Contact Magnetic Curves: Classification Results

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Overview of the talk

1. Preliminaries
   - Magnetic curves
   - Slant curves
   - Quasi-Sasakian manifolds
   - 3-dimensional case
2. Classification results for magnetic curves in:
   - Sasakian manifolds
   - Cosymplectic manifolds
   - Quasi-Sasakian manifolds
   - Quasi-Sasakian 3-manifolds

Preliminaries - magnetic curves

• The starting point in the study of magnetic curves was the classical treatment of static magnetic fields (time-independent) in $\mathbb{E}^3$.
• First results were obtained for magnetic fields on Riemannian manifolds.

We give now some general definitions.

• A closed 2-form F on a (complete) Riemannian manifold (M, g) is called a *magnetic field*.
• The *Lorentz force* of a magnetic background (M, g, F) is the skew symmetric $(1, 1)-$type tensor field Φ on M satisfying

$$g(\Phi(X), Y) = F(X, Y), \quad \forall X, Y \in \chi(M). \tag{1}$$

• A *trajectory* generated by the magnetic field F is defined as a smooth curve γ on M fulfilling the *Lorentz equation* (or *Newton equation*):

$$\nabla_{\dot\gamma}\dot\gamma = \Phi(\dot\gamma), \tag{2}$$

∇ denotes the Levi Civita connection of g.

In a physical interpretation: the magnetic curve γ describes the trajectory of a charged particle under the action of F in the magnetic background (M, g, F).

• The skew symmetry of Lorentz force yields a basic property of magnetic curves, i.e. the following conservation law: *particles evolve with constant speed (and so with constant energy) along the magnetic trajectories*.

• Property for magnetic curves: $\frac{\mathrm{d}}{\mathrm{d}t} g(\gamma', \gamma') = 0$. In particular, a magnetic curve is called normal if it has unit energy, i.e. $||\gamma'|| = 1$.
In the sequel we study only unit speed curves.

Magnetic curves and geodesics

• For trivial magnetic field $F = 0$ (the magnetic field is absent) magnetic curves correspond to geodesics of (M, g).

• Geodesics are characterized as critical points of an energy action and so they represent the trajectories for free fall particles (moving only under the influence of the gravity). Magnetic curves of (M, g, F) can be also viewed (at least locally) as the solutions of a variational principle.

• The existence and uniqueness of geodesics remain true for magnetic curves.

Magnetic curves on surfaces

M -oriented surface,
$d\sigma$-the area element: $d\sigma(X, Y) = 1$ for any p.o. orth. $\{X, Y\}$ on M.

Any magnetic field on the surface M is determined from a smooth function f (the strength) by $F = fd\sigma$.
A parallel magnetic field F, i.e. a magnetic field with constant strength $f = q$, is called uniform magnetic field.

Proposition (Barros et al. 2005)

Let $F = qd\sigma$ be a uniform magnetic field with constant strength q, on a Riemannian surface (M, g). A curve γ, with constant velocity v, hence constant energy, is a magnetic curve of (M, g, F) if and only if it has constant curvature $\kappa = q/v$. □

Uniform magnetic curves on $\mathbb{E}^2$ and $\mathbb{S}^2(r)$

On the Euclidean plane: **circles with radius** $\frac{1}{|q|}$.

On the 2-sphere: **(small) circles with radius** $\frac{r\sqrt{e}}{\sqrt{e+r^2q^2}}$ $(< r)$

(e is the energy)

Uniform magnetic curves on $\mathbb{H}^2(-r)$, $r > 0$

The situation in hyperbolic plane is quite different:
Consider the upper half-plane model for the hyperbolic plane.
The description of the flowlines are due to Comtet, 1987.

(i) if $\frac{|q|}{\sqrt{r}} > 1$: **geodesic circles**, and therefore they are closed curves;

(ii) if $\frac{|q|}{\sqrt{r}} \leq 1$: **non-closed curves** of the upper half-plane; in particular they are tangent to the boundary and they are horocycles when $|q| = \sqrt{r}$.

