Submanifolds with Parallel Mean Curvature and Biharmonic Submanifolds in Riemannian Manifolds

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Submanifolds with parallel mean curvature

Let Σ^m be a submanifold of a Riemannian manifold N.

$$\nabla^N_X Y = \nabla_X Y + \sigma(X,Y) \quad \text{(Eq. Gauss)}$$

$$\nabla^N_X V = -A_V X + \nabla^\perp_X V \quad \text{(Eq. Weingarten)}$$

Definition

If the mean curvature vector field $H = \frac{1}{m} \operatorname{trace} \sigma$ is parallel in the normal bundle, i.e., $\nabla^\perp H = 0$, then Σ^m is called a pmc submanifold. If $|H| = \text{constant}$, then Σ^m is a cmc submanifold.

Using holomorphic differentials to study pmc surfaces

- 1951 - H. Hopf - any genus zero cmc surface in 3-dimensional Euclidean space is a round sphere
- 1981 - S.-S. Chern - cmc surfaces in 3-dimensional space forms
- 2004, 2005 - U. Abresch, H. Rosenberg - cmc surfaces in $M^2(c) \times \mathbb{R}$, where $M^2(c)$ is a complete simply-connected surface with constant curvature $c \neq 0$
- 2010 - F. Torralbo, F. Urbano - pmc surfaces in $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$
- 2007, 2010 - H. Alencar, M. do Carmo, R. Tribuzy - pmc surfaces in $M^n(c) \times \mathbb{R}$, where $M^n(c)$ is an n-dimensional simply-connected space form

Pmc surfaces in complex space forms

Theorem (F.-2012)

Let Σ^2 be a pmc surface in a complex space form $N^n(c)$. Then the $(2,0)$-part of the quadratic form Q defined on Σ^2 by

$$Q(X,Y) = 8|H|^2\langle\sigma(X,Y),H\rangle + 3c\langle JX,H\rangle\langle JY,H\rangle,$$

is holomorphic.

Reduction of codimension

Theorem (F.-2012)

Let Σ^2 be a non-minimal pmc surface in a complex space form $N^n(c)$, $n \geq 3$, $c \neq 0$. Then, one of the following holds:

1. *Σ^2 is a totally real pseudo-umbilical surface; or*
2. *Σ^2 is not pseudo-umbilical and lies in a complex space form $N^5(c)$.*

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Proof.

- if H is umbilical, then Σ^2 is a totally real pseudo-umbilical surface
- if H is not umbilical, define $(J(\operatorname{Im}\sigma))^\perp = \{(J\sigma(X,Y))^\perp\}$, $(J(T\Sigma^2))^\perp = \{(JX)^\perp\}$, and

$$L = \operatorname{span}\{\operatorname{Im}\sigma \cup (J\operatorname{Im}\sigma)^\perp \cup (JT\Sigma^2)^\perp\}$$

- $\nabla^N(T\Sigma^2 \oplus L) \subset T\Sigma^2 \oplus L$, $J(T\Sigma^2 \oplus L) \subset T\Sigma^2 \oplus L$, $R^N(X,Y)Z \subset T\Sigma^2 \oplus L$
- Eschenburg, Tribuzy - 1993 $\Rightarrow$ Σ^2 lies in $N^5(c)$

□

Pmc 2-spheres in complex space forms

Theorem (F.-2012)

There are no non-minimal pmc 2-spheres with constant Kähler angle in a non-flat complex space form.

Pmc surfaces in $\mathbb{C}P^n \times \mathbb{R}$ and $\mathbb{C}H^n \times \mathbb{R}$

Let $M^n(c)$ be a complex space form.
Consider $N^{2n+1} = M^n(c) \times \mathbb{R}$ and the following tensors on N^{2n+1}:

$$\varphi = J \circ d\pi, \quad \xi = \frac{\partial}{\partial t}, \quad \eta = dt, \quad \text{and} \quad \langle , \rangle_N = \langle , \rangle_M + dt \otimes dt,$$

where $\pi : M^n(c) \times \mathbb{R} \to M^n(c)$ is the projection map and t is the standard coordinate function on the real axis.
Then $(N^{2n+1}, \varphi, \xi, \eta, \langle , \rangle_N)$ is a cosymplectic space form with constant φ-sectional curvature equal to c.

Pmc surfaces in $\mathbb{C}P^n \times \mathbb{R}$ and $\mathbb{C}H^n \times \mathbb{R}$

Theorem (F., Rosenberg - 2014)

If Σ^2 is a pmc surface in a cosymplectic space form $N^{2n+1}(c)$, then the $(2,0)$-part of the quadratic form Q, defined on Σ^2 by

$$Q(X,Y) = 8|H|^2\langle\sigma(X,Y),H\rangle - c|H|^2\eta(X)\eta(Y) + 3c\langle\varphi X,H\rangle\langle\varphi Y,H\rangle,$$

is holomorphic.

