Proprietăţi globale ale suprafeţelor biconservative

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Harmonic and biharmonic maps

Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map. We consider

- Energy functional

$$E(\varphi) = E_1(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 v_g$$

- Euler-Lagrange equation

$$\begin{aligned} \tau(\varphi) &= \tau_1(\varphi) = \operatorname{trace}_g \nabla d\varphi \\ &= 0. \end{aligned}$$

The critical points of E are called harmonic maps.

Harmonic and biharmonic maps

We consider

- Bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g$$

- Euler-Lagrange equation

$$\begin{aligned}\tau_2(\varphi) &= -\Delta^\varphi \tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi \\ &= 0.\end{aligned}$$

Critical points of E_2 are called biharmonic maps.

The biharmonic equation (G.Y. Jiang, 1986)

$$\tau_2(\varphi) = -\Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0,$$

where

$$\Delta^{\varphi} = -\operatorname{trace}_g\left(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla}\right)$$

is the rough Laplacian on sections of $\varphi^{-1}TN$ and

$$R^N(X,Y)Z = \nabla^N_X\nabla^N_Y Z - \nabla^N_Y\nabla^N_X Z - \nabla^N_{[X,Y]}Z.$$

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- is a fourth-order non-linear elliptic equation;
- any harmonic map is biharmonic;
- a non-harmonic biharmonic map is called proper biharmonic;

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Stress-energy tensor

- D. Hilbert, 1924, described a symmetric $(1,1)$ tensor field S, associated to a variational problem, conservative at critical points, i.e., S satisfies $\operatorname{div} S = 0$ at these points, and called it the stress-energy tensor.

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- P. Baird and J. Eells, 1981; A. Sanini, 1983, used the tensor

$$S = \frac{1}{2}|d\varphi|^2 g - \varphi^* h$$

that satisfies

$$\operatorname{div} S = -\langle \tau(\varphi), d\varphi \rangle$$

to study harmonic maps, since

$$\varphi = \text{harmonic} \Rightarrow \operatorname{div} S = 0.$$

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When the map is a submersion, $\operatorname{div} S = 0$ if and only if φ is harmonic.

Stress-energy tensor

Obviously, if $\varphi : M \to N$ is any isometric immersion (not necessarily minimal) then $\operatorname{div} S = 0$, since $\tau(\varphi)$ is normal.

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It is not interesting to study isometric immersions with $\operatorname{div} S = 0$.

Stress-bienergy tensor

- G.Y. Jiang, 1987, defined the stress-energy tensor S_2 of the bienergy:

$$\begin{aligned} S_2(X,Y) =& \frac{1}{2}|\tau(\varphi)|^2\langle X,Y\rangle + \langle d\varphi, \nabla\tau(\varphi)\rangle\langle X,Y\rangle \\ & - \langle d\varphi(X), \nabla_Y\tau(\varphi)\rangle - \langle d\varphi(Y), \nabla_X\tau(\varphi)\rangle, \end{aligned}$$

that satisfies

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$$\operatorname{div} S_2 = -\langle \tau_2(\varphi), d\varphi\rangle.$$

When the map is a submersion, $\operatorname{div} S_2 = 0$ if and only if φ is biharmonic.

If $\varphi : M \to N$ is an isometric immersion, then $(\operatorname{div} S_2)^\sharp = -\tau_2(\varphi)^\top$. In general, for an isometric immersion $\operatorname{div} S_2 \neq 0$.

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Proposition

Let $\varphi : M^m \to N^n$ be a submanifold of a Riemannian manifold N. Then $\tau(\varphi) = mH$, where H is the mean curvature tensor field. So, the submanifold is minimal if and only if the map φ is harmonic.

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Let $\varphi : M^m \to N^n$ be a submanifold of a Riemannian manifold N. Then the stress-bienergy tensor field is given by

$$S_2 = -\frac{m^2}{2} h^2 I + 2mA_H$$

and

$$\operatorname{trace} S_2 = m^2 h^2 \left(2 - \frac{m}{2}\right),$$

where $h = |H|$, I is of type $(1,1)$ and it is the identity tensor field, and A_H is the shape operator corresponding to H.

