# On biconservative surfaces

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# Harmonic and biharmonic maps

Let $\varphi : (M, g) \to (N, h)$ be a smooth map.

**Energy functional**

$$E(\varphi) = E_1(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 v_g$$

**Euler-Lagrange equation**

$$\begin{aligned} \tau(\varphi) = \tau_1(\varphi) &= \operatorname{trace}_g \nabla d\varphi \\ &= 0 \end{aligned}$$

Critical points of $E$:
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**Bienergy functional**

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g$$

**Euler-Lagrange equation**

$$\begin{aligned} \tau_2(\varphi) &= -\Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi \\ &= 0 \end{aligned}$$

Critical points of $E_2$:
biharmonic maps

# The biharmonic equation (Jiang, 1986)

$$\tau_2(\varphi) = -\Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0$$

where

$$\Delta^{\varphi} = -\operatorname{trace}_g \left(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla}\right)$$

is the rough Laplacian on sections of $\varphi^{-1}TN$ and

$$R^N(X,Y)Z = \nabla^N_X\nabla^N_Y Z - \nabla^N_Y\nabla^N_X Z - \nabla^N_{[X,Y]}Z$$

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- is a fourth-order non-linear elliptic equation
- any harmonic map is biharmonic
- a non-harmonic biharmonic map is called proper biharmonic
- a submanifold $M$ of a Riemannian manifold $N$ is called a biharmonic submanifold if the immersion $\varphi : M \to N$ is biharmonic ($\varphi$ is harmonic if and only if $M$ is minimal)

# Biconservative submanifolds

- D. Hilbert, 1924, described a symmetric $2$-covariant tensor $S$, associated to a variational problem, conservative at critical points, i.e., $S$ satisfies $\operatorname{div} S = 0$ at these points, and called it the stress-energy tensor

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- P. Baird and J. Eells, 1981; A. Sanini, 1983, used the tensor

$$S = \frac{1}{2}|d\varphi|^2 g - \varphi^* h$$

that satisfies

$$\operatorname{div} S = -\langle \tau(\varphi), d\varphi \rangle,$$

to study harmonic maps, since

$$\varphi = \text{harmonic} \Rightarrow \operatorname{div} S = 0$$

Obviously

$$\varphi : M \to N \quad \text{is an isometric immersion} \Rightarrow \tau(\varphi) = \text{normal} \Rightarrow \operatorname{div} S = 0$$

- G. Y. Jiang, 1987, defined the stress-energy tensor $S_2$ of the bienergy:

$$S_2(X,Y) = \frac{1}{2}|\tau(\varphi)|^2\langle X,Y\rangle + \langle d\varphi, \nabla\tau(\varphi)\rangle\langle X,Y\rangle - \langle d\varphi(X), \nabla_Y\tau(\varphi)\rangle - \langle d\varphi(Y), \nabla_X\tau(\varphi)\rangle$$

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## Remark

*If $\varphi : M \to (N,h)$ is a fixed map, then $E_2$ can be thought as a functional on the set of all Riemannian metrics on $M$. This new functional's critical points are Riemannian metrics determined by $S_2 = 0$.*

### Definition

*A submanifold* $\varphi : M \to N$ *of a Riemannian manifold* $N$ *is called a* ***biconservative submanifold*** *if* $\operatorname{div} S_2 = 0$*, i.e.,* $\tau_2(\varphi)^\top = 0$*.*

### Definition

*A submanifold* $\varphi : M \to N$ *of a Riemannian manifold* $N$ *is called a biconservative submanifold if* $\operatorname{div} S_2 = 0$, *i.e.,* $\tau_2(\varphi)^\top = 0$.

### Theorem (Balmuş-Montaldo-Oniciuc, 2012)

*A submanifold* $M$ *in a Riemannian manifold* $N$, *with second fundamental form* $\sigma$, *mean curvature vector field* $H$, *and shape operator* $A$, *is biharmonic if and only if*

$$\frac{m}{2}\nabla|H|^2 + 2\operatorname{trace} A_{\nabla^\perp_\cdot H}(\cdot) + 2\operatorname{trace}(R^N(\cdot,H)\cdot)^\top = 0$$

*and*

$$\Delta^\perp H + \operatorname{trace}\sigma(\cdot, A_H\cdot) + \operatorname{trace}(R^N(\cdot,H)\cdot)^\perp = 0,$$