Pmc surfaces in $\mathbb{C}P^n \times \mathbb{R}$ and $\mathbb{C}H^n \times \mathbb{R}$

Definition

Let $\gamma : I \subset \mathbb{R} \to M^n$ be a curve parametrized by arc-length in a Riemannian manifold. The curve γ is called a Frenet curve of osculating order r, $1 \leq r \leq n$, if there exist r orthonormal vector fields $\{E_1 = \gamma', \ldots, E_r\}$, along γ, such that

$$\nabla^M_{E_1} E_1 = \kappa_1 E_2, \quad \nabla^M_{E_1} E_i = -\kappa_{i-1} E_{i-1} + \kappa_i E_{i+1}, \quad \nabla^M_{E_1} E_r = -\kappa_{r-1} E_{r-1},$$

for $i \in \{2, \ldots, r-1\}$, where $\{\kappa_1, \ldots, \kappa_{r-1}\}$ are positive functions on I called the curvatures of γ. A Frenet curve of osculating order r is called a helix of order r if $\kappa_i = \text{constant} > 0$, $1 \leq i \leq r-1$. A helix of order 2 is called a circle and a helix of order 3 is simply a helix.

Definition

When γ is a Frenet curve in a complex space form $M^n(c)$ complex torsions are defined by $\tau_{ij} = \langle E_i, JE_j \rangle$, $1 \leq i < j \leq r$. A helix of order r is called a holomorphic helix if its complex torsions are constant.

Pmc surfaces in $\mathbb{C}P^n \times \mathbb{R}$ and $\mathbb{C}H^n \times \mathbb{R}$

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Definition

Let M be a Riemannian manifold and consider the product manifold $M \times \mathbb{R}$. A submanifold Σ^m in $M \times \mathbb{R}$ is called a vertical cylinder over Σ^{m-1} if $\Sigma^m = \pi^{-1}(\Sigma^{m-1})$, where Σ^{m-1} is a submanifold in M.

Pmc surfaces in $\mathbb{C}P^n \times \mathbb{R}$ and $\mathbb{C}H^n \times \mathbb{R}$

Proposition (F., Rosenberg - 2014)

A vertical cylinder $\Sigma^2 = \pi^{-1}(\gamma)$ in $M^n(c) \times \mathbb{R}$ is pmc with vanishing $Q^{(2,0)}$ iff $c < 0$ and γ is a circle in $M^n(c)$ with curvature $\kappa = (1/2)\sqrt{-c(1+3\tau^2)}$ and complex torsion τ.

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Remark

For any positive number κ and for any number τ, such that $|\tau| < 1$, there exists a circle with curvature κ and complex torsion τ in any complex space form. Therefore, for any $c < 0$, circles γ, as in the previous proposition, do exist.

Reduction of codimension in $M^n(c) \times \mathbb{R}$

Theorem (F., Rosenberg - 2014)

Let Σ^2 be a non-minimal pmc surface in $N^{2n+1}(c) = M^n(c) \times \mathbb{R}$, where $M^n(c)$ is a non-flat complex space form with $n \geq 2$. Then one of the following holds:

1. *Σ^2 is a pseudo-umbilical non-minimal totally real pmc surface in $M^n(c)$ and $n \geq 3$; or*
2. *Σ^2 is not pseudo-umbilical in $N^{2n+1}(c)$ and lies in $M^5(c) \times \mathbb{R}$.*

Reduction of codimension in Sasakian space forms

Theorem (F., Rosenberg - 2015)

Let Σ^2 be a non-minimal pmc surface in a Sasakian space form $N^{2n+1}(c)$, with $c \neq 1$. Then one of the following holds:

1. *Σ^2 is an integral pseudo-umbilical surface and $n \geq 3$; or*
2. *Σ^2 is not pseudo-umbilical and lies in $N^{11}(c)$.*

Pmc integral and anti-invariant surfaces

Definition

A submanifold Σ^m of a Sasakian manifold N^{2n+1} is called anti-invariant when $\varphi(T\Sigma^m) \subset N\Sigma^m$ and integral if $\eta(X) = 0$, for all vector fields X tangent to Σ^m. An integral curve is called a Legendre curve.

Remark

Any integral submanifold is anti-invariant.

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Definition

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Remark

Any integral submanifold is anti-invariant.

Theorem (F., Rosenberg - 2015)

If Σ^2 is an anti-invariant pmc surface in a Sasakian space form $N^{2n+1}(c)$, then the $(2,0)$-parts of

$$Q_1(X,Y) = 8\langle\sigma(X,Y),H\rangle - (c-1)\eta(X)\eta(Y)$$

and

$$Q_2(X,Y) = \langle\varphi X,H\rangle\langle\varphi Y,H\rangle + \eta(X)\eta(Y) - \eta(X)\langle\varphi Y,H\rangle - \eta(Y)\langle\varphi X,H\rangle,$$

are holomorphic.