Definition

A submanifold $\varphi : M^m \to N^n$ of a Riemannian manifold N is called a biharmonic submanifold if the map $\varphi : M \to N$ is biharmonic, i.e., $\tau_2(\varphi) = 0$.

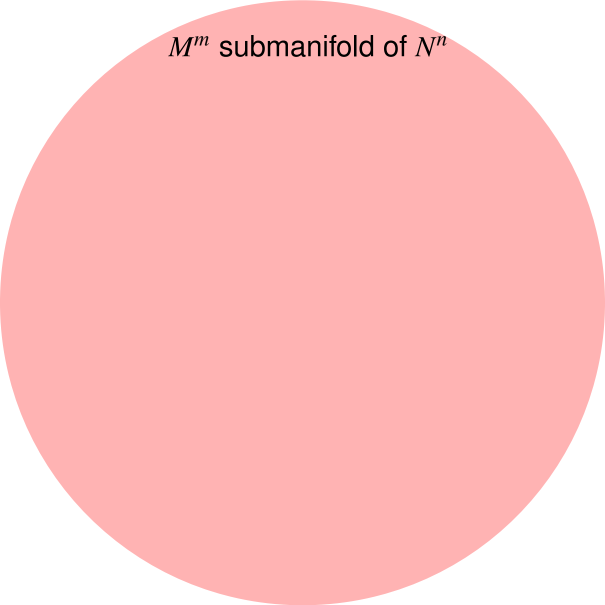
Biharmonic and biconservative submanifolds

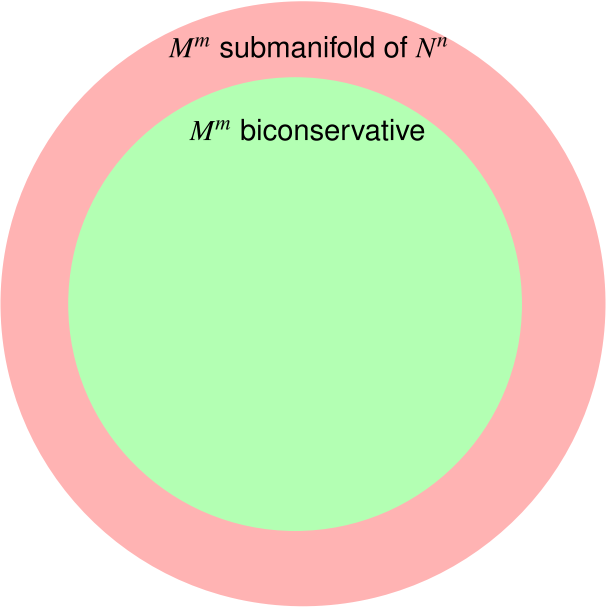
Definition

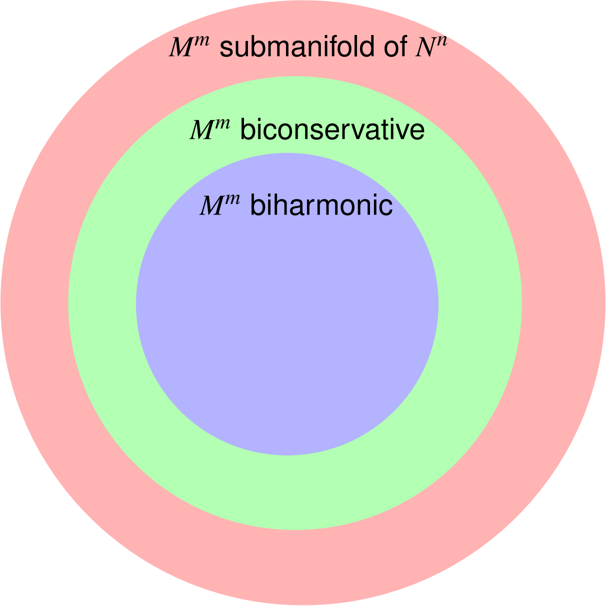
A submanifold $\varphi : M^m \to N^n$ *of a Riemannian manifold* N *is called a* *biharmonic submanifold* *if the map* $\varphi : M \to N$ *is biharmonic, i.e.,* $\tau_2(\varphi) = 0$.

