

Properties of biconservative surfaces

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Let $\varphi : (M^m, g) \rightarrow (N^n, h)$ be a smooth map. We consider

- Energy functional

$$E(\varphi) = E_1(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 v_g$$

- Euler-Lagrange equation

$$\begin{aligned} \tau(\varphi) &= \tau_1(\varphi) = \text{trace}_g \nabla d\varphi \\ &= 0. \end{aligned}$$

The critical points of E are called **harmonic maps**.

$$\tau(\varphi) = g^{ij} \left(\frac{\partial^2 \varphi^\alpha}{\partial x^i \partial x^j} - M \Gamma_{ij}^k \frac{\partial \varphi^\alpha}{\partial x^k} + N \Gamma_{\beta\sigma}^\alpha \frac{\partial \varphi^\beta}{\partial x^i} \frac{\partial \varphi^\sigma}{\partial x^j} \right) \left(\varphi^* \frac{\partial}{\partial y^\alpha} \right) = 0$$



$$g^{ij} \left(\frac{\partial^2 \varphi^\alpha}{\partial x^i \partial x^j} - M \Gamma_{ij}^k \frac{\partial \varphi^\alpha}{\partial x^k} + N \Gamma_{\beta\sigma}^\alpha \frac{\partial \varphi^\beta}{\partial x^i} \frac{\partial \varphi^\sigma}{\partial x^j} \right) = 0,$$

where $M \Gamma_{ij}^k$ are the Christoffel symbols of the domain metric g .

We consider

- Bienergy functional

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 \nu_g$$

- Euler-Lagrange equation

$$\begin{aligned} \tau_2(\varphi) &= -\Delta^\varphi \tau(\varphi) - \text{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi \\ &= 0. \end{aligned}$$

Critical points of E_2 are called **biharmonic maps**.

The biharmonic equation (G.Y. Jiang, 1986)

$$\tau_2(\varphi) = -\Delta^\varphi \tau(\varphi) - \text{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0,$$

where

$$\Delta^\varphi = -\text{trace}_g (\nabla^\varphi \nabla^\varphi - \nabla_{\nabla^\varphi}^\varphi)$$

is the **rough Laplacian** on sections of $\varphi^{-1}TN$ and

$$R^N(X, Y)Z = \nabla_X^N \nabla_Y^N Z - \nabla_Y^N \nabla_X^N Z - \nabla_{[X, Y]}^N Z.$$

- is a fourth-order non-linear elliptic equation;
- any harmonic map is biharmonic;
- a non-harmonic biharmonic map is called **proper biharmonic**;
- a submanifold $\varphi : M \rightarrow N$ of a Riemannian manifold N is called **biharmonic** if the map $\varphi : M \rightarrow N$ is biharmonic (φ is a **harmonic map** if and only if M is **minimal**).

Stress-energy tensor

- D. Hilbert, 1924, described a symmetric 2-covariant tensor S , associated to a variational problem, conservative at critical points, i.e., S satisfies $\operatorname{div} S = 0$ at these points, and called it the stress-energy tensor.
- P. Baird and J. Eells, 1981; A. Sanini, 1983, used the tensor

$$S = \frac{1}{2}|d\varphi|^2 g - \varphi^* h$$

that satisfies

$$\operatorname{div} S = -\langle \tau(\varphi), d\varphi \rangle$$

to study harmonic maps, since

$$\varphi = \text{harmonic} \Rightarrow \operatorname{div} S = 0.$$

When the map is a submersion, $\operatorname{div} S = 0$ if and only if φ is harmonic.

Obviously, if $\varphi : M \rightarrow N$ is any isometric immersion (not necessarily minimal) then $\operatorname{div} S = 0$, since $\tau(\varphi)$ is normal.

It is not interesting to study isometric immersions with $\operatorname{div} S = 0$.

Stress-bienergy tensor

- G.Y. Jiang, 1987, defined the stress-energy tensor S_2 of the bienergy:

$$S_2(X, Y) = \frac{1}{2} |\tau(\varphi)|^2 \langle X, Y \rangle + \langle d\varphi, \nabla \tau(\varphi) \rangle \langle X, Y \rangle \\ - \langle d\varphi(X), \nabla_Y \tau(\varphi) \rangle - \langle d\varphi(Y), \nabla_X \tau(\varphi) \rangle,$$

that satisfies

$$\operatorname{div} S_2 = -\langle \tau_2(\varphi), d\varphi \rangle.$$

When the map is a submersion, $\operatorname{div} S_2 = 0$ if and only if φ is biharmonic.