Integral and anti-invariant pmc surfaces

Theorem (F., Rosenberg - 2015)

An integral non-minimal pmc 2-sphere in $N^7(c)$ is a round sphere in a space form $M^3((c+3)/4)$.

Integral and anti-invariant pmc surfaces

Theorem (F., Rosenberg - 2015)

An integral non-minimal pmc 2-sphere in $N^7(c)$ is a round sphere in a space form $M^3((c+3)/4)$.

Theorem (F., Rosenberg - 2015)

There are no anti-invariant non-minimal non-pseudo-umbilical pmc 2-spheres in $N^{2n+1}(c)$.

Simons type formulas and cmc submanifolds

- 1968 - J. Simons - a formula for the Laplacian of the second fundamental form of a submanifold in a Riemannian manifold
  - for a minimal hypersurface Σ^m in $\mathbb{S}^{m+1}$:

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + |A|^2(m - |A|^2) \geq |A|^2(m - |A|^2)$$

Simons type formulas and cmc submanifolds

Theorem (Simons - 1968)

Let Σ^m be a closed minimal submanifold in $\mathbb{S}^n$. Then

$$\int_{\Sigma^m} \left(|A|^2 - \frac{m(n-m)}{2n-2m-1} \right) |A|^2 \geq 0.$$

Simons type formulas and cmc submanifolds

Theorem (Simons - 1968)

Let Σ^m *be a closed minimal submanifold in* $\mathbb{S}^n$. *Then*

$$\int_{\Sigma^m} \left(|A|^2 - \frac{m(n-m)}{2n-2m-1} \right) |A|^2 \geq 0.$$

Corollary

Let Σ^m *be a closed minimal submanifold in* $\mathbb{S}^n$ *with*

$$|A|^2 \leq \frac{m(n-m)}{2n-2m-1}.$$

Then, either Σ^m *is totally geodesic or* $|A|^2 = \frac{m(n-m)}{2n-2m-1}$.

Simons type formulas and cmc submanifolds

- 1969 - K. Nomizu, B. Smyth; 1971 - J. Erbacher; 1973 - B. Smyth - Simons type formulas for pmc submanifolds in a space form
- 1977 - S.-Y. Cheng, S.-T. Yau - a general Simons type equation for operators S, acting on a submanifold of a Riemannian manifold and satisfying $(\nabla_X S)Y = (\nabla_Y S)X$

Simons type formulas in $M^n(c)\times\mathbb{R}$

Theorem (F., Oniciuc, Rosenberg - 2013)

Let Σ^m be a submanifold of $M^n(c)\times\mathbb{R}$. If V is a normal vector field, parallel in the normal bundle, with $\operatorname{trace}A_V=\text{constant}$, then

$$
\begin{aligned}
\frac{1}{2}\Delta|A_V|^2 = & |\nabla A_V|^2 + c\{(m-|T|^2)|A_V|^2 - 2m|A_VT|^2 \\
& + 3(\operatorname{trace}A_V)\langle A_VT,T\rangle + m(\operatorname{trace}(A_NA_V))\langle V,N\rangle - (\operatorname{trace}A_V)^2 \\
& - m(\operatorname{trace}A_V)\langle H,N\rangle\langle V,N\rangle\} \\
& + \sum_{\alpha=m+1}^{n+1}\{(\operatorname{trace}A_\alpha)(\operatorname{trace}(A_V^2A_\alpha)) - (\operatorname{trace}(A_VA_\alpha))^2\},
\end{aligned}
$$

where $\{E_\alpha\}_{\alpha=m+1}^{n+1}$ is a local orthonormal frame field in the normal bundle.

Simons type formulas in $M^n(c) \times \mathbb{R}$

Theorem (F., Rosenberg - 2014)

Let Σ^m be a pmc submanifold of $M^n(c) \times \mathbb{R}$. Then we have

$$\begin{aligned}\frac{1}{2}\Delta|\sigma|^2 = & |\nabla^{\perp}\sigma|^2 + c\Big\{(m-|T|^2)|\sigma|^2 - 2m\sum_{\alpha=m+1}^{n+1}|A_\alpha T|^2 \\ & + 3m\langle\sigma(T,T),H\rangle + m|A_N|^2 - m^2\langle H,N\rangle^2 - m^2|H|^2\Big\} \\ & + \sum_{\alpha,\beta=m+1}^{n+1}\{(\operatorname{trace} A_\beta)(\operatorname{trace}(A_\alpha^2 A_\beta)) + \operatorname{trace}[A_\alpha,A_\beta]^2 \\ & - (\operatorname{trace}(A_\alpha A_\beta))^2\},\end{aligned}$$

where $\{E_\alpha\}_{\alpha=m+1}^{n+1}$ is a local orthonormal frame field in the normal bundle.