Landau Hall problem on surfaces

Landau Hall problem on a surface of revolution:
M. Barros, J.L. Cabrerizo, M. Fernández, A. Romero,
The Gauss-Landau-Hall problem on Riemannian surfaces,
J. Math. Phys. **46** (2005) 11, art. 112905.

The Landau-Hall problem on canal surfaces:
M.I. Munteanu,
The Landau Hall problem on canal surfaces,
J. Math. Analysis Appl., **414** (2014) 2, 725–733.

Killing magnetic curves on Riemannian $3-$manifolds

In the case of a **3-dimensional Riemannian manifold** (M, g), 2-forms and vector fields may be identified via the Hodge star operator $\star$ and the volume form dv_g of the manifold.

Thus, magnetic fields mean **divergence free vector fields**.
In particular, Killing vector fields define an important class of magnetic fields, called Killing magnetic fields.

Killing vector field V:

$$g(\nabla_Y V, Z) + g(\nabla_Z V, Y) = 0$$

Killing vector fields $\longrightarrow$ divergence free vector fields.

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One can define on M the cross product of two vector fields

$$g(X \times Y, Z) = dv_g(X, Y, Z).$$

V is a Killing vector field on M:

$F_V = dv_g(V, -, -)$ the corresponding Killing magnetic field.

Then, the Lorentz force corresponding to F_V is $\boxed{\Phi(X) = V \times X}$.
Consequently, the Lorentz equation can be written as

$$\nabla_{\gamma'}\gamma' = V \times \gamma'.$$

- S.L. Druţă Romaniuc and M.I. Munteanu, *Magnetic curves corresponding to Killing magnetic fields in $\mathbb{E}^3$*, J. Math. Phys. **52**, 113506 (2011).
- M.I. Munteanu and A.I. Nistor, *The classification of Killing magnetic curves in $\mathbb{S}^2 \times \mathbb{R}$*, J. Geom. Phys. **62**, (2012) 170 – 182.

The classification of magnetic curves in $\mathbb{H}^2 \times \mathbb{R}$ was done in:

- A. I. Nistor, *Motion of charged particles in a Killing magnetic field in* $\mathbb{H}^2 \times \mathbb{R}$, Rend. Sem. Mat. Univ. Politec. Torino, 2016.

Preliminaries - Frenet curves

According to [Blair2002], a curve γ in an $m-$dimensional Riemannian manifold (M, g) is a *Frenet curve of osculating order r*, $r \geq 1$ if there exists an orthonormal frame of dimension r along γ, namely $\{T = \dot{\gamma}, \nu_1, \ldots, \nu_{r-1}\}$, such that

$$\begin{aligned}
&\nabla_T T = \kappa_1 \nu_1, \\
&\nabla_T \nu_1 = -\kappa_1 T + \kappa_2 \nu_2, \\
&\nabla_T \nu_j = -\kappa_j \nu_{j-1} + \kappa_{j+1} \nu_{j+1}, \quad j = 2, \ldots, r-2 \\
&\nabla_T \nu_{r-1} = -\kappa_{r-1} \nu_{r-2},
\end{aligned} \tag{3}$$

where $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ are positive C^∞ functions of s. Moreover, κ_j is called the *j-th curvature of* γ.

D.E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Math. 203, 2002, Birkhäuser, Boston-Basel-Berlin.

Preliminaries - Quasi-Sasakian manifolds

"Sasakian manifolds have often been considered the odd-dimensional analogues of Kaehler manifolds. However, if M^{2n} is a Kaehler manifold, $M^{2n} \times \mathbb{R}$ can be considered an odd-dimensional analogue, but $M^{2n} \times \mathbb{R}$ carries a natural cosymplectic (quasi-Sasakian of rank 1) structure. Thus, in a certain sense, quasi-Sasakian manifolds are better analogues of Kaehler manifolds."

