Biconservative surfaces in space forms

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Conferinţa Internaţională a Şcoliilor Doctorale din cadrul
Universităţii "Alexandru Ioan Cuza" din Iaşi
16 Decembrie, 2015

Harmonic and biharmonic maps

Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map. We consider

- Energy functional

$$E(\varphi) = E_1(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 v_g$$

- Euler-Lagrange equation

$$\begin{aligned} \tau(\varphi) &= \tau_1(\varphi) = \operatorname{trace}_g \nabla d\varphi \\ &= 0. \end{aligned}$$

The critical points of E are called harmonic maps.

Harmonic and biharmonic maps

$$\tau(\varphi) = g^{ij}\left(\frac{\partial^2\varphi^\alpha}{\partial x^i \partial x^j} - {}^M\Gamma_{ij}^k \frac{\partial \varphi^\alpha}{\partial x^k} + {}^N\Gamma_{\beta\sigma}^\alpha \frac{\partial \varphi^\beta}{\partial x^i}\frac{\partial \varphi^\sigma}{\partial x^j}\right)\left(\varphi^* \frac{\partial}{\partial y^\alpha}\right) = 0$$

$$\Updownarrow$$

$$g^{ij}\left(\frac{\partial^2\varphi^\alpha}{\partial x^i \partial x^j} - {}^M\Gamma_{ij}^k \frac{\partial \varphi^\alpha}{\partial x^k} + {}^N\Gamma_{\beta\sigma}^\alpha \frac{\partial \varphi^\beta}{\partial x^i}\frac{\partial \varphi^\sigma}{\partial x^j}\right) = 0,$$

where ${}^M\Gamma_{ij}^k$ are the Christoffel symbols of the domain metric g.

Harmonic and biharmonic maps

We consider

- Bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g$$

- Euler-Lagrange equation

$$\begin{aligned}\tau_2(\varphi) &= -\Delta^\varphi \tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi \\ &= 0.\end{aligned}$$

Critical points of E_2 are called biharmonic maps.

The biharmonic equation (G.Y. Jiang, 1986)

$$\tau_2(\varphi) = -\Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0,$$

where

$$\Delta^{\varphi} = -\operatorname{trace}_g\left(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla}\right)$$

is the rough Laplacian on sections of $\varphi^{-1}TN$ and

$$R^N(X,Y)Z = \nabla^N_X\nabla^N_Y Z - \nabla^N_Y\nabla^N_X Z - \nabla^N_{[X,Y]}Z.$$

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- is a fourth-order non-linear elliptic equation;
- any harmonic map is biharmonic;
- a non-harmonic biharmonic map is called proper biharmonic;
- a submanifold $\varphi : M \to N$ of a Riemannian manifold N is called biharmonic if the map $\varphi : M \to N$ is biharmonic (φ is a harmonic map if and only if M is minimal).

Stress-energy tensor

- D. Hilbert, 1924, described a symmetric 2-covariant tensor S, associated to a variational problem, conservative at critical points, i.e., S satisfies $\operatorname{div} S = 0$ at these points, and called it the stress-energy tensor.

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- P. Baird and J. Eells, 1981; A. Sanini, 1983, used the tensor

$$S = \frac{1}{2}|d\varphi|^2 g - \varphi^* h$$

that satisfies

$$\operatorname{div} S = -\langle \tau(\varphi), d\varphi \rangle$$

to study harmonic maps, since

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When the map is a submersion, $\operatorname{div} S = 0$ if and only if φ is harmonic.

Stress-energy tensor

Obviously, if $\varphi : M \to N$ is any isometric immersion (not necessarily minimal) then $\operatorname{div} S = 0$, since $\tau(\varphi)$ is normal.

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It is not interesting to study isometric immersions with $\operatorname{div} S = 0$.