Gap theorems in $M^n(c) \times \mathbb{R}$

Theorem (F., Rosenberg - 2014)

Let Σ^m be a complete non-minimal pmc submanifold in $M^n(c) \times \mathbb{R}$, $n > m \geq 3$, $c > 0$. If the angle between H and ξ is constant and

$$|\sigma|^2 + \frac{2c(2m+1)}{m}|T|^2 \leq 2c + \frac{m^2}{m-1}|H|^2,$$

then Σ^m is a totally umbilical cmc hypersurface in $M^{m+1}(c)$.

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then Σ^m is a totally umbilical cmc hypersurface in $M^{m+1}(c)$.

Proof.

- $\{E_{m+1}=H/|H|,E_{m+2},\ldots,E_{n+1}\}$ - orthonormal frame field in the normal bundle and

$$\mathscr{A}=\sum_{\alpha=m+2}^{n+1}A_{E_\alpha}$$

- Simons type formula $\Rightarrow \Delta|\mathscr{A}|^2\geq(m-3)|\mathscr{A}|^2$
- Omori-Yau Maximum Principle $\Rightarrow |\mathscr{A}|=0$

□

Gap theorems in $M^n(c) \times \mathbb{R}$

Theorem (F., Rosenberg - 2014)

Let Σ^2 be a complete non-minimal pmc surface in $M^n(c) \times \mathbb{R}$, $n > 2$, $c > 0$, such that the angle between H and ξ is constant and

$$|\sigma|^2 + 3c|T|^2 \leq 4|H|^2 + 2c \qquad \left(\Leftrightarrow K \geq \frac{1}{2}c|T|^2\right).$$

Then, either

1. *Σ^2 is pseudo-umbilical and lies in $M^n(c)$; or*
2. *Σ^2 is a torus in $M^3(c)$.*

The Abresch-Rosenberg differential

Theorem (Abresch, Rosenberg - '04; Alencar, do Carmo, Tribuzy - '10)

Let Σ^2 be a pmc surface in $M^n(c) \times \mathbb{R}$. Then, the $(2,0)$-part of the quadratic form Q defined by

$$Q(X,Y) = 2\langle A_H X, Y\rangle - c\langle X, \xi\rangle\langle Y, \xi\rangle,$$

is holomorphic.

The Abresch-Rosenberg differential

Proposition (Batista - 2010)

The operator S *defined on a pmc surface* Σ^2 *by*

$$S = \frac{1}{|H|}A_H - \frac{c}{2|H|}\langle T, \cdot\rangle T + \left(\frac{c}{4|H|}|T|^2 - |H|\right)\mathrm{Id},$$

is symmetric, traceless, and satisfies $(\nabla_X S)Y = (\nabla_Y S)X$. *Moreover,* $S = 0$ *iff* $Q^{(2,0)} = 0$.

The Abresch-Rosenberg differential

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$$S = \frac{1}{|H|}A_H - \frac{c}{2|H|}\langle T, \cdot\rangle T + \left(\frac{c}{4|H|}|T|^2 - |H|\right)\mathrm{Id},$$

is symmetric, traceless, and satisfies $(\nabla_X S)Y = (\nabla_Y S)X$*. Moreover,* $S = 0$ *iff* $Q^{(2,0)} = 0$*.*

Theorem (Batista - 2010; F., Rosenberg - 2011)

On a pmc surface Σ^2 *in* $M^n(c) \times \mathbb{R}$ *with Gaussian curvature* K *we have*

$$\frac{1}{2}\Delta|S|^2 = 2K|S|^2 + |\nabla S|^2.$$

Pmc surfaces in $M^n(c) \times \mathbb{R}$

Theorem (Alencar, do Carmo, Tribuzy - ’10; F., Rosenberg - ’11)

Let Σ^2 *be a complete non-minimal pmc surface with* $K \geq 0$ *in* $M^n(c) \times \mathbb{R}$*,* $c \neq 0$. *Then one of the following holds*:

1. $K = 0$;
2. Σ^2 *is a minimal surface of a totally umbilical hypersurface of* $M^n(c)$;
3. Σ^2 *is a cmc surface in a* 3*-dimensional totally umbilical submanifold of* $M^n(c)$;
4. Σ^2 *lies in* $M^4(c) \times \mathbb{R} \subset \mathbb{R}^5 \times \mathbb{R}$*, and there exists a plane* π *such that the level lines of the height function* $p \in \Sigma^2 \to \langle x(p), \xi \rangle$ *are curves lying in planes parallel to* π.

Cmc surfaces with finite total curvature

Definition

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Theorem (Batista, Cavalcante, F. - 2014)

Let Σ^2 be a complete non-minimal cmc surface in $M^2(c) \times \mathbb{R}$ with

$$\int_{\Sigma^2} |S|^2 \, dv < +\infty.$$

Then the function $u = |S|$ goes to zero uniformly at infinity.

