Constant mean curvature biharmonic surfaces

Dorel Fetcu

Gheorghe Asachi Technical University of Iaşi, Romania

Brest, France, May 2017

Harmonic and biharmonic maps

Let $\varphi : (M,g) \to (N,h)$ be a smooth map.

Energy functional

$$E(\varphi) = E_1(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 v_g$$

Euler-Lagrange equation

$$\begin{aligned}\tau(\varphi) = \tau_1(\varphi) &= \operatorname{trace}_g \nabla d\varphi \\ &= 0\end{aligned}$$

Critical points of E:
harmonic maps

Harmonic and biharmonic maps

Let $\varphi : (M,g) \to (N,h)$ be a smooth map.

Energy functional

$$E(\varphi) = E_1(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 v_g$$

Euler-Lagrange equation

$$\begin{aligned}\tau(\varphi) = \tau_1(\varphi) &= \operatorname{trace}_g \nabla d\varphi \\ &= 0\end{aligned}$$

Critical points of E:
harmonic maps

Bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g$$

Euler-Lagrange equation

$$\begin{aligned}\tau_2(\varphi) &= \Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi \\ &= 0\end{aligned}$$

Critical points of E_2:
biharmonic maps

The biharmonic equation

(Jiang - 1986)

$$\tau_2(\varphi) = \Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0$$

where

$$\Delta^{\varphi} = \operatorname{trace}_g \left(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla}\right)$$

is the rough Laplacian on sections of $\varphi^{-1}TN$

The biharmonic equation

(Jiang - 1986)

$$\tau_2(\varphi) = \Delta^\varphi \tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0$$

where

$$\Delta^\varphi = \operatorname{trace}_g \left(\nabla^\varphi \nabla^\varphi - \nabla^\varphi_\nabla\right)$$

is the rough Laplacian on sections of $\varphi^{-1}TN$

- it is a fourth-order non-linear elliptic equation
- any harmonic map is biharmonic
- a non-harmonic biharmonic map is proper-biharmonic
- a submanifold $i : M \to N$ is a biharmonic submanifolds if the immersion i is biharmonic

Biharmonic submanifolds

Theorem (Balmuş, Montaldo, Oniciuc - 2012)

A submanifold Σ^m in a Riemannian manifold N is biharmonic iff

$$\begin{cases} -\Delta^\perp H + \operatorname{trace} \sigma(\cdot, A_H \cdot) + \operatorname{trace}(R^N(\cdot, H)\cdot)^\perp = 0 \\ \frac{m}{2} \operatorname{grad} |H|^2 + 2 \operatorname{trace} A_{\nabla^\perp_\cdot H}(\cdot) + 2 \operatorname{trace}(R^N(\cdot, H)\cdot)^\top = 0, \end{cases}$$

where $\Delta^\perp$ is the Laplacian in the normal bundle.

CMC biharmonic immersions in $\mathbb{S}^n$

Theorem (Loubeau, Oniciuc - 2016)

Let D be a small disk about the origin in the Euclidean plane $\mathbb{R}^2$ and $\phi : D \to \mathbb{S}^n$ be a CMC proper-biharmonic immersion with mean curvature $h = |H| \in (0,1)$. Then n is odd, $n \geq 5$, and ϕ extends uniquely to a CMC proper-biharmonic immersion of $\mathbb{R}^2$ into $\mathbb{S}^n$.

CMC biharmonic immersions in $\mathbb{S}^n$

Theorem (Loubeau, Oniciuc - 2016)

Let D be a small disk about the origin in the Euclidean plane $\mathbb{R}^2$ and $\phi : D \to \mathbb{S}^n$ be a CMC proper-biharmonic immersion with mean curvature $h = |H| \in (0,1)$. Then n is odd, $n \geq 5$, and ϕ extends uniquely to a CMC proper-biharmonic immersion of $\mathbb{R}^2$ into $\mathbb{S}^n$.

Remark

The above result also gives the explicit expression of

$$\psi = i \circ \phi : \mathbb{R}^2 \to \mathbb{R}^n$$

The structure theorem in $\mathbb{S}^5$

Theorem (Loubeau, Oniciuc - 2016)

For a given $h \in (0,1)$ *there is unique one-parameter family of CMC proper-biharmonic surfaces* $\phi_{h,\rho} = \phi_\rho : \mathbb{R}^2 \to \mathbb{S}^5$ *with mean curvature* h *and* $\rho \in [0, (1/2)\arccos((h-1)/(1+h))]$*.*

The structure theorem in $\mathbb{S}^5$

Theorem (Loubeau, Oniciuc - 2016)