Stress-bienergy tensor

- G.Y. Jiang, 1987, defined the stress-energy tensor S_2 of the bienergy:

$$\begin{aligned} S_2(X,Y) =& \frac{1}{2}|\tau(\varphi)|^2\langle X,Y\rangle + \langle d\varphi,\nabla\tau(\varphi)\rangle\langle X,Y\rangle \\ & - \langle d\varphi(X),\nabla_Y\tau(\varphi)\rangle - \langle d\varphi(Y),\nabla_X\tau(\varphi)\rangle, \end{aligned}$$

that satisfies

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\end{aligned}
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that satisfies

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When the map is a submersion, $\operatorname{div} S_2 = 0$ if and only if φ is biharmonic.

If $\varphi : M \to N$ is an isometric immersion, then $(\operatorname{div} S_2)^\sharp = -\tau_2(\varphi)^\top$. In general, for an isometric immersion $\operatorname{div} S_2 \neq 0$.

Definition

A submanifold $\varphi : M \to N$ of a Riemannian manifold N is called a ***biconservative submanifold*** *if $\operatorname{div} S_2 = 0$, i.e. $\tau_2(\varphi)^\top = 0$.*

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Theorem (Loubeau-Montaldo-Oniciuc, 2008)

A submanifold M^m *in a Riemannian manifold* N^n, *with second fundamental form* B, *mean curvature vector field* H, *and shape operator* A, *is biharmonic if and only if*

$$\frac{m}{2} \operatorname{grad} |H|^2 + 2 \operatorname{trace} A_{\nabla^\perp_\cdot H}(\cdot) + 2 \operatorname{trace}(R^N(\cdot, H)\cdot)^\top = 0$$

and

$$\Delta^\perp H + \operatorname{trace} B(\cdot, A_H \cdot) + \operatorname{trace}(R^N(\cdot, H)\cdot)^\perp = 0,$$

where $\Delta^\perp$ *is the Laplacian in the normal bundle and* R^N *is the curvature tensor of* N.

The case of biconservative hypersurfaces

Theorem

If M^m is a hypersurface in a Riemannian manifold N^{m+1}, then M is biharmonic if and only if

$$2A(\operatorname{grad} f) + f \operatorname{grad} f - 2f(\operatorname{Ricci}^N(\eta))^\top = 0$$

and

$$\Delta f + f|A|^2 - f \operatorname{Ricci}^N(\eta, \eta) = 0,$$

where η is a unit normal vector field of M in N and $f = \operatorname{trace} A$ is the mean curvature function.

Let M^2 be a surface in a three-dimensional space form $N^3(c)$ of constant sectional curvature c. Then M^2 is a biconservative surface in $N^3(c)$ if and only if

$$A(\operatorname{grad} f) = -\frac{f}{2}\operatorname{grad} f.$$

Remark

Any CMC surface in $N^3(c)$ is biconservative.

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For simplicity, we restrict ourselves to biconservative surfaces with $\operatorname{grad} f \neq 0$ at any point of M and $f > 0$. We note that there are examples of complete biconservative surfaces with f positive and $\operatorname{grad} f \neq 0$ at any point of an open dense subset of M.

The explicit parametric equations of the biconservative surfaces in $\mathbb{R}^3$, in $\mathbb{S}^3$ and in $\mathbb{H}^3$ were determined in the article
"Surfaces in three-dimensional space forms with divergence-free stress-bienergy tensor",
by R. Caddeo, S. Montaldo, C. Oniciuc, and P. Piu.
Our aim is to study the biconservative surfaces from *intrinsic point of view*.

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Our aim is to study the biconservative surfaces from *intrinsic point of view*.

More precisely, we will try to find some properties of an abstract Riemannian surface which admits a biconservative immersion in $N^3(c)$. Then, we will determine the necessary and sufficient conditions such that an abstract Riemannian surface to admit a biconservative immersion in $N^3(c)$.

The link between the biconservative surfaces and Ricci surfaces

Theorem (Caddeo-Montaldo-Oniciuc-Piu, 2014)

Let M^2 be a biconservative surface in a space form $N^3(c)$. Assume that $f(p) > 0$ and $(\operatorname{grad} f)(p) \neq 0$ at any point p of M. Then the Gaussian curvature K of M satisfies the extrinsic condition

$$K = \det A + c = -\frac{3f^2}{4} + c$$

and the intrinsic condition

$$(c-K)\Delta K - |\operatorname{grad} K|^2 - \frac{8}{3} K (c-K)^2 = 0,$$

where Δ is the Laplace-Beltrami operator on M.