[D.E. Blair, The Theory of Quasi-Sasakian Structures, PhD Thesis, University of Illinois, 1966]

• A differentiable manifold M^{2n+1} is said to have an *almost contact metric structure* (φ, ξ, η, g) if it admits a field φ of endomorphisms of tangent spaces, a vector field ξ, a 1-form η s.t.

$$\eta(\xi) = 1, \; \varphi^2 = -\mathrm{I} + \eta \otimes \xi, \; \varphi\xi = 0, \; \eta \circ \varphi = 0,$$

and a compatible Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$\forall X, Y \in \mathfrak{X}(M^{2n+1})$.

• *The fundamental 2-form* of the almost contact metric structure (φ, ξ, η, g) is defined as a 2-form Ω on $(M^{2n+1}, \varphi, \xi, \eta, g)$ by

$$\Omega(X, Y) = g(\varphi X, Y), \tag{4}$$

for all $X, \; Y \in \mathfrak{X}(M^{2n+1})$.

• An almost contact metric manifold M^{2n+1} is said to be *normal* if the normality tensor

$$S = N_{\varphi}(X, Y) + 2d\eta(X, Y)\xi$$

vanishes, where N_{φ} is the *Nijenhuis tensor field* of φ defined by

$$N_{\varphi}(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y],$$

for any $X, Y \in \mathfrak{X}(M^{2n+1})$.

- A normal almost contact metric manifold with Ω closed is called a *quasi-Sasakian manifold*.
- If $\Omega = d\eta$, then $(M^{2n+1}, \varphi, \xi, \eta, g)$ is called a *contact metric manifold*.
- A normal contact metric manifold is called a *Sasakian manifold*.
- A *cosymplectic manifold* is defined as a normal almost contact metric manifold with both η and Ω closed.

According to [BlairPhD], the types of quasi-Sasakian manifolds range from the case of cosymplectic manifolds, $d\eta = 0$ ($rank = 1$) to the case of contact manifolds, $\eta \wedge (d\eta)^n \neq 0$ ($rank = 2n + 1$).
Recall that the rank of the quasi-Sasakian structure is the rank of the 1-form η, i.e. η has $rank = 2p$ if $(d\eta)^p \neq 0$ and $\eta \wedge (d\eta)^p = 0$, and has $rank = 2p + 1$ if $\eta \wedge (d\eta)^p \neq 0$ and $(d\eta)^{p+1} = 0$, see [BlairPhD].

Recall now the definition of a Kaehler manifold.

An almost complex manifold (B^{2k}, J, g_B) endowed with the almost complex structure J and the Riemannian metric g_B such that

$$g_B(JX, Y) = -g_B(X, JY)$$

is called an almost Hermitian manifold.

If the Nijenhuis tensor field of the structure J vanishes, then the manifold is Hermitian.

A *Kaehler manifold* is a Hermitian manifold with closed fundamental 2-form, defined as

$$\Omega_B(X, Y) = g_B(JX, Y).$$

quasi-Sasakian manifolds as product manifolds

We consider the quasi-Sasakian manifolds which can be written locally as a product of a Sasakian and a Kaeher manifold.

- $(N^{2p+1}, \varphi, \xi, \eta, g_N)$ a Sasakian manifold
- (B^{2k}, J, g_B) a Kaehler manifold
- $M = N^{2p+1} \times B^{2k}$ is the product manifold endowed with the quasi-Sasakian structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g)$ defined as:

$$\tilde{\varphi} = \varphi + J, \quad \tilde{\xi} = (\xi, 0), \quad \tilde{\eta} = \eta, \quad g = g_N + g_B. \tag{4}$$

Let γ be a smooth curve parametrized by arc-length s in the quasi-Sasakian manifold $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g)$ given as

$$\gamma : I \to M, \quad \gamma(s) = (\gamma_N(s), \gamma_B(s)), \tag{5}$$

such that γ_N is a smooth curve in the Sasakian manifold N^{2p+1} and γ_B is a smooth curve in the Kaehler manifold B^{2k}.

Preliminaries - Slant curves

Let $M = (M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold and $\gamma(s)$ a smooth curve in M parametrized by arclength.

• The *contact angle* of γ is defined as the angle $\theta(s) \in [0, \pi]$ made by γ with the trajectories of ξ, that is we have

$$\cos\theta(s) = g(\gamma'(s), \xi).$$

• The curve $\gamma(s)$ in M is said to be a *slant curve* if the contact angle θ is constant.