Cmc surfaces with finite total curvature

Theorem (Batista, Cavalcante, F. - 2014)

Let Σ^2 be a complete cmc surface in $M^2(c) \times \mathbb{R}$ such that

$$\int_{\Sigma^2} |S|^2 \, dv < +\infty.$$

Then we have

1. *If $c > 0$ and $|H| > \sqrt{c}/2$, then Σ^2 is compact.*
2. *If $c < 0$ and $|H| > (\sqrt{\sqrt{5}+2}/2)\sqrt{-c}$, then Σ^2 is compact.*

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$$\int_{\Sigma^2} |S|^2\, dv < +\infty.$$

Then we have

1. *If $c>0$ and $|H|>\sqrt{c}/2$, then Σ^2 is compact.*
2. *If $c<0$ and $|H|>(\sqrt{\sqrt{5}+2}/2)\sqrt{-c}$, then Σ^2 is compact.*

Proof.

- $K\geq c(1-|T|^2)+|H|^2-\frac{1}{2}|S|^2-\frac{c^2}{16|H|^2}-\frac{|c|}{2\sqrt{2}|H|}|S|$
  $\Rightarrow \limsup_{\Sigma^2} K>0 \Rightarrow K^-$ has compact support $\Rightarrow \int_{\Sigma^2}|K^-|\, dv<+\infty$
- White - 1987: $\int_{\Sigma^2}|K^-|\, dv<+\infty \Rightarrow \int_{\Sigma^2} K^+\, dv<+\infty$
- $K^+\geq k>0$ on $\Sigma^2\setminus\Omega$, where $\Omega=$ compact, $k=\mathrm{constant}(c,|H|)\Rightarrow \mathrm{Vol}(\Sigma^2\setminus\Omega)<+\infty$
- Frensel - 1996: $\Sigma^2=$ complete non-compact surface in $M^n(c)\times\mathbb{R}\Rightarrow \mathrm{Vol}(\Sigma^2)=+\infty$

□

Harmonic and biharmonic maps

Let $\varphi : (M,g) \to (N,h)$ be a smooth map.

Energy functional

$$E(\varphi) = E_1(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 v_g$$

Euler-Lagrange equation

$$\begin{aligned}\tau(\varphi) = \tau_1(\varphi) &= \operatorname{trace}_g \nabla d\varphi \\ &= 0\end{aligned}$$

Critical points of E:
harmonic maps

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Critical points of E:
harmonic maps

Bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g$$

Euler-Lagrange equation

$$\begin{aligned} \tau_2(\varphi) &= -\Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi \\ &= 0 \end{aligned}$$

Critical points of E_2:
biharmonic maps

General biharmonicity results

The biharmonic equation (Jiang, 1986)

$$\tau_2(\varphi) = -\Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0$$

- is a fourth-order non-linear elliptic equation
- any harmonic map is biharmonic
- a non-harmonic biharmonic map is called proper biharmonic
- a submanifold M of a Riemannian manifold N is called a biharmonic submanifold if the immersion $\varphi : M \to N$ is biharmonic (φ is harmonic iff M is minimal)

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Theorem (Balmuş, Montaldo, Oniciuc - 2012)

A submanifold Σ^m in a Riemannian manifold N is biharmonic iff

$$\begin{cases} -\Delta^{\perp} H + \operatorname{trace}\sigma(\cdot, A_H\cdot) + \operatorname{trace}(R^N(\cdot,H)\cdot)^{\perp} = 0 \\ \frac{m}{2}\operatorname{grad}|H|^2 + 2\operatorname{trace} A_{\nabla^{\perp}_{\cdot} H}(\cdot) + 2\operatorname{trace}(R^N(\cdot,H)\cdot)^{\top} = 0. \end{cases}$$

Pmc biharmonic submanifolds in $\mathbb{S}^n \times \mathbb{R}$

Theorem (F., Oniciuc, Rosenberg - 2013)

Let Σ^m be a complete pmc proper-biharmonic submanifold in $\mathbb{S}^n \times \mathbb{R}$, with $m \geq 2$, such that

$$|H|^2 > C(m) = \frac{(m-1)(m^2+4)+(m-2)\sqrt{(m-1)(m-2)(m^2+m+2)}}{2m^3}$$

and $|\sigma|$ is bounded. Then $m < n$, $|H| = 1$, and Σ^m is a minimal submanifold of a small hypersphere $\mathbb{S}^{n-1}(2) \subset \mathbb{S}^n$.

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and $|\sigma|$ is bounded. Then $m < n$, $|H| = 1$, and Σ^m is a minimal submanifold of a small hypersphere $\mathbb{S}^{n-1}(2) \subset \mathbb{S}^n$.

Proof.