*where* $\Delta^\perp$ *is the Laplacian in the normal bundle.*

### Corollary

*If $M$ is a hypersurface in a Riemannian manifold $N$, then $M$ is biharmonic if and only if*

$$2A(\nabla f) + f\nabla f - 2f(\mathrm{Ricci}^N(\eta))^\top = 0$$

*and*

$$-\Delta f - f|A|^2 + f\,\mathrm{Ricci}^N(\eta,\eta) = 0,$$

*where $\eta$ is the unit normal of $M$ in $N$ and $f = \mathrm{trace}\,A$ is the mean curvature function.*

# Biconservative surfaces in $3$-dimensional space forms

### Corollary

*A surface $M^2$ in a space form $N^3(c)$ is biconservative if and only if*

$$A(\nabla f) = -\frac{f}{2}\nabla f.$$

### Remark

*Any CMC surface in $N^3(c)$ is biconservative.*

# Biconservative surfaces in $3$-dimensional space forms

## Theorem (Caddeo-Montaldo-Oniciuc-Piu, 2014)

*Let $M^2$ be a non-CMC biconservative surface in a space form $N^3(c)$. There exists an open subset $U \subset M$ such that, on $U$, the Gaussian curvature $K$ of $M$ satisfies*

$$K = \det A + c = -\frac{3f^2}{4} + c$$

*and*

$$(c-K)\Delta K - |\nabla K|^2 - \frac{8}{3}K(c-K)^2 = 0,$$

*where $\Delta$ is the Laplace-Beltrami operator on $M$.*

## Remark

*For a non-CMC biconservative surface in $N^3(c)$ we have $c - K > 0$.*

# Ricci surfaces

## Definition

*A Riemannian surface $(M^2, g)$ with Gaussian curvature $K$ is said to satisfy the Ricci condition if $c - K > 0$ and the metric $(c-K)^{1/2}g$ is flat, where $c \in \mathbb{R}$ is a constant. In this case, $(M^2, g)$ is called a Ricci surface.*

- G. Ricci-Curbastro, 1895, proved that, when $c = 0$, a surface satisfying the Ricci condition can be locally isometrically embedded in $\mathbb{R}^3$ as a minimal surface
- H. B. Lawson, 1970, generalized this result to CMC surfaces in space forms $N^3(c)$

# Ricci surfaces

Proposition (A. Moroianu-S. Moroianu, 2014)

*Let $(M^2, g)$ be a Riemannian surface such that its Gaussian curvature $K$ satisfies $c - K > 0$, where $c \in \mathbb{R}$ is a constant. Then, the following conditions are equivalent:*

(i) $(c-K)\Delta K - |\nabla K|^2 - 4K(c-K)^2 = 0;$

(ii) $\Delta \log(c-K) + 4K = 0;$

(iii) *the metric* $(c-K)^{1/2} g$ *is flat.*

*Moreover, if $c = 0$, then we also have a fourth equivalent condition:*

(iv) *the metric* $(-K)g$ *has constant Gaussian curvature equal to* $1$.

### Proposition (F.-Nistor-Oniciuc, 2015)

*Let $(M^2, g)$ be a Riemannian surface such that its Gaussian curvature $K$ satisfies $c - K > 0$, where $c \in \mathbb{R}$ is a constant. Then, the following conditions are equivalent:*

(i) $(c-K)\Delta K - |\nabla K|^2 - \frac{8}{3}K(c-K)^2 = 0;$

(ii) $\Delta \log(c-K) + (8/3)K = 0;$

(iii) *the metric* $(c-K)^{3/4} g$ *is flat.*

*Moreover, if $c = 0$, then we also have a fourth equivalent condition:*

(iv) *the metric* $(-K)g$ *has constant Gaussian curvature equal to* $1/3$.

### Theorem (F.-Nistor-Oniciuc, 2015)

*Let* $(M^2, g)$ *be a Riemannian surface with negative Gaussian curvature* $K$ *that satisfies*

$$K\Delta K + |\nabla K|^2 + \frac{8}{3}K^3 = 0.$$

*Then* $(M^2, (-K)^{1/2}g)$ *is a Ricci surface in* $\mathbb{R}^3$*.*

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### Corollary

*Let $(M^2, g)$ be a biconservative surface in $\mathbb{R}^3$, where $g$ is the induced metric on $M$. If $f(p) > 0$ and $(\nabla f)(p) \neq 0$ at any point $p \in M$, where $f$ is the mean curvature function, then $(M^2, (-K)^{1/2}g)$ is a Ricci surface.*