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where Δ is the Laplace-Beltrami operator on M.

Remark

We have $c - K(p) > 0$ at any point p of M.

Ricci surfaces

Definition

An abstract Riemannian surface (M^2, g) with Gaussian curvature K is said to satisfy the Ricci condition if $c - K > 0$ and the metric $(c-K)^{1/2}g$ is flat, where $c \in \mathbb{R}$ is a constant. In this case, (M^2, g) is called a Ricci surface.

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- G. Ricci-Curbastro, 1895, proved that, when $c = 0$, a surface satisfying the Ricci condition can be locally isometrically embedded in $\mathbb{R}^3$ as a minimal surface. More precisely, we have:

Theorem

Let (M^2, g) *be an abstract Riemannian surface and suppose there exists a minimal immersion* $\psi : (M^2, g) \to \mathbb{R}^3$*. Then* $K \leq 0$ *and* $K = 0$ *only at isolated points. Moreover, away from the points where* $K = 0$*, we have:*

$$\Delta \log(-K) + 4K = 0 (\Leftrightarrow (-K)^{1/2} g \text{ is flat}).$$

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Theorem (Ricci)

Let (M^2, g) *be a Riemannian surface such that* $K < 0$ *and*

$$\Delta \log(-K) + 4K = 0.$$

Then, locally, there exists a one-parameter family of minimal emmbedings $\psi_\theta : (M^2, g) \to \mathbb{R}^3$*, where* $\theta \in \mathbb{S}^1$*.*

- H. B. Lawson, 1970, generalized the result to surfaces in space forms $N^3(c)$, i.e. a surface satisfying the Ricci condition can be locally isometrically embedded in $N^3(c)$ as a minimal surface.

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Proposition (A. Moroianu-S. Moroianu, 2014)

Let (M^2, g) be a Riemannian surface such that its Gaussian curvature K satisfies $c - K > 0$, where $c \in \mathbb{R}$ is a constant. Then, the following conditions are equivalent:

(a) $\Delta \log(c - K) + 4K = 0$;

(b) *the metric $(c - K)^{1/2} g$ is flat;*

(c) $(c - K)\Delta K - |\operatorname{grad} K|^2 - 4K(c - K)^2 = 0$.

Moreover, if $c = 0$, then we also have a fourth equivalent condition:

(d) *the metric $(-K)g$ has constant Gaussian curvature equal to 1.*

Proposition (Fetcu-N.-Oniciuc, 2015)

Let (M^2, g) be a Riemannian surface such that its Gaussian curvature K satisfies $c-K>0$, where $c\in\mathbb{R}$ is a constant. Then, the following conditions are equivalent:

(a) $(c-K)\Delta K - |\operatorname{grad} K|^2 - \frac{8}{3}K(c-K)^2 = 0$ *;*

(b) $\Delta \log(c-K) + (8/3)K = 0$ *;*

(c) *the metric* $(c-K)^{3/4}g$ *is flat.*

Moreover, if $c=0$, then we also have a fourth equivalent condition:

(d) *the metric* $(-K)g$ *has constant Gaussian curvature equal to* $1/3$*.*

Theorem

Let (M^2, g) be an abstract Riemannian surface with negative Gaussian curvature K that satisfies

$$K\Delta K + |\operatorname{grad} K|^2 + \frac{8}{3}K^3 = 0. \tag{1}$$

Then $(M^2, (-K)^{1/2}g)$ is a Ricci surface with $c = 0$.