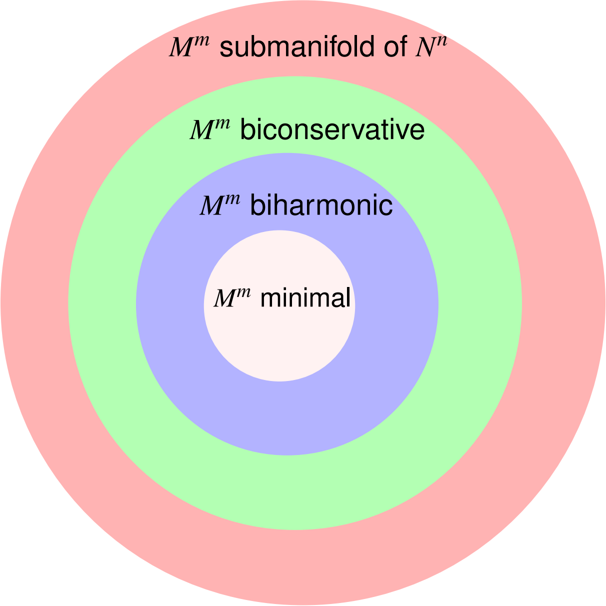
Definition

A submanifold $\varphi : M^m \to N^n$ *of a Riemannian manifold* N *is called a* *biconservative submanifold* *if* $\operatorname{div} S_2 = 0$*, i.e.,* $\tau_2(\varphi)^\top = 0$.


M^m submanifold of N^n


M^m submanifold of N^n
M^m biconservative


M^m submanifold of N^n
M^m biconservative
M^m biharmonic


M^m submanifold of N^n
M^m biconservative
M^m biharmonic
M^m minimal

Theorem (Loubeau-Montaldo-Oniciuc, 2008)

A submanifold $\varphi : M^m \to N^n$ *of a Riemannian manifold* N*, with second fundamental form* B*, mean curvature vector field* H*, and shape operator* A*, is biharmonic if and only if*

$$\frac{m}{2}\operatorname{grad}|H|^2 + 2\operatorname{trace} A_{\nabla^{\perp}_{\cdot} H}(\cdot) + 2\operatorname{trace}(R^N(\cdot, H)\cdot)^{\top} = 0$$

and

$$\Delta^{\perp} H + \operatorname{trace} B(\cdot, A_H \cdot) + \operatorname{trace}(R^N(\cdot, H)\cdot)^{\perp} = 0,$$

where $\Delta^{\perp}$ *is the Laplacian in the normal bundle and* R^N *is the curvature tensor field of* N*.*

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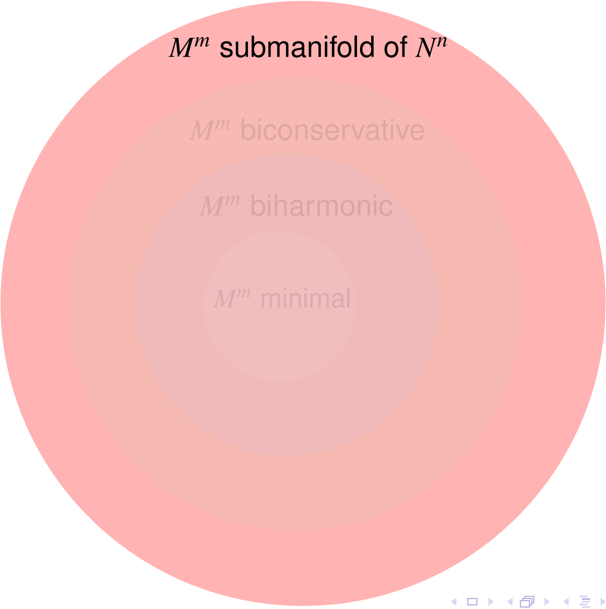
Properties of submanifolds