### Theorem (F.-Nistor-Oniciuc, 2015)

*Let $(M^2, g)$ be a biconservative surface in a space form $N^3(c)$ with induced metric $g$ and Gaussian curvature $K$. If $f(p) > 0$ and $(\nabla f)(p) \neq 0$ at any point $p \in M$, where $f$ is the mean curvature function, then, on an open dense set, $(M^2, (c-K)^r g)$ is a Ricci surface in $N^3(c)$, where $r$ is a locally defined function that satisfies*

$$K + \Delta\left(\frac{1}{4}\log(c - K_r) + \frac{r}{2}\log(c-K)\right) = 0,$$

*with the Gaussian curvature $K_r$ of $(c-K)^r g$ given by*

$$K_r = (c-K)^{-r}\left(\frac{3-4r}{3}K + \frac{1}{2}\log(c-K)\Delta r + (c-K)^{-1} g(\nabla r, \nabla K)\right).$$

# The characterization theorem

**Theorem (F.-Nistor-Oniciuc, 2015)**

*Let $(M^2, g)$ be a Riemannian surface and $c \in \mathbb{R}$ a constant. Then $M$ can be locally isometrically embedded in a space form $N^3(c)$ as a biconservative surface with positive mean curvature having the gradient different from zero at any point $p \in M$ if and only if the Gaussian curvature $K$ satisfies $c - K(p) > 0$, $(\nabla K)(p) \neq 0$, and its level curves are circles in $M$ with curvature $\kappa = (3|\nabla K|)/(8(c - K))$.*

## Theorem (F.-Nistor-Oniciuc, 2015)

*Let $(M^2,g)$ be a Riemannian surface with Gaussian curvature $K$ satisfying $(\nabla K)(p)\neq 0$ and $c-K(p)>0$ at any point $p\in M$, where $c\in\mathbb{R}$ is a constant. Let $X_1=(\nabla K)/|\nabla K|$ and $X_2\in C(TM)$ be two vector fields on $M$ such that $\{X_1(p),X_2(p)\}$ is a positively oriented orthonormal basis at any point $p\in M$. If level curves of $K$ are circles in $M$ with constant curvature*

$$\kappa=\frac{3X_1K}{8(c-K)}=\frac{3|\nabla K|}{8(c-K)},$$

*then, for any point $p_0\in M$, there exists a parametrization $X=X(u,s)$ of $M$ in a neighborhood $U\subset M$ of $p_0$ positively oriented such that*

(a) *the curve $u\to X(u,0)$ is an integral curve of $X_1$ with $X(0,0)=p_0$ and $s\to X(u,s)$ is an integral curve of $X_2$, for any $u$;*

(b) $K(u,s)=(K\circ X)(u,s)=(K\circ X)(u,0)=K(u)$;

(c) $g_{11}(u,s)=\frac{9}{64}\left(\frac{K'(u)}{c-K(u)}\right)^2 s^2+1,\ g_{12}(u,s)=-\frac{3K'(u)}{8(c-K(u))}s,\ g_{22}(u,s)=1$;

(d) $24(c-K)K''+33(K')^2+64K(c-K)^2=0$;

(e) $X_1=X_u-g_{12}X_s,\ X_2=X_s,\ \nabla_{X_1}X_1=\nabla_{X_1}X_2=0,\ \nabla_{X_2}X_2=-\frac{3X_1K}{8(c-K)}X_1,\ \nabla_{X_2}X_1=\frac{3X_1K}{8(c-K)}X_2$, *and, therefore, the integral curves of $X_1$ are geodesics.*

## Remark

$$24(c-K)K'' + 33(K')^2 + 64K(c-K)^2 = 0$$

$$\Leftrightarrow$$

$$(c-K)\Delta K - |\nabla K|^2 - \frac{8}{3}K(c-K)^2 = 0$$

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## Remark

$$(u,s)\to\left(u,(c-K)^{3/8}s\right)=(u,v)\Rightarrow g=du^2+(c-K)^{-\frac{3}{4}}dv^2$$

$$(u,v)\to\left(\int_{u_0}^{u}(c-K)^{3/8}du,v\right)=(\tilde{u},\tilde{v})\Rightarrow g=(c-K)^{-\frac{3}{4}}\left(d\tilde{u}^2+d\tilde{v}^2\right)$$