Theorem

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Then $\left(M^2, (-K)^{1/2}g\right)$ *is a Ricci surface with* $c = 0$*.*

Corollary

Let (M^2, g) *be a biconservative surface in* $\mathbb{R}^3$*, where* g *is the induced metric on* M*. If* $f(p) > 0$ *and* $(\operatorname{grad} f)(p) \neq 0$ *at any point* $p \in M$*, where* f *is the mean curvature function, then* $\left(M^2, (-K)^{1/2}g\right)$ *is a Ricci surface with* $c = 0$*.*

Theorem

Let (M^2,g) be a biconservative surface in a space form $N^3(c)$ with induced metric g and Gaussian curvature K. If $f(p)>0$ and $(\mathrm{grad} f)(p)\neq 0$ at any point $p\in M$, where f is the mean curvature function, then, on an open dense subset of M^2, $(c-K)^r g$ is a Ricci metric, where r is a locally defined function that satisfies

$$K+\Delta\left(\frac{1}{4}\log(c-K_r)+\frac{r}{2}\log(c-K)\right)=0,$$

with the Gaussian curvature K_r of $(c-K)^r g$ given by

$$K_r=(c-K)^{-r}\left(\frac{3-4r}{3}K+\frac{1}{2}\log(c-K)\Delta r+(c-K)^{-1}g(\mathrm{grad}\, r,\mathrm{grad}\, K)\right).$$

The intrinsic characterization

Recall that a surface M^2 in a space form $N^3(c)$ is biconservative if and only if

$$A(\operatorname{grad} f) = -\frac{f}{2} \operatorname{grad} f.$$

Theorem

Let (M^2, g) be an abstract Riemannian surface with Gaussian curvature K satisfying $(\operatorname{grad} K)(p) \neq 0$ and $c - K(p) > 0$ at any point $p \in M$, where $c \in \mathbb{R}$ is a constant. Let $X_1 = (\operatorname{grad} K)/|\operatorname{grad} K|$ and $X_2 \in C(TM)$ be two vector fields on M such that $\{X_1(p), X_2(p)\}$ is a positively oriented orthonormal basis at any point $p \in M$. If level curves of K are circles in M with constant curvature

$$\kappa = \frac{3X_1 K}{8(c-K)} = \frac{3|\operatorname{grad} K|}{8(c-K)},$$

then, for any point $p_0 \in M$, there exists a parametrization $X = X(u,s)$ of M in a neighborhood $U \subset M$ of p_0 positively oriented such that

Theorem

(a) *the curve* $u \to X(u,0)$ *is an integral curve of* X_1 *with* $X(0,0) = p_0$ *and* $s \to X(u,s)$ *is an integral curve of* X_2, *for any* u;

(b) $K(u,s) = (K \circ X)(u,s) = (K \circ X)(u,0) = K(u)$;

(c) $g_{11}(u,s) = \frac{9}{64}\left(\frac{K'(u)}{c-K(u)}\right)^2 s^2 + 1$, $g_{12}(u,s) = -\frac{3K'(u)}{8(c-K(u))}s$, $g_{22}(u,s) = 1$;

(d) $24(c-K)K'' + 33(K')^2 + 64K(c-K)^2 = 0$;

(e) $X_1 = X_u - g_{12}X_s$, $X_2 = X_s$, $\nabla_{X_1}X_1 = \nabla_{X_1}X_2 = 0$, $\nabla_{X_2}X_2 = -\frac{3X_1K}{8(c-K)}X_1$, $\nabla_{X_2}X_1 = \frac{3X_1K}{8(c-K)}X_2$, *and, therefore, the integral curves of* X_1 *are geodesics.*

Remark

$$24(c-K)K'' + 33(K')^2 + 64K(c-K)^2 = 0$$

$$\Updownarrow$$

$$(c-K)\Delta K - |\operatorname{grad} K|^2 - \frac{8}{3}K(c-K)^2 = 0$$

Remark

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$$\Updownarrow$$

$$(c-K)\Delta K - |\operatorname{grad} K|^2 - \frac{8}{3}K(c-K)^2 = 0$$

Remark

$$(u,s) \to \left(u, (c-K)^{3/8}s\right) = (u,v) \Rightarrow g = du^2 + (c-K)^{-\frac{3}{4}}dv^2$$

$$(u,v) \to \left(\int_{u_0}^{u} (c-K(\tau))^{3/8} d\tau, v\right) = (\tilde{u},\tilde{v}) \Rightarrow g = (c-\tilde{K}(\tilde{u}))^{-\frac{3}{4}}\left(d\tilde{u}^2 + d\tilde{v}^2\right),$$

where $\tilde{K}(\tilde{u}) = K(u(\tilde{u}))$.