• Slant curves of contact angle $\pi/2$ are called (almost) *Legendre curves* or *almost contact curves*.

Quasi-Sasakian 3-manifolds

Let M be a quasi-Sasakian 3-manifold. The following statements hold true:

- $\mathrm{rank} M = 1$ if and only if M is cosymplectic.
- There are no quasi-Sasakian 3-manifolds with $\mathrm{rank} M = 2$ ([Blair67]).
- $\mathrm{rank} M = 3$ if and only if η is a contact form on M.

Typical examples of cosymplectic 3-manifolds are: the Euclidean 3-space $\mathbb{E}^3$ and product manifolds $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.

D.E. Blair, *The theory of quasi-Sasakian structures*, J. Differ. Geom. **1** (1967), 331–345.

All the eight model spaces of Thurston geometry admit homogeneous almost contact structure naturally associated to the metric.

In particular, other than Sol_3, the naturally associated almost contact structures are normal.

- space forms and product spaces are cosymplectic.
- The unit 3-sphere $\mathbb{S}^3$, the Heisenberg group Nil_3 and the universal covering $\widetilde{\mathrm{SL}_2\mathbb{R}}$ of the special linear group $\mathrm{SL}_2\mathbb{R}$ equipped with the compatible normal contact metric structure are Sasakian space forms. In particular, Nil_3 is identified with the Sasakian space form $\mathbb{R}^3(-3)$.
- The hyperbolic 3-space $\mathbb{H}^3$ equipped with the compatible normal contact metric structure is a Kenmotsu manifold.
- The space Sol_3 equipped with naturally associated almost contact structure is a non-Sasakian contact metric 3-manifold.

Thus the six model spaces $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil_3, $\widetilde{\mathrm{SL}_2\mathbb{R}}$ are quasi-Sasakian.

Z. Olszak, *Normal almost contact metric manifolds of dimension three*, Ann. Pol. Math. **47** (1986), 41–50.

Olszak studied quasi-Sasakian 3-manifolds and obtained the following fundamental facts.

Proposition Olszak

Let M be an almost contact metric 3-manifold. Then M is quasi-Sasakian if and only if M satisfies

$$(\nabla_X \varphi)Y = \alpha\big(g(X,Y)\xi - \eta(Y)X\big) \tag{6}$$

for some function α satisfying $d\alpha(\xi) = 0$. □

On a quasi-Sasakian 3-manifold, we have

$$\nabla_X \xi = -\alpha\varphi X.$$

Note that on a quasi-Sasakian manifold of arbitrary odd dimension, ξ is a Killing vector field, especially, $\nabla_\xi \xi = 0$.

Contact magnetic fields

$(M^{2n+1}, \varphi, \xi, \eta, g)$: a contact metric manifold
We can define a magnetic field on M^{2n+1} by

$$F_q(X, Y) = q\Omega(X, Y)$$

where q is a real constant.
We call F_q the *contact magnetic field* with the *strength q*.
We assume $q \neq 0$.
The *Lorentz force* ϕ_q associated to the contact magnetic field F_q

$$\phi_q = q\varphi$$

Magnetic curves in Sasakian manifolds

By definition, Sasakian manifolds: quasi-Sasakian manifolds of rank $2n+1$.

The magnetic curves associated to the contact magnetic field defined by the fundamental 2-form of a Sasakian manifold have **maximum order** 3.

Magnetic curves in Sasakian manifolds

Theorem (S.L. Druta-Romaniuc, J. Inoguchi, M.I. Munteanu, A.I.N.)