- Σ^m is biharmonic iff $\langle H, \xi \rangle = 0 \Rightarrow A_H T = 0$
- $\phi_H = A_H - |H|^2 \,\mathrm{Id}$
- Σ^m is biharmonic $\Rightarrow |\phi_H|^2 = (m - |T|^2)|H|^2 - m|H|^4$
- $\frac{1}{2}\Delta|\phi_H|^2 \geq m|\phi_H|^2\Big(-\frac{m-2}{\sqrt{m(m-1)}}|\phi_H| + 2|H|^2 - |T|^2\Big)$
- Omori-Yau Maximum Principle $\Rightarrow \phi_H = 0 \Rightarrow T = 0$

□

Pmc biharmonic submanifolds in $\mathbb{S}^n \times \mathbb{R}$

Theorem (F., Oniciuc, Rosenberg - 2013)

Let Σ^2 be a pmc proper-biharmonic surface in $\mathbb{S}^n(c) \times \mathbb{R}$. Then either

1. *Σ^2 is a minimal surface of a small hypersphere $\mathbb{S}^{n-1}(2c) \subset \mathbb{S}^n(c)$; or*
2. *Σ^2 is $($an open part of$)$ a vertical cylinder $\pi^{-1}(\gamma)$, where γ is a circle in $\mathbb{S}^2(c)$ with curvature equal to $\sqrt{c}$, i.e., γ is biharmonic in $\mathbb{S}^2(c)$.*

Biharmonic Legendre curves

Let $(N^{2n+1}, \varphi, \xi, \eta, \langle , \rangle)$ be a Sasakian space form with constant φ-sectional curvature c.
Let $\gamma : I \to N$ be a Frenet curve of osculating order r with Frenet frame field $\{E_1 = T = \gamma', E_2, \dots, E_r\}$.

(F., Oniciuc - 2009)

- proper-biharmonic Legendre curves exist only when $c > -3$
- classification results for biharmonic Legendre curves: two cases as $c = 1$ or $c \neq 1$
    - $c \neq 1$: three subcases as $E_2 \parallel \varphi T$, $E_2 \perp \varphi T$, or $\langle E_2, \varphi T \rangle \neq 0, 1,$ or -1
- explicit equations of all biharmonic Legendre curves in $\mathbb{S}^{2n+1}(1)$ and all such curves with $E_2 \parallel \varphi T$ or $E_2 \perp \varphi T$ in $\mathbb{S}^{2n+1}(c)$, $c \geq -3$, $c \neq 1$

Biharmonic Legendre curves in $\mathbb{S}^{2n+1}(1)$

Theorem (F., Oniciuc - 2009)

Let $\gamma : I \to \mathbb{S}^{2n+1}(1)$, $n \geq 2$, be a proper-biharmonic Legendre curve parametrized by arc length. Then the equation of γ in $\mathbb{R}^{2n+2}$ is either

$$\gamma(s) = \frac{1}{\sqrt{2}} \cos\left(\sqrt{2}s\right) e_1 + \frac{1}{\sqrt{2}} \sin\left(\sqrt{2}s\right) e_2 + \frac{1}{\sqrt{2}} e_3,$$

where $\{e_i, \mathscr{I} e_j\}_{i,j=1}^3$ are constant unit vectors orthogonal to one another, or

$$\gamma(s) = \frac{1}{\sqrt{2}} \cos(As) e_1 + \frac{1}{\sqrt{2}} \sin(As) e_2 + \frac{1}{\sqrt{2}} \cos(Bs) e_3 + \frac{1}{\sqrt{2}} \sin(Bs) e_4,$$

where $A = \sqrt{1+\kappa_1}$, $B = \sqrt{1-\kappa_1}$, $\quad \kappa_1 \in (0,1)$ and $\{e_i\}_{i=1}^4$ are constant unit vectors orthogonal to one another, with $\langle e_1, \mathscr{I} e_3\rangle = \langle e_1, \mathscr{I} e_4\rangle = 0$, $\langle e_2, \mathscr{I} e_3\rangle = \langle e_2, \mathscr{I} e_4\rangle = 0$, $A\langle e_1, \mathscr{I} e_2\rangle + B\langle e_3, \mathscr{I} e_4\rangle = 0$.

Biharmonic non-Legendre curves in $N^{2n+1}(1)$

Theorem (F. - 2010)

Let $\gamma : I \to N^{2n+1}(1)$ be a non-Legendre Frenet curve of osculating order r. Then γ is proper-biharmonic if and only if either it is a circle with $\kappa_1 = 1$ or a helix with $\kappa_1^2 + \kappa_2^2 = 1$.