Theorem

Let D be an open subset of $\mathbb{R}^2$ and $c \in \mathbb{R}$ a constant. Consider (u,s) the usual cartesian coordinates on $\mathbb{R}^2$ and let $K = K(u)$ be a function on D such that $K'(u) > 0$, $c - K(u) > 0$ and

$$24(c-K)K'' + 33(K')^2 + 64K(c-K)^2 = 0.$$

Define a Riemannian metric $g = g_{11}du^2 + 2g_{12}duds + g_{22}ds^2$ on D by

$$g_{11}(u,s) = \frac{9}{64}\left(\frac{K'(u)}{c-K(u)}\right)^2 s^2 + 1, \quad g_{12}(u,s) = -\frac{3K'(u)}{8(c-K(u))}s, \quad g_{22}(u,s) = 1.$$

Then K is the Gaussian curvature of g and its level curves, i.e., the curves $s \to (u,s)$, are circles in (D,g) with curvature $\kappa(u) = 3K'(u)/(8(c-K(u)))$.

Now, we will give a more explicit form for the metrics g for which $(\operatorname{grad} K)(p) \neq 0$, $c - K(p) > 0$ at any point $p \in M$, where $c \in \mathbb{R}$ is a constant and the level curves of K are circles with constant curvature

$$\kappa = \frac{3|\operatorname{grad} K|}{8(c-K)}.$$

Remark

The level curves of K are circles with constant curvature

$$\kappa = \frac{3|\operatorname{grad} K|}{8(c-K)}$$

if and only if $X_2X_1K = 0$ and $\nabla_{X_2}X_2 = -\frac{3X_1K}{8(c-K)}X_1$, where $X_1 = \frac{\operatorname{grad} K}{|\operatorname{grad} K|}$ and $X_2 \in C(TM)$ are two vector fields on M such that $\{X_1(p), X_2(p)\}$ is a positively oriented orthonormal basis at any point $p \in M$.

Proposition

Let (M^2, g) be a Riemannian surface with Gaussian curvature K satisfying $(\operatorname{grad} K)(p) \neq 0$ and $c - K(p) > 0$ at any point $p \in M$, where $c \in \mathbb{R}$ is a constant. Let $X_1 = \frac{\operatorname{grad} K}{|\operatorname{grad} K|}$ and $X_2 \in C(TM)$ be two vector fields on M such that $\{X_1(p), X_2(p)\}$ is a positively oriented orthonormal basis at any point $p \in M$. Then $X_2 X_1 K = 0$ and $\nabla_{X_2} X_2 = -\frac{3X_1 K}{8(c-K)} X_1$ if and only if the Riemannian metric, g, can be locally, written as $g = e^{2\varphi(u)}(du^2 + dv^2)$, where $(W; u, v)$ is a positive isothermal chart, and φ satisfies

$$8ce^{2\varphi}\varphi' + 2\varphi'\varphi'' + 3\varphi''' = 0$$

and $K' = e^{-2\varphi}(2\varphi'\varphi'' - \varphi''') > 0$ and $c - K = c + e^{-2\varphi}\varphi'' > 0$.

Corollary

Let (M^2, g) be a Riemannian surface with Gaussian curvature K satisfying $(\operatorname{grad} K)(p) \neq 0$ and $K(p) < 0$ at any point $p \in M$. Let $X_1 = \frac{\operatorname{grad} K}{|\operatorname{grad} K|}$ and $X_2 \in C(TM)$ be two vector fields on M such that $\{X_1(p), X_2(p)\}$ is a positively oriented orthonormal basis at any point $p \in M$. Then $X_2 X_1 K = 0$ and $\nabla_{X_2} X_2 = \frac{3X_1 K}{8K} X_1$ if and only if the Riemannian metric, g, can be locally, written as

$$g = e^{6\log(\cosh(\frac{u}{3}))+\beta}(du^2 + dv^2) = e^{\beta} e^{6\log(\cosh(\frac{u}{3}))}(du^2 + dv^2),$$

where $(W; u, v)$ is a positive isothermal chart, $u > 0$ and $\beta \in \mathbb{R}$ is a constant.