## Theorem

*Let $M^2$ be a surface and $c \in \mathbb{R}$ a constant. Consider a fixed point $p_0 \in M$, a parametrization $X = X(u,s)$ of $M$ on a neighborhood $U \subset M$ of $p_0$ positively oriented, and $K = K(u)$ a function on $M$ such that $K'(u) > 0$ and $c - K(u) > 0$, for any $u$, and*

$$24(c-K)K'' + 33(K')^2 + 64K(c-K)^2 = 0.$$

*Define a Riemannian metric $g = g_{11}du^2 + 2g_{12}duds + g_{22}ds^2$ on $U$ by*

$$g_{11}(u,s) = \frac{9}{64}\left(\frac{K'(u)}{c-K(u)}\right)^2 s^2 + 1, \quad g_{12}(u,s) = -\frac{3K'(u)}{8(c-K(u))}s, \quad g_{22}(u,s) = 1.$$

*Then $K$ is the Gaussian curvature of $g$ and its level curves, i.e., the curves $s \to X(u,s)$, are circles in $M$ with curvature $\kappa = 3K'(u)/(8(c-K(u)))$.*

# PMC biconservative surfaces in $M^n(c)\times\mathbb{R}$

### Theorem

*A submanifold $\Sigma^m$ in a Riemannian manifold $\bar{M}$ is biconservative if and only if*

$$\frac{m}{2}\nabla|H|^2 + 2\,\mathrm{trace}\,A_{\nabla^\perp_{\cdot} H}(\cdot) + 2\,\mathrm{trace}(\bar{R}(\cdot,H)\cdot)^\top = 0.$$

### Definition

If the mean curvature vector field $H$ of a surface $\Sigma^2$ in a Riemannian manifold $\bar{M}$ is parallel in the normal bundle, i.e., $\nabla^\perp H = 0$, then $\Sigma^2$ is called a PMC surface.

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## Remark

*In a space of constant curvature, a PMC submanifold is biconservative.*

## Corollary

*Let $\Sigma^2$ be a non-minimal PMC surface in $\bar{M} = M^n(c)\times\mathbb{R}$, where $M^n(c)$ is either $\mathbb{S}^n$ or $\mathbb{H}^n$. Then $\Sigma^2$ is biconservative if and only if*

$$\langle H, N\rangle T = 0,$$

*where $T$ and $N$ are the tangent and the normal components of $\xi$, respectively.*

## Theorem (F.-Oniciuc-Pinheiro, 2015)

*Let $\Sigma^2$ be a PMC biconservative surface with mean curvature vector field $H$ in $\bar{M}=M^n(c)\times\mathbb{R}$, $c=\pm1$ and $H\neq 0$. Then either*

1. *$\Sigma^2$ either is a minimal surface of an umbilical hypersurface of $M^n(c)$ or it is a CMC surface in a $3$-dimensional umbilical submanifold of $M^n(c)$; or*
2. *$\Sigma^2$ is a vertical cylinder over a circle in $M^2(c)$ with curvature $\kappa=2|H|$; or*
3. *$\Sigma^2$ lies in $\mathbb{S}^4\times\mathbb{R}\subset\mathbb{R}^5\times\mathbb{R}$ and, as a surface in $\mathbb{R}^5\times\mathbb{R}$, is locally given by*

$$X(u,v)=\frac{1}{a}\{C_3+\sin\theta(D_1\cos(au)+D_2\sin(au))\}+(u\cos\theta+b)\xi+\frac{1}{\kappa}(C_1(\cos v-1)+C_2\sin v),$$

*where $\theta\in(0,\pi/2)$ is a constant, $a=\sqrt{1+\sin^2\theta}$, $b$ is a real constant,*

$$\kappa=\sqrt{1+4|H|^2+\sin^2\theta},$$

*$C_1$ and $C_2$ are two constant orthonormal vectors in $\mathbb{R}^5\times\mathbb{R}$ such that $C_1\perp\xi$ and $C_2\perp\xi$, $C_3$ is a unit constant vector such that $\langle C_3,C_1\rangle=a/\kappa\in(0,1)$, $C_3\perp C_2$, and $C_3\perp\xi$, and $D_1$ and $D_2$ are two constant orthonormal vectors in the orthogonal complement of $\operatorname{span}\{C_1,C_2,C_3,\xi\}$ in $\mathbb{R}^5\times\mathbb{R}$.*