The CMC proper-biharmonic immersion $\phi_{h,\rho} : \mathbb{R}^2 \to \mathbb{S}^5$ quotients to a torus if and only if either

(a) $\rho = 0$ *and*

$$h = \frac{1-b}{1+b},$$

where $b = r^2/t^2$, $r,t \in \mathbb{N}^*$, *with* $r < t$ *and* $(r,t) = 1$; *or*

(b) $\rho \in (0, (1/2)\arccos((h-1)/(1+h))]$ *is a constant depending on* a *and* b *and*

$$h = \frac{1-(a-b)^2}{1+(a-b)^2+2(a+b)},$$

where $a = p^2/q^2$ *and* $b = r^2/t^2$, *with* $p,q,r,t \in \mathbb{N}^*$, *such that* $0 \le b-a < 1$.

The structure theorem in $\mathbb{S}^5$

Theorem (Loubeau, Oniciuc - 2016)

The CMC proper-biharmonic immersion $\phi_{h,\rho} : \mathbb{R}^2 \to \mathbb{S}^5$ quotients to a torus if and only if either

(a) $\rho = 0$ *and*

$$h = \frac{1-b}{1+b},$$

where $b = r^2/t^2$, $r,t \in \mathbb{N}^*$, *with* $r < t$ *and* $(r,t) = 1$; *or*

(b) $\rho \in (0, (1/2)\arccos((h-1)/(1+h))]$ *is a constant depending on* a *and* b *and*

$$h = \frac{1-(a-b)^2}{1+(a-b)^2+2(a+b)},$$

where $a = p^2/q^2$ *and* $b = r^2/t^2$, *with* $p,q,r,t \in \mathbb{N}^*$, *such that* $0 \leq b-a < 1$.

Remark

The corresponding lattices $\Lambda_{\phi_{h,0}}$ *and* $\Lambda_{\phi_{h,\rho}}$ *were also explicitly determined.*

A class of CMC biharmonic rectangular tori

Theorem (F., Loubeau, Oniciuc - 2016)

Consider a rectangular lattice $\Lambda = \{(2\pi k, 2\pi l\theta) : k, l \in \mathbb{Z}\}$ *and the torus* $T^2 = \mathbb{R}^2/\Lambda$*, where* $\theta \in \mathbb{R}_+^*$*. Then* T^2 *admits a proper-biharmonic immersion in* $\mathbb{S}^n$ *with constant mean curvature* $h \in (0,1)$ *iff*

$$\theta^2 = (q_1^2 + q_2^2)/2 \quad \text{and} \quad n \in \{5, 7\},$$

where $q_1, q_2 \in \mathbb{N}$ *and* $q_1 < q_2$*. In this case*

$$h = \frac{q_2^2 - q_1^2}{2(q_1^2 + q_2^2)},$$

with $q_1 \geq 0$ *when* $n = 5$ *and* $q_1 > 0$ *when* $n = 7$*.*

A class of CMC biharmonic rectangular tori

Theorem (F., Loubeau, Oniciuc - 2016)

Consider a rectangular lattice $\Lambda = \{(2\pi k, 2\pi l\theta) : k, l \in \mathbb{Z}\}$ *and the torus* $T^2 = \mathbb{R}^2/\Lambda$*, where* $\theta \in \mathbb{R}^*_+$*. Then* T^2 *admits a proper-biharmonic immersion in* $\mathbb{S}^n$ *with constant mean curvature* $h \in (0,1)$ *iff*

$$\theta^2 = (q_1^2 + q_2^2)/2 \quad \text{and} \quad n \in \{5,7\},$$

where $q_1, q_2 \in \mathbb{N}$ *and* $q_1 < q_2$*. In this case*

$$h = \frac{q_2^2 - q_1^2}{2(q_1^2 + q_2^2)},$$

with $q_1 \geq 0$ *when* $n = 5$ *and* $q_1 > 0$ *when* $n = 7$*.*

Remark

The proof relies on explicitly finding all admissible CMC proper-biharmonic immersions of T^2 *in* $\mathbb{S}^n$*.*

A class of CMC biharmonic rectangular tori

Remark

For $0 < q_1^2 < q_2^2$ the same torus can be immersed in $\mathbb{S}^5$ and $\mathbb{S}^7$ as a CMC proper-biharmonic surface, with the same constant mean curvature.

Remark

The same rectangular torus can be immersed in $\mathbb{S}^5$ (or $\mathbb{S}^7$) as a CMC proper-biharmonic surface in different ways with different mean curvatures.