Biharmonic non-Legendre curves in $N^{2n+1}(c)$, $c \neq 1$

(F. - 2010)

- classification results when $\eta(T) = \text{constant}$, $E_2 \parallel \varphi T$, or $E_2 \perp \varphi T$
- explicit equations for proper-biharmonic non-Legendre curves with $E_2 \parallel \varphi T$ or $E_2 \perp \varphi T$ in the generalized Heisenberg group $\mathbb{R}^{2n+1}(-3)$

Biharmonic anti-invariant submanifolds

Theorem (F., Oniciuc - 2009)

Let $(N^{2n+1},\varphi,\xi,\eta,\langle,\rangle)$ be a Sasakian space form with constant φ-sectional curvature c and let $f:\Sigma^r\to N$ be an r-dimensional integral submanifold of N, $1\leq r\leq n$. Consider

$$F:\widetilde{\Sigma}=I\times\Sigma^r\to N,\quad F(t,p)=\phi_t(p)=\phi_p(t),$$

where $I=\mathbb{S}^1$ or $I=\mathbb{R}$ and $\{\phi_t\}_{t\in I}$ is the flow of the vector field ξ. Then $F:(\widetilde{\Sigma},\langle,\rangle_{\widetilde{\Sigma}}=dt^2+f^*\langle,\rangle)\to N$ is a Riemannian immersion which is proper-biharmonic if and only if Σ^r is a proper-biharmonic submanifold of N.

Biharmonic anti-invariant submanifolds

Theorem (F., Oniciuc - 2009)

Let $\widetilde{\Sigma}^2$ be a surface of $N^{2n+1}(c)$ invariant under the flow-action of the characteristic vector field ξ. Then $\widetilde{\Sigma}^2$ is proper-biharmonic iff it is locally given by $x(t,s)=\phi_t(\gamma(s))$, where γ is a proper-biharmonic Legendre curve.

Biharmonic hypersurfaces

Definition

Let $N^{2n+1}(c)$ be a Sasakian space form and $\pi : N^{2n+1}(c) \to \bar{N}^n(c+3)$ be the Boothby-Wang fibration. Consider $\bar{\Sigma}^{m-1}$ a submanifold in $\bar{N}$. Then $\Sigma^m = \pi^{-1}(\bar{\Sigma}^{m-1})$ is called a Hopf cylinder in N.

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(F., Oniciuc - 2009)

- proper-biharmonic Hopf cylinders $\Sigma^{2n} = \pi^{-1}(\bar{\Sigma}^{2n-1})$ exist only when $c > -3$
- non-existence results when $\bar{\Sigma}^{2n-1}$ is a homogeneous real hypersurface of type B, C, D, or E in $\mathbb{C}P^n(c+3)$ (according to Takagi's classification)
- characterization of proper-biharmonic Hopf cylinders Σ^{2n} when $\bar{\Sigma}^{2n-1}$ is of type A, in terms of principal curvatures of $\bar{\Sigma}^{2n-1}$

Biharmonic integral $\mathscr{C}$-parallel submanifolds

Definition

An integral submanifold Σ^m in a Sasakian space form is called $\mathscr{C}$-parallel if $\nabla^\perp\sigma$ is parallel to the characteristic vector field ξ.

Biharmonic integral $\mathscr{C}$-parallel submanifolds

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An integral submanifold Σ^m in a Sasakian space form is called $\mathscr{C}$-parallel if $\nabla^{\perp}\sigma$ is parallel to the characteristic vector field ξ.

(F., Oniciuc -2012)

- a classification result for 3-dimensional biharmonic integral $\mathscr{C}$-parallel submanifolds in $N^7(c)$
- explicit equation of 3-dimensional flat biharmonic integral $\mathscr{C}$-parallel submanifolds in $\mathbb{S}^7(1)$ (they are products of three helices)

Biharmonic submanifolds in complex space forms

(F., Loubeau, Montaldo, Oniciuc - 2010)

- characterization results for:
  - biharmonic submanifolds $\bar{\Sigma}$ in $\mathbb{C}P^n$ with JH parallel or normal to $\bar{\Sigma}$
  - biharmonic Hopf cylinders
  - biharmonic submanifolds of Clifford type
- classification results for biharmonic curves in terms of their curvatures and complex torsions (the complete classification of biharmonic curves in $\mathbb{C}P^2$)

Biharmonic curves in $\mathbb{C}P^2$

Theorem (F., Loubeau, Montaldo, Oniciuc - 2010)

A proper-biharmonic curve in $\mathbb{C}P^2$ is a holomorphic circle or a holomorphic helix of order 4.

Theorem (F., Loubeau, Montaldo, Oniciuc - 2010)

A proper-biharmonic holomorphic helix $\bar{\gamma}$ of order 4 in $\mathbb{C}P^2$ is of class I_3, if $\bar{\tau}_{12} < 0$ and $\bar{\tau}_{23} < 0$, or I_4, if $\bar{\tau}_{12} > 0$ and $\bar{\tau}_{23} > 0$ (according to Adachi-Maeda classification). Conversely, for any

$$\alpha_0 \in (\pi/2, \arccos(-(2-\sqrt{3})/\sqrt{2})) \cup (3\pi/2, \pi + \arccos(-(2-\sqrt{3})/\sqrt{2}))$$

there exist two proper-biharmonic holomorphic helices of order 4 and class I_3 or I_4, with $\bar{\kappa}_1 = \bar{\kappa}_1(\alpha_0)$, $\quad \bar{\kappa}_2 = \bar{\kappa}_2(\alpha_0)$, $\quad \bar{\kappa}_3 = \bar{\kappa}_3(\alpha_0)$.