Corollary

Let (M^2, g) *be a Riemannian surface with Gaussian curvature* K *satisfying* $(\operatorname{grad} K)(p) \neq 0$ *and* $1 - K(p) > 0$ *at any point* $p \in M$*. Let* $X_1 = \frac{\operatorname{grad} K}{|\operatorname{grad} K|}$ *and* $X_2 \in C(TM)$ *be two vector fields on* M *such that* $\{X_1(p), X_2(p)\}$ *is a positively oriented orthonormal basis at any point* $p \in M$*. Then* $X_2 X_1 K = 0$ *and* $\nabla_{X_2} X_2 = -\frac{3 X_1 K}{8(1-K)} X_1$ *if and only if the Riemannian metric,* g*, can be locally, written as* $g = e^{2\varphi(u)}(du^2 + dv^2)$*, where* $(W; u, v)$ *is a positive isothermal chart, and* $u = u(\varphi)$ *satisfies*

$$u = u(\varphi) = \int_{\varphi_0}^{\varphi} \frac{d\tau}{\sqrt{\frac{B}{3} e^{-\frac{2}{3}\tau} - e^{2\tau} + A}},$$

forall $\varphi \in I$*, and* $A > 0, B < 0$ *and* $\frac{B}{3} e^{-\frac{2}{3}\varphi} - e^{2\varphi} + A > 0$*, forall* $\varphi \in I$*, where* I *is some interval.*

Corollary

Let (M^2, g) *be a Riemannian surface with Gaussian curvature* K *satisfying* $(\operatorname{grad} K)(p) \neq 0$ *and* $-1-K(p) > 0$ *at any point* $p \in M$*. Let* $X_1 = \frac{\operatorname{grad} K}{|\operatorname{grad} K|}$ *and* $X_2 \in C(TM)$ *be two vector fields on* M *such that* $\{X_1(p), X_2(p)\}$ *is a positively oriented orthonormal basis at any point* $p \in M$*. Then* $X_2 X_1 K = 0$ *and* $\nabla_{X_2} X_2 = -\frac{3X_1 K}{8(-1-K)} X_1$ *if and only if the Riemannian metric,* g*, can be locally, written as* $g = e^{2\varphi(u)}(du^2 + dv^2)$*, where* $(W; u, v)$ *is a positive isothermal chart, and* $u = u(\varphi)$ *satisfies*

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forall $\varphi \in I$*, and* $A \in \mathbb{R}, B < 0$ *and* $\frac{B}{3}e^{-\frac{2}{3}\varphi} + e^{2\varphi} + A > 0$*, forall* $\varphi \in I$*, where* I *is some interval.*

Theorem

Let (M^2, g) be an abstract Riemannian surface and $c \in \mathbb{R}$ a constant. Then M can be locally isometrically embedded in a space form $N^3(c)$ as a biconservative surface with positive mean curvature having the gradient different from zero at any point $p \in M$ if and only if the Gaussian curvature K satisfies $c - K(p) > 0$, $(\operatorname{grad} K)(p) \neq 0$, and its level curves are circles in M with curvature $\kappa = (3|\operatorname{grad} K|)/(8(c-K))$.

Rigidity

Proposition

Let (M^2, g) be an abstract Riemannian surface and $c \in \mathbb{R}$ a constant. If M admits two biconservative immersions in $N^3(c)$ such that their mean curvatures are positive with gradients different from zero at any point of M, then the two immersions differ by an isometry of $N^3(c)$.

Remark

Not as in the case of minimal immersions.

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Acknowledgement

This work was co-funded by the European Social Fund through Sectorial Operational Programme Human Resources Development 2007-2013, project number POSDRU$/187/1.5/S/155397$, project title "Towards a New Generation of Elite Researchers through Doctoral Scholarships ".