Remark

A rectangular torus with both sides of length less than $1/\sqrt{2}$ cannot be immersed in a sphere $\mathbb{S}^n$ as a CMC proper-biharmonic surface.

CMC biharmonic square tori in $\mathbb{S}^n$

Theorem (F., Loubeau, Oniciuc - 2016)

Consider a square lattice $\Lambda = \{(2\pi ka, 2\pi la) : k, l \in \mathbb{Z}\}$ *and the torus* $T^2 = \mathbb{R}^2/\Lambda$, *where* $a \in \mathbb{R}^*_+$. *Then we have*

(a) T^2 *admits a proper-biharmonic immersion in* $\mathbb{S}^n$, $n \equiv 3 \pmod 4$, *with constant mean curvature* $h \in (0,1)$ *iff*

$$4a^2 = p_1^2 + q_1^2 + p_2^2 + q_2^2, \quad h = \frac{p_2^2 + q_2^2 - p_1^2 - q_1^2}{p_1^2 + q_1^2 + p_2^2 + q_2^2},$$

and

$$7 \leq n \leq r_2(p_1^2 + q_1^2) + r_2(p_2^2 + q_2^2) - 1,$$

where $r_2(p)$ *is the number of representations of* $p \in \mathbb{N}$ *as the sum of two squares of integers and* $p_1, q_1, p_2, q_2 \in \mathbb{N}$ *such that* $0 < p_1^2 + q_1^2 < p_2^2 + q_2^2$.

(b) *If* $4a^2 = p^2 + q^2$, *where* $p, q \in \mathbb{N}$ *such that* $0 < p < q$, *then* T^2 *admits a CMC proper-biharmonic immersion in* $\mathbb{S}^n$ *with* $h = (q^2 - p^2)/(p^2 + q^2)$ *for any odd* n, $5 \leq n \leq r_2(p^2) + r_2(q^2) - 1$.

CMC biharmonic square tori in $\mathbb{S}^n$

Remark

One obtains that $a \geq \sqrt{3}/2$. *Moreover, if* $a = \sqrt{3}/2$, *the corresponding square torus can be immersed only in* $\mathbb{S}^7$, *in a unique manner.*

Remark

While any positive integer can be written as a sum of four squares (not necessarily satisfying the condition in the theorem), a positive integer can be written as a sum of two squares if and only if each of its prime factors of the form $4p-1$ *occurs with an even power in its prime factorization.*

CMC biharmonic square tori in $\mathbb{S}^n$

As positive integers p and q can be chosen such that $r_2(p^2)+r_2(q^2)$ is arbitrarily large, we have the following

Theorem (F., Loubeau, Oniciuc - 2016)

For any sphere $\mathbb{S}^n$, with n odd, $n \geq 5$, there exists a square torus that can be immersed in $\mathbb{S}^n$ as a CMC proper-biharmonic surface.

Surfaces with parallel mean curvature

Let Σ be a surface of a Riemannian manifold N

$$\nabla^N_X Y = \nabla_X Y + \sigma(X,Y) \quad \text{(Eq. Gauss)}$$

$$\nabla^N_X V = -A_V X + \nabla^\perp_X V \quad \text{(Eq. Weingarten)}$$

Definition

If the mean curvature vector field $H = \frac{1}{2}\operatorname{trace}\sigma$ is parallel in the normal bundle, i.e., $\nabla^\perp H = 0$, then Σ is called a PMC surface.

Curves in complex space forms

- A curve $\gamma : I \subset \mathbb{R} \to N^n(c)$ parametrized by arc-length is called a Frenet curve of osculating order r, $1 \leq r \leq 2n$, if there exist r orthonormal vector fields $\{E_1 = \gamma', \ldots, E_r\}$ along γ such that

$$\nabla^N_{E_1} E_1 = \kappa_1 E_2, \quad \nabla^N_{E_1} E_i = -\kappa_{i-1} E_{i-1} + \kappa_i E_{i+1}, \ldots, \quad \nabla^N_{E_1} E_r = -\kappa_{r-1} E_{r-1}$$

  where the functions $\kappa_i > 0$ are the curvatures of γ
- $\kappa_i = \text{constant} > 0$: helix of order r
  - $r = 2$: circle
  - $r = 3$: helix
- the complex torsions of γ are $\tau_{ij} = \langle E_i, JE_j \rangle$, $1 \leq i < j \leq r$
- a helix of order r is a holomorphic helix if $\tau_{ij} = \text{constant}$