If $\varphi : M \rightarrow N$ is an isometric immersion, then $(\operatorname{div} S_2)^\# = -\tau_2(\varphi)^\top$.
In general, for an isometric immersion $\operatorname{div} S_2 \neq 0$.

Definition

A submanifold $\varphi : M \rightarrow N$ of a Riemannian manifold N is called a *biconservative submanifold* if $\operatorname{div} S_2 = 0$, i.e. $\tau_2(\varphi)^\top = 0$.

Theorem (Loubeau-Montaldo-Oniciuc, 2008)

A submanifold M^m in a Riemannian manifold N^n , with second fundamental form B , mean curvature vector field H , and shape operator A , is biharmonic if and only if

$$\frac{m}{2} \operatorname{grad} |H|^2 + 2 \operatorname{trace} A_{\nabla^\perp H}(\cdot) + 2 \operatorname{trace}(R^N(\cdot, H)\cdot)^\top = 0$$

and

$$\Delta^\perp H + \operatorname{trace} B(\cdot, A_H \cdot) + \operatorname{trace}(R^N(\cdot, H)\cdot)^\perp = 0,$$

where Δ^\perp is the Laplacian in the normal bundle and R^N is the curvature tensor of N .

Theorem

If M^m is a hypersurface in a Riemannian manifold N^{m+1} , then M is biharmonic if and only if

$$2A(\operatorname{grad}f) + f \operatorname{grad}f - 2f(\operatorname{Ricci}^N(\eta))^\top = 0$$

and

$$\Delta f + f|A|^2 - f \operatorname{Ricci}^N(\eta, \eta) = 0,$$

where η is a unit normal vector field of M in N and $f = \operatorname{trace}A$ is the mean curvature function.

Let M^2 be a surface in a three-dimensional space form $N^3(c)$ of constant sectional curvature c . Then M^2 is a **biconservative surface** in $N^3(c)$ if and only if

$$A(\operatorname{grad} f) = -\frac{f}{2} \operatorname{grad} f.$$

Remark

Any CMC surface in $N^3(c)$ is biconservative.

We are interested in biconservative surfaces which are not CMC, i.e. $\text{grad}f \neq 0$.

For simplicity, we restrict ourselves to biconservative surfaces with $\text{grad}f \neq 0$ at any point of M and $f > 0$. We note that there are examples of **complete** biconservative surfaces with f positive and $\text{grad}f \neq 0$ at any point of an open dense subset of M .

The explicit parametric equations of the biconservative surfaces in \mathbb{R}^3 , in \mathbb{S}^3 and in \mathbb{H}^3 were determined in the article

"Surfaces in three-dimensional space forms with divergence-free stress-bienergy tensor",

by R. Caddeo, S. Montaldo, C. Oniciuc, and P. Piu.

Our aim is to study the biconservative surfaces from *intrinsic point of view*.

More precisely, we will try to find some properties of an abstract Riemannian surface which admits a biconservative immersion in $N^3(c)$. Then, we will determine the necessary and sufficient conditions such that an abstract Riemannian surface to admit a biconservative immersion in $N^3(c)$.

The link between the biconservative surfaces and Ricci surfaces

Theorem (Caddeo-Montaldo-Oniciuc-Piu, 2014)

Let M^2 be a biconservative surface in a space form $N^3(c)$. Assume that $f(p) > 0$ and $(\text{grad}f)(p) \neq 0$ at any point p of M . Then the Gaussian curvature K of M satisfies the extrinsic condition

$$K = \det A + c = -\frac{3f^2}{4} + c$$

and the intrinsic condition

$$(c - K)\Delta K - |\text{grad}K|^2 - \frac{8}{3}K(c - K)^2 = 0,$$

where Δ is the Laplace-Beltrami operator on M .

Remark

We have $c - K(p) > 0$ at any point p of M .