Properties of submanifolds

Proposition

Let $\varphi : M^m \to N^n$ be a submanifold of a Riemannian manifold N. We have

$$\operatorname{trace} \nabla A_H = \frac{m}{2} \operatorname{grad}\left(h^2\right) + \operatorname{trace} A_{\nabla^\perp_{\cdot} H}(\cdot) + \operatorname{trace}\left(R^N(\cdot, H)\cdot\right)^T . \qquad (1)$$

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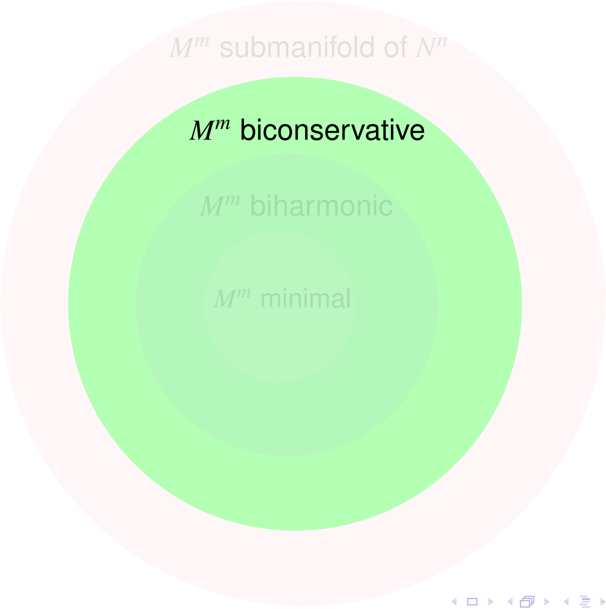
Corollary

Let $\varphi : M^m \to N^n(c)$ be a submanifold of a Riemannian manifold N which has the constant sectional curvature equal to $c \in \mathbb{R}$. We have

$$\operatorname{trace} \nabla A_H = \frac{m}{2} \operatorname{grad}\left(h^2\right) + \operatorname{trace} A_{\nabla^\perp_{\cdot} H}(\cdot).$$

Properties of biconservative submanifolds



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If $\varphi : M^m \to N^n$ is a submanifold of a Riemannian manifold N then the following relations are equivalent:

1. *M is a biconservative submanifold;*

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1. M *is a biconservative submanifold;*
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3. $\frac{m}{2} \operatorname{grad}\left(h^2\right) + 2 \operatorname{trace} A_{\nabla^\perp_{\cdot} H}(\cdot) + 2 \operatorname{trace} \left(R^N(\cdot, H)\cdot\right)^T = 0$;

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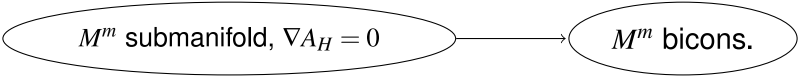
4. $2 \operatorname{trace} \nabla A_H - \frac{m}{2} \operatorname{grad}\left(h^2\right) = 0$.

Our first objective



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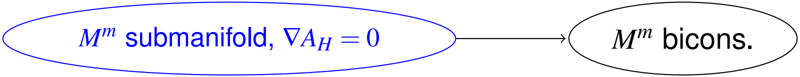

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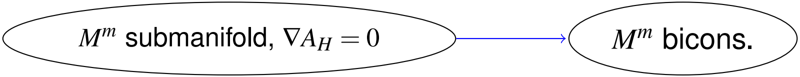

Our first objective

M^m submanifold, $\nabla A_H = 0$ $\longrightarrow$ M^m bicons.