### Remark

*There are no PMC biconservative surfaces in $M^3(c)\times\mathbb{R}$, where $c=\pm1$, that do not lie in $M^3(c)$ nor are vertical cylinders.*

### Remark

*Surfaces given by the third case of the theorem lie in the Riemannian product of a small hypersphere of $\mathbb{S}^4$ with $\mathbb{R}$.*

### Remark

*It can be proved that PMC biconservative surfaces in $M^4(c)\times\mathbb{R}$, with $c\neq0$ an arbitrary constant, that are not pseudo-umbilical nor vertical cylinders exist only when $c>0$.*

### Remark

*Surfaces described in the third case of the theorem are not biharmonic.*

# CMC biconservative surfaces in $M^3(c)\times\mathbb{R}$

## Theorem (F.-Oniciuc-Pinheiro, 2015)

*Let $\Sigma^2$ be a CMC biconservative surface in $M^3(c)\times\mathbb{R}$, $c\neq 0$, with mean curvature vector field $H\neq 0$ orthogonal to $\xi$ and $|T|\in(0,1)$. Then $\Sigma^2$ is flat and it is locally given by $X=X(u,v)$, where $X:D\subset\mathbb{R}^2\to M^3(c)\times\mathbb{R}$ is an isometric immersion, $D$ is an open set in $\mathbb{R}^2$, and either*

1. *$\Sigma^2$ is pseudo-umbilical, $c<0$, $|H|^2=-c(1-|T|^2)$, the integral curves of $X_u$ are helices such that $\langle X_u,\xi\rangle=|T|$, with curvatures $\kappa_1^1=|H|$ and $\kappa_2^1=\sqrt{-c}|T|$, and the integral curves of $X_v$ are circles such that $\langle X_v,\xi\rangle=0$, with curvature $\kappa_1^2=|H|$; or*
2. *$|H|^2>-c(1-|T|^2)$ and the integral curves of $X_u$ and $X_v$ are helices in $M^3(c)\times\mathbb{R}$ satisfying*

$$\langle X_u,\xi\rangle=a \quad \text{and} \quad \langle X_v,\xi\rangle=b,$$

*where $a,b\in\mathbb{R}$, $0<a^2+b^2=|T|^2<1$, and $|H|^2+c(1-a^2-b^2)>0$, and with curvatures*

$$\kappa_1^1=|H|+\sqrt{|H|^2+c(1-a^2-b^2)} \quad \text{and} \quad \kappa_2^1=\frac{|a|}{\sqrt{1-a^2-b^2}}\kappa_1^1,$$

$$\kappa_1^2=\left||H|-\sqrt{|H|^2+c(1-a^2-b^2)}\right| \quad \text{and} \quad \kappa_2^2=\frac{|b|}{\sqrt{1-a^2-b^2}}\kappa_1^2,$$

*respectively.*

## Remark

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## Theorem (F.-Oniciuc-Pinheiro, 2015)

*If $\Sigma^2$ is a CMC biharmonic surface in $M^3(c)\times\mathbb{R}$, $c\neq 0$, with mean curvature vector field $H\neq 0$ orthogonal to $\xi$ and $|T|\in(0,1)$, then $c>0$, $b^2>a^2$, and $\Sigma^2$ is one of the non-pseudo-umbilical CMC biconservative surfaces in the previous theorem, with*

$$|H|^2=\frac{c(1-a^2-b^2)(b^2-a^2)^2}{4(1-a^2)(1-b^2)}.$$

# CMC biconservative surfaces in Hadamard manifolds

Theorem (F.-Oniciuc-Pinheiro, 2015)

*Let $\Sigma^2$ be a non-minimal CMC biconservative surface in a Riemannian manifold $\bar{M}$. Then*

$$-\frac{1}{2}\Delta|\phi_H|^2 = 2K|\phi_H|^2 + |\nabla\phi_H|^2,$$

*where $\phi_H = A_H - |H|^2 \operatorname{Id}$ is the traceless part of $A_H$.*

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**Corollary**

*Let $\Sigma^2$ be a CMC biconservative surface in a Riemannian manifold $\bar{M}$ and assume that $\Sigma^2$ is compact and $K \geq 0$. Then $\nabla A_H = 0$ and $\Sigma^2$ is pseudo-umbilical or flat.*