Definition

An abstract Riemannian surface (M^2, g) with Gaussian curvature K is said to satisfy the **Ricci condition** if $c - K > 0$ and the metric $(c - K)^{1/2}g$ is flat, where $c \in \mathbb{R}$ is a constant. In this case, (M^2, g) is called a **Ricci surface**.

- **G. Ricci-Curbastro, 1895**, proved that, when $c = 0$, a surface satisfying the Ricci condition can be locally isometrically embedded in \mathbb{R}^3 as a minimal surface. More precisely, we have:

Theorem

Let (M^2, g) be an abstract Riemannian surface and suppose there exists a minimal immersion $\psi : (M^2, g) \rightarrow \mathbb{R}^3$. Then $K \leq 0$ and $K = 0$ only at isolated points. Moreover, away from the points where $K = 0$, we have:

$$\Delta \log(-K) + 4K = 0 (\Leftrightarrow (-K)^{1/2} g \text{ is flat}).$$

Theorem (Ricci)

Let (M^2, g) be a Riemannian surface such that $K < 0$ and

$$\Delta \log(-K) + 4K = 0.$$

Then, locally, there exists a one-parameter family of minimal embeddings $\psi_\theta : (M^2, g) \rightarrow \mathbb{R}^3$, where $\theta \in \mathbb{S}^1$.

- **H. B. Lawson, 1970**, generalized the result to surfaces in space forms $N^3(c)$, i.e. a surface satisfying the Ricci condition can be locally isometrically embedded in $N^3(c)$ as a minimal surface.

Proposition (A. Moroianu-S. Moroianu, 2014)

Let (M^2, g) be a Riemannian surface such that its Gaussian curvature K satisfies $c - K > 0$, where $c \in \mathbb{R}$ is a constant. Then, the following conditions are equivalent:

- (a) $\Delta \log(c - K) + 4K = 0$;
- (b) *the metric $(c - K)^{1/2}g$ is flat;*
- (c) $(c - K)\Delta K - |\text{grad} K|^2 - 4K(c - K)^2 = 0$.

Moreover, if $c = 0$, then we also have a fourth equivalent condition:

- (d) *the metric $(-K)g$ has constant Gaussian curvature equal to 1.*

Proposition (Fetcu-N.-Oniciuc, 2015)

Let (M^2, g) be a Riemannian surface such that its Gaussian curvature K satisfies $c - K > 0$, where $c \in \mathbb{R}$ is a constant. Then, the following conditions are equivalent:

- (a) $(c - K)\Delta K - |\text{grad} K|^2 - \frac{8}{3}K(c - K)^2 = 0$;
- (b) $\Delta \log(c - K) + (8/3)K = 0$;
- (c) the metric $(c - K)^{3/4}g$ is flat.

Moreover, if $c = 0$, then we also have a fourth equivalent condition:

- (d) the metric $(-K)g$ has constant Gaussian curvature equal to $1/3$.

Theorem

Let (M^2, g) be an abstract Riemannian surface with negative Gaussian curvature K that satisfies

$$K\Delta K + |\text{grad } K|^2 + \frac{8}{3}K^3 = 0. \quad (1)$$

Then $(M^2, (-K)^{1/2}g)$ is a Ricci surface with $c = 0$.

Corollary

Let (M^2, g) be a biconservative surface in \mathbb{R}^3 , where g is the induced metric on M . If $f(p) > 0$ and $(\text{grad } f)(p) \neq 0$ at any point $p \in M$, where f is the mean curvature function, then $(M^2, (-K)^{1/2}g)$ is a Ricci surface with $c = 0$.

Theorem

Let (M^2, g) be a biconservative surface in a space form $N^3(c)$ with induced metric g and Gaussian curvature K . If $f(p) > 0$ and $(\text{grad}f)(p) \neq 0$ at any point $p \in M$, where f is the mean curvature function, then, on an open dense subset of M^2 , $(c - K)^r g$ is a Ricci metric, where r is a locally defined function that satisfies

$$K + \Delta \left(\frac{1}{4} \log(c - K_r) + \frac{r}{2} \log(c - K) \right) = 0,$$

with the Gaussian curvature K_r of $(c - K)^r g$ given by

$$K_r = (c - K)^{-r} \left(\frac{3 - 4r}{3} K + \frac{1}{2} \log(c - K) \Delta r + (c - K)^{-1} g(\text{grad} r, \text{grad} K) \right).$$