Pmc surfaces

Let $\bar{\Sigma}^2$ be a pmc surface in a complex space form $N^n(c)$.

- The $(2,0)$-part of the quadratic form Q defined on $\bar{\Sigma}^2$ by

$$Q(X,Y) = 8|H|^2\langle A_H X, Y\rangle + 3c\langle X,T\rangle\langle Y,T\rangle,$$

  where T is the tangent part of JH, is holomorphic
- consider

$$S = 8|H|^2 A_H + 3c\langle T,\cdot\rangle T - \left(\frac{3c}{2}|T|^2 + 8|H|^4\right)\mathrm{Id}$$

- $$\langle SX,Y\rangle = Q(X,Y) - \frac{\operatorname{trace} Q}{2}\langle X,Y\rangle$$

- $$\frac{1}{2}\Delta|S|^2 = 2K|S|^2 + |\nabla S|^2$$

Pmc surfaces

Theorem (F., Pinheiro - 2015)

Let $\bar{\Sigma}^2$ be a complete non-minimal pmc surface with $K \geq 0$ in $N^n(c)$, $c \neq 0$. Then one of the following holds:

1. *the surface is flat;*
2. *there exists a point $p \in \bar{\Sigma}^2$ such that $K(p) > 0$ and $Q^{(2,0)}$ vanishes on $\bar{\Sigma}^2$.*

Remark

For a surface $\bar{\Sigma}^2$ as in the above theorem we have $\nabla S = 0$.

Pmc biharmonic surfaces

Proposition (F., Pinheiro - 2015)

If $\bar{\Sigma}^2$ is a pmc biharmonic surface in $N^n(c)$, then $N^n(c) = \mathbb{C}P^n(c)$.

Proposition (F., Pinheiro - 2015)

Let $\bar{\Sigma}^2$ be a complete pmc proper-biharmonic surface in $\mathbb{C}P^n(c)$.

- *If $T \neq 0$, then $\bar{\Sigma}^2$ is totally real and $\nabla T = 0$. Moreover, if $K \geq 0$, then $K = 0$ and $\nabla A_H = 0$.*
- *If $T = 0$ and $K \geq 0$, then $n \geq 3$ and $\bar{\Sigma}^2$ is pseudo-umbilical and totally real. Moreover, $|H| = \sqrt{c}/2$.*

The classification of pmc biharmonic surfaces

Theorem (F., Pinheiro - 2015)

Let $\bar{\Sigma}^2$ be a complete pmc proper-biharmonic surface with $K \geq 0$ in $\mathbb{C}P^n(c)$. Then $\bar{\Sigma}^2$ is totally real and either

1. *$\bar{\Sigma}^2$ is pseudo-umbilical and $|H| = \sqrt{c}/2$; or*
2. *$\bar{\Sigma}^2$ is a complete Lagrangian pmc proper-biharmonic surface in $\mathbb{C}P^2(c)$; or*
3. *$\bar{\Sigma}^2$ is the product between a holomorphic circle and a holomorphic helix of order 4 in $\mathbb{C}P^3(c)$. Moreover, such curves always exist and are unique up to holomorphic isometries.*

The classification of pmc biharmonic surfaces (proof of the third case)

- $JH \neq$ tangent ($|T| < |H|$) $\Rightarrow$
  $L = \mathrm{span}\left\{E_3 = JE_1, E_4 = JE_2, E_5 = \frac{1}{|N|}JN, E_6 = \frac{1}{|N|}N\right\} \subset N\bar{\Sigma}^2$ is parallel,
  $J(T\bar{\Sigma}^2 \oplus L) = T\bar{\Sigma}^2 \oplus L$ (where $N = (JH)^{\perp}$)
  Eschenburg, Tribuzy - 1993 $\Rightarrow \bar{\Sigma}^2$ lies in $\mathbb{C}P^3(c)$
- $\bar{\Sigma}^2 =$ totally real and proper-biharmonic, Ricci eq., $K = 0 \Rightarrow |H| = \frac{\sqrt{c}}{3}$ and

$$A_3 = \frac{1}{2}\sqrt{\frac{c}{3}}\begin{pmatrix} -\frac{11}{3} & 0 \\ 0 & 1 \end{pmatrix}, \quad A_4 = \frac{1}{2}\sqrt{\frac{c}{3}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_5 = -\frac{1}{2}\sqrt{\frac{5c}{3}}\begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}, \quad A_6 = 0 \qquad (1)$$

- consider $\{E_1 = T/|T|, E_2\}$ a global orthonormal frame on $\bar{\Sigma}^2$
- $\nabla E_1 = \nabla E_2 = 0$, de Rham Decomposition Theorem $\Rightarrow \bar{\Sigma}^2 = \gamma_1 \times \gamma_2$
- (1) and Adachi, Maeda classification of holomorphic helices $\Rightarrow$ case (3)