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**Corollary**

*Let $\Sigma^2$ be a non-minimal CMC biconservative surface in a Riemannian manifold $\bar{M}$, with sectional curvature bounded from below by a constant $K_0$, such that $\mu = \sup_{\Sigma^2}(|\sigma|^2 - (1/|H|)^2|A_H|^2) < +\infty$. Then*

$$\Delta|\phi_H| \leq a|\phi_H|^3 + b|\phi_H|,$$

*where $a$ and $b$ are constants depending on $K_0$, $|H|$, and $\mu$.*

### Theorem (F.-Oniciuc-Pinheiro, 2015)

*Let $\Sigma^2$ be a complete non-minimal CMC biconservative surface in a Hadamard manifold $\bar{M}$, with sectional curvature bounded from below by a constant $K_0 < 0$, such that the norm of its second fundamental form $\sigma$ is bounded and*

$$\int_{\Sigma^2} |\phi_H|^2 \, dv < +\infty.$$

*Then the function $u = |\phi_H|$ goes to zero uniformly at infinity. More exactly, there exist positive constants $C_0$ and $C_1$, depending on $K_0$, $|H|$, and $\mu = \sup_{\Sigma^2}(|\sigma|^2 - (1/|H|)^2|A_H|^2)$, and a positive radius $R_{\Sigma^2}$, determined by $C_1 \int_{E(R_{\Sigma^2})} u^2 \, dv \leq 1$, such that*

$$||u||_{\infty,E(2R)} \leq C_0 \int_{\Sigma^2} u^2 \, dv,$$

*for all $R \geq R_{\Sigma^2}$. Moreover, there exist some positive constants $D_0$ and $E_0$, depending on $K_0$, $|H|$, and $\mu$, such that the inequality $\int_{\Sigma^2} u^2 \, dv \leq D_0$ implies*

$$||u||_\infty \leq E_0 \int_{\Sigma^2} u^2 \, dv.$$

### Corollary

*Let $\Sigma^2$ be a complete non-minimal CMC biconservative surface in a $3$-dimensional Hadamard manifold $\bar{M}$, with sectional curvature bounded from below by a constant $K_0 < 0$, such that*

$$\int_{\Sigma^2} |\phi_H|^2 \, dv < +\infty.$$

*Then the function $u = |\phi_H|$ goes to zero uniformly at infinity. More exactly, there exist positive constants $C_0$ and $C_1$, depending on $K_0$ and $|H|$, and a positive radius $R_{\Sigma^2}$, determined by $C_1 \int_{E(R_{\Sigma^2})} u^2 \, dv \leq 1$, such that*

$$||u||_{\infty, E(2R)} \leq C_0 \int_{\Sigma^2} u^2 \, dv,$$

*for all $R \geq R_{\Sigma^2}$. Moreover, there exist some positive constants $D_0$ and $E_0$, depending on $K_0$ and $|H|$, such that the inequality $\int_{\Sigma^2} u^2 \, dv \leq D_0$ implies*

$$||u||_\infty \leq E_0 \int_{\Sigma^2} u^2 \, dv.$$

**Theorem (F.-Oniciuc-Pinheiro, 2015)**

*Let $\Sigma^2$ be a complete non-minimal CMC biconservative surface in a Hadamard manifold $\bar{M}$, with sectional curvature bounded from below by a constant $K_0 < 0$, such that the norm of its second fundamental form $\sigma$ is bounded,*

$$\int_{\Sigma^2} |\phi_H|^2 \, dv < +\infty,$$

*and $|H|^2 > (\mu - 2K_0)/2$, where $\mu = \sup_{\Sigma^2}(|\sigma|^2 - (1/|H|^2)|A_H|^2)$. Then $\Sigma^2$ is compact.*

### Corollary

*Let $\Sigma^2$ be a complete non-minimal CMC biconservative surface in a $3$-dimensional Hadamard manifold $\bar{M}$, with sectional curvature bounded from below by a constant $K_0 < 0$, such that*

$$\int_{\Sigma^2} |\phi_H|^2 \, dv < +\infty,$$

*and $|H|^2 > -K_0$. Then $\Sigma^2$ is compact.*

# References


- D. Fetcu, S. Nistor, and C. Oniciuc, **On biconservative surfaces in $3$-dimensional space forms**, Comm. Anal. Geom., to appear.
- D. Fetcu, C. Oniciuc, and A. L. Pinheiro, **CMC biconservative surfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$**, J. Math. Anal. Appl. 425 (2015), 588–609.