The intrinsic characterization

Recall that a surface M^2 in a space form $N^3(c)$ is biconservative if and only if

$$A(\operatorname{grad}f) = -\frac{f}{2} \operatorname{grad}f.$$

Theorem

Let (M^2, g) be an abstract Riemannian surface with Gaussian curvature K satisfying $(\operatorname{grad}K)(p) \neq 0$ and $c - K(p) > 0$ at any point $p \in M$, where $c \in \mathbb{R}$ is a constant. Let $X_1 = (\operatorname{grad}K)/|\operatorname{grad}K|$ and $X_2 \in C(TM)$ be two vector fields on M such that $\{X_1(p), X_2(p)\}$ is a positively oriented orthonormal basis at any point $p \in M$. If level curves of K are circles in M with constant curvature

$$\kappa = \frac{3X_1K}{8(c-K)} = \frac{3|\operatorname{grad}K|}{8(c-K)},$$

then, for any point $p_0 \in M$, there exists a parametrization $X = X(u, s)$ of M in a neighborhood $U \subset M$ of p_0 positively oriented such that

Theorem

- (a) *the curve $u \rightarrow X(u, 0)$ is an integral curve of X_1 with $X(0, 0) = p_0$ and $s \rightarrow X(u, s)$ is an integral curve of X_2 , for any u ;*
- (b) $K(u, s) = (K \circ X)(u, s) = (K \circ X)(u, 0) = K(u)$;
- (c) $g_{11}(u, s) = \frac{9}{64} \left(\frac{K'(u)}{c-K(u)} \right)^2 s^2 + 1$, $g_{12}(u, s) = -\frac{3K'(u)}{8(c-K(u))} s$, $g_{22}(u, s) = 1$;
- (d) $24(c - K)K'' + 33(K')^2 + 64K(c - K)^2 = 0$;
- (e) $X_1 = X_u - g_{12}X_s$, $X_2 = X_s$, $\nabla_{X_1}X_1 = \nabla_{X_1}X_2 = 0$, $\nabla_{X_2}X_2 = -\frac{3X_1K}{8(c-K)}X_1$,
 $\nabla_{X_2}X_1 = \frac{3X_1K}{8(c-K)}X_2$, *and, therefore, the integral curves of X_1 are geodesics.*

Remark

$$24(c - K)K'' + 33(K')^2 + 64K(c - K)^2 = 0$$

\Updownarrow

$$(c - K)\Delta K - |\text{grad } K|^2 - \frac{8}{3}K(c - K)^2 = 0$$

Remark

$$(u, s) \rightarrow (u, (c - K)^{3/8}s) = (u, v) \Rightarrow g = du^2 + (c - K)^{-3/4}dv^2$$

$$(u, v) \rightarrow \left(\int_{u_0}^u (c - K(\tau))^{3/8} d\tau, v \right) = (\tilde{u}, \tilde{v}) \Rightarrow g = (c - \tilde{K}(\tilde{u}))^{-3/4} (d\tilde{u}^2 + d\tilde{v}^2),$$

where $\tilde{K}(\tilde{u}) = K(u(\tilde{u}))$.

Theorem

Let D be an open subset of \mathbb{R}^2 and $c \in \mathbb{R}$ a constant. Consider (u, s) the usual cartesian coordinates on \mathbb{R}^2 and let $K = K(u)$ be a function on D such that $K'(u) > 0$, $c - K(u) > 0$ and

$$24(c - K)K'' + 33(K')^2 + 64K(c - K)^2 = 0.$$

Define a Riemannian metric $g = g_{11}du^2 + 2g_{12}duds + g_{22}ds^2$ on D by

$$g_{11}(u, s) = \frac{9}{64} \left(\frac{K'(u)}{c - K(u)} \right)^2 s^2 + 1, \quad g_{12}(u, s) = -\frac{3K'(u)}{8(c - K(u))}s, \quad g_{22}(u, s) = 1.$$

Then K is the Gaussian curvature of g and its level curves, i.e., the curves $s \rightarrow (u, s)$, are circles in (D, g) with curvature $\kappa(u) = 3K'(u)/(8(c - K(u)))$.