PMC surfaces in complex space forms

Let Σ be a PMC surface in a complex space form $N^n(c)$

- (F. - 2012) The $(2,0)$-part of the quadratic form Q defined on Σ by
$$Q(X,Y) = 8|H|^2\langle A_H X, Y\rangle + 3c\langle X, T\rangle\langle Y, T\rangle,$$
where T is the tangent part of JH, is holomorphic
- consider
$$S = 8|H|^2 A_H + 3c\langle T, \cdot\rangle T - \left(\frac{3c}{2}|T|^2 + 8|H|^4\right)\mathrm{Id}$$
- then
$$\langle SX, Y\rangle = Q(X,Y) - \frac{\operatorname{trace} Q}{2}\langle X, Y\rangle$$
- and (F., Pinheiro - 2015)
$$\frac{1}{2}\Delta|S|^2 = 2K|S|^2 + |\nabla S|^2$$

PMC surfaces in complex space forms

Theorem (F., Pinheiro - 2015)

Let Σ be a complete non-minimal PMC surface with $K \geq 0$ in $N^n(c)$, $c \neq 0$. Then one of the following holds:

1. *the surface is flat;*
2. *there exists a point $p \in \Sigma$ such that $K(p) > 0$ and $Q^{(2,0)}$ vanishes.*

Remark

For a surface Σ as in the above theorem we have $\nabla S = 0$.

The general conditions for biharmonicity

Theorem (Balmuş, Montaldo, Oniciuc - 2012)

A submanifold Σ^m in a Riemannian manifold N is biharmonic iff

$$\begin{cases} -\Delta^{\perp} H + \operatorname{trace} \sigma(\cdot, A_H \cdot) + \operatorname{trace}(R^N(\cdot, H)\cdot)^{\perp} = 0 \\ \frac{m}{2} \operatorname{grad} |H|^2 + 2 \operatorname{trace} A_{\nabla^{\perp}_{\cdot} H}(\cdot) + 2 \operatorname{trace}(R^N(\cdot, H)\cdot)^{\top} = 0, \end{cases}$$

where $\Delta^{\perp}$ is the Laplacian in the normal bundle.

PMC biharmonic surfaces

Proposition

Let Σ be a PMC surface in a complex space form $N^n(c)$. Then Σ is biharmonic iff

$$\operatorname{trace}\sigma(\cdot, A_H\cdot) = \frac{\rho}{4}\left(2H - 3(JT)^\perp\right) \quad \textit{and} \quad (JT)^\top = 0$$

where T is the tangent part of JH.

PMC biharmonic surfaces

Proposition

Let Σ be a PMC surface in a complex space form $N^n(c)$. Then Σ is biharmonic iff

$$\operatorname{trace}\sigma(\cdot, A_H\cdot) = \frac{\rho}{4}\left(2H - 3(JT)^\perp\right) \quad \textit{and} \quad (JT)^\top = 0$$

where T is the tangent part of JH.

Remark

If Σ is a PMC proper-biharmonic surface in $N^n(c)$, then $N^n(c) = \mathbb{C}P^n(c)$, since

$$0 < |A_H|^2 = \frac{c}{4}\left(2|H|^2 + 3|T|^2\right)$$

PMC biharmonic surfaces

Proposition (F., Pinheiro - 2015)

Let Σ be a complete PMC proper-biharmonic surface in $\mathbb{C}P^n(c)$.

- *If $T \neq 0$, then Σ is totally real and $\nabla T = 0$. Moreover, if $K \geq 0$, then $K = 0$ and $\nabla A_H = 0$.*
- *If $T = 0$ and $K \geq 0$, then $n \geq 3$ and Σ is pseudo-umbilical and totally real. Moreover, $|H| = \sqrt{c}/2$.*

The classification theorem

Theorem (F., Pinheiro - 2015)

Let Σ be a complete PMC proper-biharmonic surface with $K \geq 0$ in $\mathbb{C}P^n(c)$. Then Σ is totally real and either

1. *Σ is pseudo-umbilical and $|H| = \sqrt{c}/2$; or*
2. *Σ is a complete Lagrangian PMC proper-biharmonic surface in $\mathbb{C}P^2(c)$; or*
3. *Σ is the product between two particular curves in $\mathbb{C}P^3(c)$, a holomorphic circle and a holomorphic helix of order 4. Moreover, these curves always exist and are unique up to holomorphic isometries.*

References

- D. Fetcu, E. Loubeau, and C. Oniciuc, *Biharmonic tori in spheres*, Differential Geom. Appl., to appear.
- D. Fetcu and A. L. Pinheiro, *Biharmonic surfaces with parallel mean curvature in complex space forms*, Kyoto J. Math. 55 (2015), 837–855.