Let $(N^{2p+1}, \varphi, \xi, \eta, g_N)$ be a Sasakian manifold and consider F_{q_N}, $q_N \neq 0$, the contact magnetic field on N^{2p+1}. Then γ is a normal magnetic curve associated to F_{q_N} in N^{2p+1} if and only if γ belongs to the following list:

a) *geodesics, obtained as integral curves of ξ;*

b) *non-geodesic φ-circles of curvature $\kappa_1 = \sqrt{q^2 - 1}$, for $|q| > 1$, and of constant contact angle $\theta = \arccos \frac{1}{q}$;*

c) *Legendre φ-curves in M^{2n+1} with curvatures $\kappa_1 = |q|$ and $\kappa_2 = 1$, i.e. 1-dimensional integral submanifolds of the contact distribution;*

d) *φ-helices of order 3 with axis ξ, having curvatures $\kappa_1 = |q| \sin \theta$ and $\kappa_2 = |q \cos \theta - 1|$, where $\theta \neq \frac{\pi}{2}$ is the constant contact angle.*

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Magnetic curves in Kaehler manifolds

Magnetic curves in Kaehler manifolds were intensively studied by many authors, e.g.

T. Adachi, *Kähler Magnetic Flow for a Manifold of Constant Holomorphic Sectional Curvature*, Tokyo J. Math. 18 (1995) 2, 473–483.

A. Comtet, *On the Landau levels on the hyperbolic plane*, Ann. of Phys. 173 (1987), 185–209.

D. Kalinin, *Trajectories of charged particles in Kähler magnetic fields*, Rep. Math. Phys., 39 (1997), 299–309.

and it was shown that they are circles, thus, they have **maximum order 2**.

Magnetic curves in cosymplectic manifolds

Cosymplectic manifolds: quasi-Sasakian manifolds of rank 1

Theorem (S.L. Druta-Romaniuc, J. Inoguchi, M.I. Munteanu, A.I.N.)

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a cosymplectic manifold. Then γ is a normal magnetic curve corresponding to the contact magnetic field F_{q_M}, $q_M \neq 0$, on M^{2n+1}, if and only if γ is given by one of the following cases:

a) *geodesics, obtained as integral curves of ξ;*

b) *Legendre circles of curvature $\kappa_1 = |q_M|$;*

c) *φ−helices of order 3, with curvatures $\kappa_1 = |q_M| \sin\theta_M$, $\kappa_2 = |q_M \cos\theta_M|$, where $\theta_M \neq \frac{\pi}{2}$ is the constant contact angle of γ.*

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c) *$\varphi-$helices of order 3, with curvatures $\kappa_1 = |q_M| \sin\theta_M$, $\kappa_2 = |q_M \cos\theta_M|$, where $\theta_M \neq \frac{\pi}{2}$ is the constant contact angle of γ.*

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b) *Legendre circles of curvature $\kappa_1 = |q_M|$;*

c) *$\varphi-$helices of* **order** *3, with curvatures $\kappa_1 = |q_M| \sin \theta_M$, $\kappa_2 = |q_M \cos \theta_M|$, where $\theta_M \neq \frac{\pi}{2}$ is the constant contact angle of γ.*

Magnetic curves in quasi-Sasakian manifolds

Theorem (M.I. Munteanu, A.I.N.)

The magnetic curves corresponding to the contact magnetic field defined by the fundamental 2-form of a quasi-Sasakian manifold of arbitrary dimension have **maximum order 5***.*

Idea of the proof

• $(M = N^{2p+1} \times B^{2k}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g)$ - a quasi-Sasakian manifold
• the fundamental 2-form Ω on M is closed and it defines a *contact magnetic field*,

$$F_q(X, Y) = q\Omega(X, Y), \tag{7}$$

where $X, Y \in \mathfrak{X}(M)$ and q is a real constant called the *strength* of F_q. The Lorentz force ϕ_q associated to F_q has the expression

$$\phi_q = q\tilde{\varphi}.$$

Hence, the curve γ parametrized by arc-length in (5) is a *normal magnetic curve* corresponding to the magnetic field F_q if and only of it is a solution of the *Lorentz equation*:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = q\tilde{\varphi}\dot{\gamma}. \tag{8}$$