Now, we will give a more explicit form for the metrics g for which $(\text{grad } K)(p) \neq 0$, $c - K(p) > 0$ at any point $p \in M$, where $c \in \mathbb{R}$ is a constant and the level curves of K are circles with constant curvature

$$\kappa = \frac{3|\text{grad } K|}{8(c - K)}.$$

Remark

The level curves of K are circles with constant curvature

$$\kappa = \frac{3|\text{grad } K|}{8(c - K)}$$

if and only if $X_2 X_1 K = 0$ and $\nabla_{X_2} X_1 = -\frac{3X_1 K}{8(c - K)} X_1$, where $X_1 = \frac{\text{grad } K}{|\text{grad } K|}$ and $X_2 \in C(TM)$ are two vector fields on M such that $\{X_1(p), X_2(p)\}$ is a positively oriented orthonormal basis at any point $p \in M$.

Proposition

Let (M^2, g) be a Riemannian surface with Gaussian curvature K satisfying $(\text{grad } K)(p) \neq 0$ and $c - K(p) > 0$ at any point $p \in M$, where $c \in \mathbb{R}$ is a constant. Let $X_1 = \frac{\text{grad } K}{|\text{grad } K|}$ and $X_2 \in C(TM)$ be two vector fields on M such that $\{X_1(p), X_2(p)\}$ is a positively oriented orthonormal basis at any point $p \in M$. Then $X_2 X_1 K = 0$ and $\nabla_{X_2} X_2 = -\frac{3X_1 K}{8(c-K)} X_1$ if and only if the Riemannian metric, g , can be locally, written as $g = e^{2\varphi(u)}(du^2 + dv^2)$, where $(W; u, v)$ is a positive isothermal chart, and φ satisfies

$$8ce^{2\varphi} \varphi' + 2\varphi' \varphi'' + 3\varphi''' = 0$$

and $K' = e^{-2\varphi}(2\varphi' \varphi'' - \varphi''') > 0$ and $c - K = c + e^{-2\varphi} \varphi'' > 0$.

Corollary

Let (M^2, g) be a Riemannian surface with Gaussian curvature K satisfying $(\text{grad } K)(p) \neq 0$ and $K(p) < 0$ at any point $p \in M$. Let $X_1 = \frac{\text{grad } K}{|\text{grad } K|}$ and $X_2 \in C(TM)$ be two vector fields on M such that $\{X_1(p), X_2(p)\}$ is a positively oriented orthonormal basis at any point $p \in M$. Then $X_2 X_1 K = 0$ and $\nabla_{X_2} X_2 = \frac{3X_1 K}{8K} X_1$ if and only if the Riemannian metric, g , can be locally, written as

$$g = e^{6 \log(\cosh(\frac{u}{3})) + \beta} (du^2 + dv^2) = e^\beta e^{6 \log(\cosh(\frac{u}{3}))} (du^2 + dv^2),$$

where $(W; u, v)$ is a positive isothermal chart, $u > 0$ and $\beta \in \mathbb{R}$ is a constant.

Corollary

Let (M^2, g) be a Riemannian surface with Gaussian curvature K satisfying $(\text{grad } K)(p) \neq 0$ and $1 - K(p) > 0$ at any point $p \in M$. Let $X_1 = \frac{\text{grad } K}{|\text{grad } K|}$ and $X_2 \in C(TM)$ be two vector fields on M such that $\{X_1(p), X_2(p)\}$ is a positively oriented orthonormal basis at any point $p \in M$. Then $X_2 X_1 K = 0$ and $\nabla_{X_2} X_2 = -\frac{3X_1 K}{8(1-K)} X_1$ if and only if the Riemannian metric, g , can be locally, written as $g = e^{2\varphi(u)}(du^2 + dv^2)$, where $(W; u, v)$ is a positive isothermal chart, and $u = u(\varphi)$ satisfies

$$u = u(\varphi) = \int_{\varphi_0}^{\varphi} \frac{d\tau}{\sqrt{\frac{B}{3}e^{-\frac{2}{3}\tau} - e^{2\tau} + A}},$$

for all $\varphi \in I$, and $A > 0, B < 0$ and $\frac{B}{3}e^{-\frac{2}{3}\varphi} - e^{2\varphi} + A > 0$, for all $\varphi \in I$, where I is some interval.