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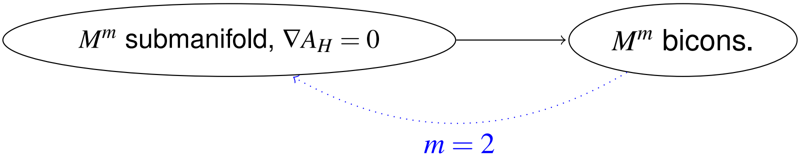

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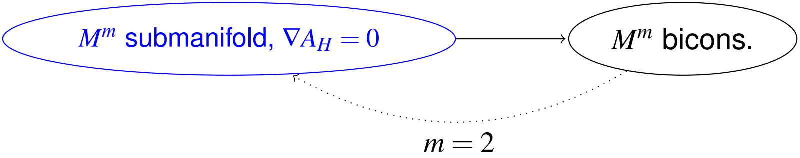

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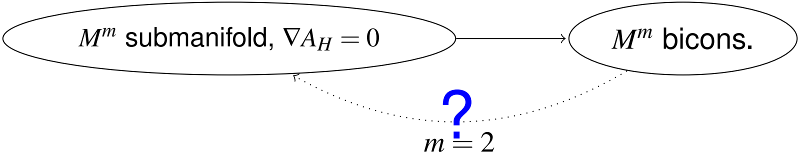

Properties of submanifolds with $\nabla A_H = 0$

Proposition

Let $\varphi : M^m \to N^n$ be a submanifold of a Riemannian manifold N and let $\lambda_1 \geq \cdots \geq \lambda_m$ be the eigenvalues functions of A_H, i.e., the principal curvatures of A_H. If $\nabla A_H = 0$, then:

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4. $\operatorname{trace} A_{\nabla^\perp_{\cdot} H}(\cdot) = -\operatorname{trace}\left(R^N(\cdot, H)\cdot\right)^T$.

Properties of submanifolds with $\nabla A_H = 0$

Corollary

Let $\varphi : M^m \to N^n(c)$ be a submanifold of a Riemannian manifold N which has the constant sectional curvature equal to $c \in \mathbb{R}$. If $\nabla A_H = 0$, then

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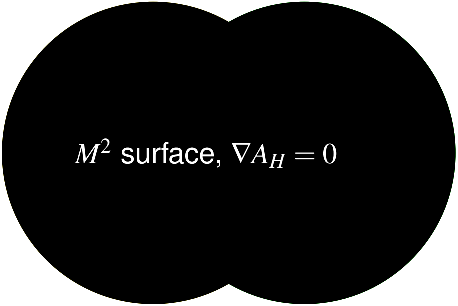
2. $\operatorname{trace} A_{\nabla^{\perp}_{\cdot} H}(\cdot) = 0$.

Properties of surfaces with $\nabla A_H = 0$

Proposition

Let $\varphi : M^2 \to N^n$ be a surface. If $\nabla A_H = 0$, then M is flat or M is pseudoumbilical.



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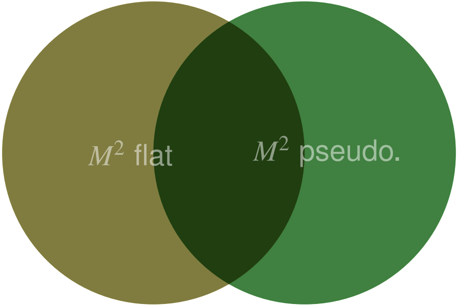

Definition

A symmetric tensor field T of type $(1,1)$ on (M^m, g) is called Codazzi if

$$(\nabla T)(X,Y) = (\nabla T)(Y,X),$$

for any $X, Y \in C(TM)$.

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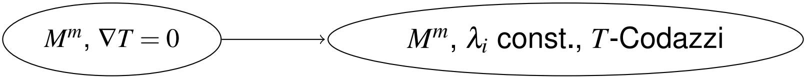
Remark

If $\nabla T = 0$ then T is Codazzi.

Proposition

If (M^m, g) is a Riemannian manifold and T is s symmetric tensor field of type $(1,1)$. Consider $\lambda_1 \geq \cdots \geq \lambda_m$ the eigenvalues functions of T. If $\nabla T = 0$, then λ_i are constant functions on M and, obviously, T is Codazzi.