Using the product structure of the quasi-Sasakian manifold,

$$\begin{aligned}\nabla_{\dot\gamma}\dot\gamma &= \nabla^N_{\gamma'_N}\gamma'_N + \nabla^B_{\gamma'_B}\gamma'_B \\ &= m_N^2\nabla^N_{\dot\gamma_N}\dot\gamma_N + m_B^2\nabla^B_{\dot\gamma_B}\dot\gamma_B \\ &= m_N^2 q_N\varphi\dot\gamma_N + m_B^2 q_B J\dot\gamma_B.\end{aligned} \tag{9}$$

and combining with the right hand side:

$$\begin{aligned} q\tilde\varphi\dot\gamma &= q(\varphi\gamma'_N + J\gamma'_B) \\ &= m_N q\varphi\dot\gamma_N + m_B qJ\dot\gamma_B.\end{aligned} \tag{10}$$

one gets that the magnetic curves γ_N and γ_B correspond to contact magnetic fields of strengths $q_N = \frac{q}{m_N}$ in the Sasakian manifold N^{2p+1} and $q_B = \frac{q}{m_B}$ in the Kaehler manifold B^{2k}, respectively.
Moreover, γ_N has maximum order 3 on the Sasakian manifold N^{2p+1} γ_B of order 2 on the Kaehler manifold B^{2k}.

Magnetic curves in quasi-Sasakian 3-manifolds

Let γ be a normal magnetic trajectory in a quasi-Sasakian 3-manifold M with respect to the Lorentz force $q\varphi$. Namely, γ satisfies

$$\nabla_{\gamma'}\gamma' = q\ \varphi\gamma'.$$

The first fundamental result is the following one.

Proposition

Every contact magnetic curve on a quasi-Sasakian 3-manifold is a slant curve. $\square$

Curvature and torsion

On an arbitrary oriented Riemannian 3-manifold one can canonically define a *cross product* $\times$ of two vector fields $X, Y \in \mathfrak{X}(M)$:

$$g(X \times Y, Z) = dv_g(X, Y, Z), \text{ for any } Z \in \mathfrak{X}(M),$$

where dv_g denotes the volume form defined by g.

• When M is an almost contact metric 3-manifold, the cross product is given by the formula

$$X \times Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X + \eta(X)\varphi Y.$$

Note that for a unitary vector field X orthogonal to ξ, the basis $\{X, \varphi X, \xi\}$ is considered to be positively oriented. Then we have

$$\xi \times \gamma' = \varphi\gamma'.$$

Curvature and torsion

Take the Frenet frame field (T, N, B) along γ. By definition $T = \gamma'$.

- Hence, the magnetic equation is written as

$$\nabla_{\gamma'}\gamma' = q\xi \times \gamma' = \kappa N.$$

Consequently, we get:

- γ has constant curvature $\kappa = |q| \sin\theta$;
- and the torsion of γ is given by $\tau = \alpha + q\cos\theta$.

Recall that a magnetic curve in a Sasakian (respectively a cosymplectic) manifold is a helix. Unlike these situations, a magnetic curve on a quasi-Sasakian 3-manifold is *not*, in general, a helix.

In order to sustain this remark, we give the following example:

Example

We consider the following quasi-Sasakian 3-manifold introduced by J. Wełyczko, *On Legendre curves in 3-dimensional normal almost contact metric manifolds*, Soochow J. Math. **33**(2007), no. 4, 929-937.

Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0\}$ be a half space. We equip M with a Riemannian metric g defined by

$$g = x^2(dx^2 + dy^2) + \eta \otimes \eta, \text{ where } \eta = dz + 2xdy.$$

Then we can take a global orthonormal frame field

$$e_1 = \frac{1}{x}\frac{\partial}{\partial x}, \; e_2 = \frac{1}{x}\frac{\partial}{\partial y} - 2\frac{\partial}{\partial z}, \; e_3 = \frac{\partial}{\partial z}.$$

The Lie brackets satisfy

$$[e_1, e_2] = -\frac{1}{x^2}e_2 - \frac{2}{x^2}e_3, \; [e_2, e_3] = [e_3, e_1] = 0.$$

We define an endomorphism field φ by

$$\varphi e_1 = e_2, \ \varphi e_2 = -e_1, \ \varphi e_3 = 0$$

and put $\xi = e_3$. Then (φ, ξ, η, g) is an almost contact metric structure on M. One can check that $(M, \varphi, \xi, \eta, g)$ satisfies

$$\nabla_X \xi = \frac{1}{x^2}\, \varphi X,$$

where ∇ is the Levi-Civita connection of g.
Thus, M is a quasi-Sasakian manifold with $\alpha = -1/x^2 < 0$.