Corollary

Let (M^2, g) be a Riemannian surface with Gaussian curvature K satisfying $(\text{grad } K)(p) \neq 0$ and $-1 - K(p) > 0$ at any point $p \in M$. Let $X_1 = \frac{\text{grad } K}{|\text{grad } K|}$ and $X_2 \in C(TM)$ be two vector fields on M such that $\{X_1(p), X_2(p)\}$ is a positively oriented orthonormal basis at any point $p \in M$. Then $X_2 X_1 K = 0$ and $\nabla_{X_2} X_2 = -\frac{3X_1 K}{8(-1-K)} X_1$ if and only if the Riemannian metric, g , can be locally, written as $g = e^{2\varphi(u)}(du^2 + dv^2)$, where $(W; u, v)$ is a positive isothermal chart, and $u = u(\varphi)$ satisfies

$$u = u(\varphi) = \int_{\varphi_0}^{\varphi} \frac{d\tau}{\sqrt{\frac{B}{3}e^{-\frac{2}{3}\tau} + e^{2\tau} + A}},$$

for all $\varphi \in I$, and $A \in \mathbb{R}, B < 0$ and $\frac{B}{3}e^{-\frac{2}{3}\varphi} + e^{2\varphi} + A > 0$, for all $\varphi \in I$, where I is some interval.

Theorem

Let (M^2, g) be an abstract Riemannian surface and $c \in \mathbb{R}$ a constant. Then M can be locally isometrically embedded in a space form $N^3(c)$ as a biconservative surface with positive mean curvature having the gradient different from zero at any point $p \in M$ if and only if the Gaussian curvature K satisfies $c - K(p) > 0$, $(\text{grad} K)(p) \neq 0$, and its level curves are circles in M with curvature $\kappa = (3|\text{grad} K|)/(8(c - K))$.

Proposition

Let (M^2, g) be an abstract Riemannian surface and $c \in \mathbb{R}$ a constant. If M admits two biconservative immersions in $N^3(c)$ such that their mean curvatures are positive with gradients different from zero at any point of M , then the two immersions differ by an isometry of $N^3(c)$.

Remark

Not as in the case of minimal immersions.

Theorem

Let $(\mathbb{R}^2, Ce^{6\log(\cosh \frac{u}{3})}(du^2 + dv^2))$ be a Riemannian surface, where C is a positive constant. Then we have:

(a) \mathbb{R}^2 with this metric is complete;

(b) $K(u, v) = -\frac{1}{6C(\cosh \frac{u}{3})^5}$, $K'(u) = \frac{5}{18C} \frac{\sinh \frac{u}{3}}{(\cosh \frac{u}{3})^6}$ and therefore $\text{grad} K \neq 0$ at any point of $\mathbb{R}^2 \setminus O_v$, which is an open dense subset of \mathbb{R}^2 .

Theorem

$$(c) X : (\mathbb{R}^2, Ce^{6\log(\cosh \frac{u}{3})}(du^2 + dv^2)) \rightarrow \mathbb{R}^3,$$




$$X(u, v) = (\sigma_1(u) \cos v, \sigma_2(u) \sin v, \sigma_2(u))$$

is a biconservative Riemannian immersion in \mathbb{R}^3 , where

$$\sigma_1(u) = C^{\frac{1}{2}} \left(\cosh \frac{u}{3} \right)^3, \forall u \in \mathbb{R}$$

and

$$\sigma_2(u) = \begin{cases} \frac{3}{2} C^{\frac{1}{2}} [(\sinh \frac{u}{3})(\cosh \frac{u}{3}) + \log(\sinh \frac{u}{3} + \cosh \frac{u}{3})] & , u > 0 \\ 0 & , u = 0 \\ -\frac{3}{2} C^{\frac{1}{2}} [-(\sinh \frac{u}{3})(\cosh \frac{u}{3}) + \log(-\sinh \frac{u}{3} + \cosh \frac{u}{3})] & , u < 0 \end{cases}$$

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