **203**, Birkhäuser, 2002.

- The Reeb vector field is given by

$$\xi = \cos(pz)\ \partial_x + \sin(pz)\ \partial_y, \tag{5}$$

- The contact distribution is spanned by

$$X = -\sin(pz)\ \partial_x + \cos(pz)\ \partial_y\ , \quad Y = \frac{\partial}{\partial z}\ .$$

Second result

This example is essentially different from the previous one.

Since η is invariant under translations in each coordinate by $\frac{2\pi}{p}$, it descends to a contact form on the 3-torus $\mathbb{T}^3 = \mathbb{R}^3/(\frac{2\pi}{p}\mathbb{Z})^3$.

The contact structure η is not regular. More precisely, for each value of z, the Reeb vector field ξ induces a flow on the 2-torus defined by $z = constant$. Depending on the value of z, the flow is a rational or irrational flow on $\mathbb{T}^2$. See Blair's book.

Actually, Blair proved that no torus $\mathbb{T}^{2n+1}$ can carry a regular contact structure.
On the other hand, the contact form η is not K-contact.

Second result

We should define an almost contact structure and a compatible Riemannian metric.
We have η and ξ.

- The $(1,1)$-type tensor field φ maps the contact distribution into itself,

$$\varphi X = \alpha X + \beta Y,\ \varphi Y = -\alpha Y - \frac{\alpha^2+1}{\beta} X\ , \varphi\xi = 0, \qquad (4)$$

where α and β are smooth functions, $\beta \neq 0$.

- The metric g is obtained from the contact condition $d\eta(X,Y) = g(\varphi X, Y)$,

$$g(X,X) = -\tfrac{p\beta}{2}\ ,\ g(X,Y) = \tfrac{p\alpha}{2}\ ,\ g(Y,Y) = -\tfrac{p(\alpha^2+1)}{2\beta}$$
$$g(X,\xi) = 0\ ,\ g(Y,\xi) = 0\ ,\ g(\xi,\xi) = 1. \qquad (5)$$

As g is a Riemannian metric we must have $p\beta < 0$.

Rmk. For $\alpha = 0$, $\beta = \varepsilon$ and $p = -2\varepsilon$, where $\varepsilon = \pm 1$, the metric g is the Euclidean metric on $\mathbb{R}^3$.

Second result

In this section we consider $\alpha = 0$, $\beta = -1$ and $p = 2$, hence we are still working with the Euclidean metric on $\mathbb{R}^3$.
Some of the integral curves of ξ are closed and others are not.

Due to its periodicity, η and all the objects obtained from it, may be defined on the 3-torus $\mathbb{T}^3$.

These properties motivate us to study periodicity of magnetic curves on $\mathbb{T}^3$, associated to contact magnetic field $d\eta$ (restricted to $\mathbb{T}^3$) or, equivalently, to the vector field $\xi = \cos(2z)\partial_x + \sin(2z)\partial_y$.

Remark that ξ is divergence free **but not Killing**.

We are looking for normal magnetic curves corresponding to the magnetic field $F_q = q\Omega$, where $q \in \mathbb{R}$ is called the *strength*.

Second result

Let $\gamma : I \longrightarrow \mathbb{R}^3$, $\gamma(s) = (x(s), y(s), z(s))$ be a curve parametrized by arc length. This means that

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 1. \tag{4}$$

On a contact metric manifold, the Lorentz equation of a magnetic trajectory writes as

$$\nabla_{\dot{\gamma}}\dot{\gamma} = q\varphi\dot{\gamma}, \tag{5}$$

where ∇ is the Levi-Civita connection on the manifold. In our situation, ∇ denotes the flat connection of $\mathbb{E}^3$.

In the coordinates (x, y, z), the endomorphism φ may be expressed as

$$\begin{cases} \varphi\partial_x = \ \sin(2z)\frac{\partial}{\partial z}, \\ \varphi\partial_y = -\cos(2z)\frac{\partial}{\partial z}, \\ \varphi\frac{\partial}{\partial z} = -\sin(2z)\partial_x + \cos(2z)\partial_y. \end{cases} \tag{6}$$

Second result

Thus, the Lorentz equation becomes

$$\begin{cases} \ddot{x} = -q \sin(2z)\dot{z}, \\ \ddot{y} = \ \ q \cos(2z)\dot{z}, \\ \ddot{z} = q\big(\sin(2z)\dot{x} - \cos(2z)\dot{y}\big). \end{cases} \tag{4}$$

We study this system of ordinary differential equations in order to find periodic solutions.

Second result

Theorem

Let $\eta = \cos(2z)dx + \sin(2z)dy$ be a contact form on the flat 3-dimensional torus $\mathbb{T}^3 = \mathbb{R}^3/(\pi\mathbb{Z})^3$ with canonical coordinates (x, y, z). Then, the closed normal magnetic curves associated to the magnetic field $F_q = q d\eta$, $q \neq 0$, belong to the following list:

(i) *the horizontal line in* $\mathbb{R}^3$

$$\gamma(s) = (x_0 \pm s\cos(2z_0), y_0 \pm s\sin(2z_0), z_0)\ ,\ \ x_0, y_0 \in \mathbb{R},$$

if $\tan(2z_0) \in \mathbb{Q}$;

(ii) *any parallel to the* y*-axis of the form*

$$\gamma(s) = \left(x_0, y_0 \pm s, \frac{(2l+1)\pi}{4}\right),\ \ l \in \mathbb{Z}\ ,\ \ x_0, y_0 \in \mathbb{R};$$

Second result

Theorem

(iii) *the helix*

$$\begin{aligned}\gamma(s) \;=\; &\Big(a \pm \tfrac{q}{4\sqrt{\zeta}} \sin 2(z_0 \pm s\sqrt{\zeta}),\\ &b \mp \tfrac{q}{4\sqrt{\zeta}} \cos 2(z_0 \pm s\sqrt{\zeta}),\, z_0 \pm s\sqrt{\zeta}\Big),\end{aligned}$$

where $a, b, z_0 \in \mathbb{R}$, $\zeta = 1 - \frac{1}{4}q^2$, $q \in (-2, 2)$;

(iv) *the special curve*

$$\begin{aligned}\gamma(s) \;=\; &\big((\lambda + \tfrac{q}{2} - \tfrac{q}{B^2})s + E(As + c, B),\\ &-\tfrac{q}{AB^2}\mathrm{dn}(As + c, B), \mathrm{am}(As + c, B)\big),\end{aligned}$$

if the condition

$$\frac{K}{\pi A}\left(\lambda + \frac{q}{2} - \frac{q}{B^2}\right) \in \mathbb{Q}. \tag{4}$$

is satisfied, where A, B *are certain positive constants,* $c \in \mathbb{R}$ *and* $E(u)$ *is a particular function involving elliptic integrals and Jacobi elliptic functions.*

Some references

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- J. Inoguchi, M.I. Munteanu, A.I. Nistor, *Magnetic curves in quasi-Sasakian 3−manifolds*, preprint.
- M.I. Munteanu, A.I. Nistor, *A note on magnetic curves on* $\mathbb{S}^{2n+1}$, C.R. Acad. Sci. Paris, Ser. I **352** (2014) 447–449.
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Thank you for attention !

Supported by: Research project PN-II-RU-TE-2014-4-0004, 272/01.10.2015, CNCS - UEFISCDI, Director: Professor Dorel Fetcu