Let us classify magnetic curves in Wełyczko's space.

The magnetic equation of Wełyczko's space writes as a system of three second order differential equations:

$$\begin{cases} \dfrac{\dot{x}^2}{x} - \dfrac{5\dot{y}^2}{x} - \dfrac{2\dot{y}\dot{z}}{x^2} + \ddot{x} & = & -q\dot{y} \\ \dfrac{6\dot{x}\dot{y}}{x} + \dfrac{2\dot{x}\dot{z}}{x^2} + \ddot{y} & = & q\dot{x} \\ -10\dot{x}\dot{y} - \dfrac{4\dot{x}\dot{z}}{x} + \ddot{x} & = & -2qx\dot{x}. \end{cases} \tag{11}$$

We denoted by dot $(\cdot)$ the derivative with respect to the arclength parameter s.

Solving (11), we get the following coordinate functions of a contact magnetic curve in M:

$$x(s)^2 = c_0 + 2\sin\theta \int_0^s \cos u(t)dt, \tag{12}$$

where c_0 is a positive constant.

$$y(s) = y_0 + \sin\theta \int_0^s \frac{\sin u(t)}{x(t)} dt, \;\; y_0 \in \mathbb{R}. \tag{13}$$

$$z(s) = z_0 + s\cos\theta - 2\sin\theta \int_0^s \sin u(t)dt, \;\; z_0 \in \mathbb{R}. \tag{14}$$

The key point is to obtain u, which is a solution of the following integro-differential equation:

$$2\cos\theta + \sin\theta \sin u(s) + \big(-q + \dot{u}(s)\big)\Big[c_0 + 2\sin\theta \int_0^s \cos u(t)dt\Big] = 0.$$

Particular case: $u = u_0$(const.)

The coordinate functions of the contact magnetic curve are:
$x(s) = \sqrt{c_0}$,
$y(s) = y_0 + \epsilon \frac{\sin\theta}{\sqrt{c_0}}\ s$,
$z(s) = z_0 + (\cos\theta - 2\epsilon\sin\theta)s$.
Thus, along this magnetic curve, α is constant.
Hence this magnetic curve is a helix.
For $\theta = \frac{\pi}{2}$, that is γ is a Legendre magnetic curve, and for $\epsilon = 1$ we obtain

$$\gamma(s) = \left(\sqrt{c_0}, y_0 + \frac{s}{\sqrt{c_0}}, z_0 - 2s\right).$$

Its strength is $q = \frac{1}{c_0}$.
This magnetic curve is a helix with $\kappa = \tau = 1/c_0 > 0$.

Some references

- S.L. Druţă Romaniuc, J. Inoguchi, M.I. Munteanu, A.I. Nistor, *Magnetic curves in Sasakian manifolds*, J. Nonlinear Math. Phys., **22** (2015) 3, 428–447.
- S.L. Druţă Romaniuc, J. Inoguchi, M.I. Munteanu, A.I. Nistor, *Magnetic curves in cosymplectic manifolds*, Reports on Math. Physics, **78** (2016) 1, 33–48.
- J. Inoguchi, M.I. Munteanu, A.I. Nistor, *Magnetic curves in quasi-Sasakian* 3*−manifolds*, preprint.
- M.I. Munteanu, A.I. Nistor, *A note on magnetic curves on* $\mathbb{S}^{2n+1}$, C.R. Acad. Sci. Paris, Ser. I **352** (2014) 447–449.
- M.I. Munteanu, A.I. Nistor, *On some closed magnetic curves on a* 3*−torus*, Math. Phys. Analysis Geom. **20** (2017)2, art.8.
- A.I. Nistor, *New examples of F-planar curves*, preprint.

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