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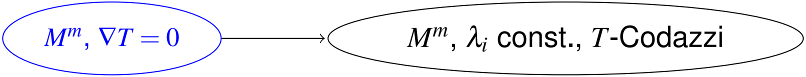

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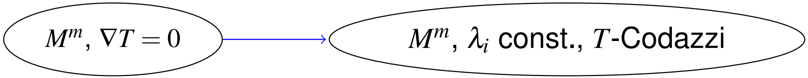

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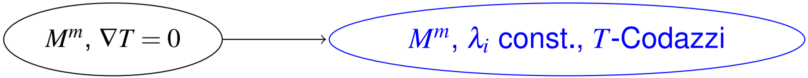

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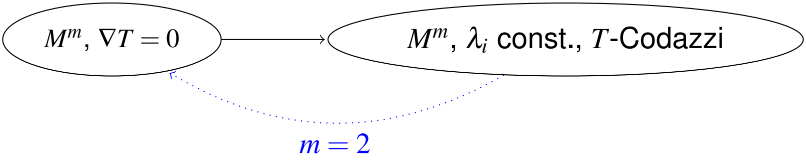

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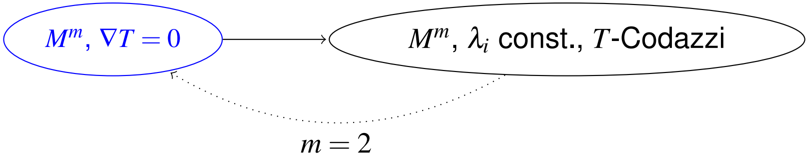

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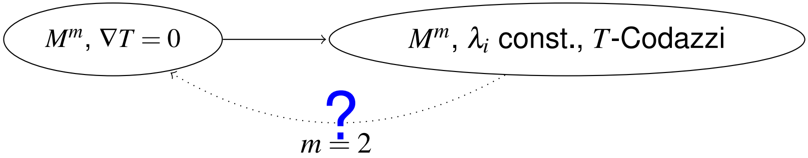

Properties of surfaces with A_H Codazzi tensor field

Proposition

Let (M^2, g) be a surface and let T be a symmetric tensor field of type $(1,1)$. Consider $\lambda_1 \geq \lambda_2$ the eigenvalues functions of T. If λ_1 and λ_2 are constant functions on M and T is Codazzi, then $\nabla T = 0$.
Moreover, if $\lambda_1 > \lambda_2$, then (M^2, g) is flat.

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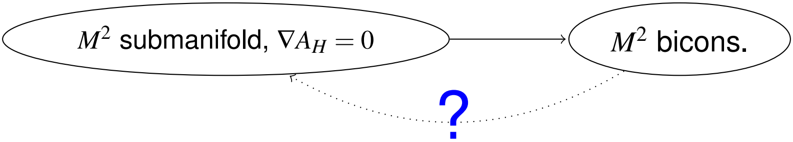
Moreover, if $\lambda_1 > \lambda_2$, then (M^2, g) is flat.

Corollary

Let $\varphi : M^2 \to N^n$ be a surface and $\lambda_1 \geq \lambda_2$ the principal curvatures of A_H. If λ_1 and λ_2 are constant functions on M and A_H is Codazzi, then $\nabla A_H = 0$.

To effectively find the converse that we have already mentioned, we need an integral formula.
We know that a biconservative surface is characterized by $\operatorname{div} S_2 = 0$.
First, we will give a formula for $\Delta^R T$, where T is a symmetric tensor field of type $(1,1)$ that satisfies $\operatorname{div} T = 0$.



Rough Laplacian $\Delta^R S_2$

Proposition (N.-B., 2016)

Let (M^2, g) *be a surface and let* T *be a symmetric tensor field of type* $(1,1)$*. We assume that* $\operatorname{div} T = 0$*. Then*

$$\Delta^R T = -2KT + tKI + (\Delta t)I + \nabla \operatorname{grad} t, \tag{2}$$

where $\Delta^R T = -\operatorname{trace}\left(\nabla^2 T\right)$*,* $t = \operatorname{trace} T$ *and* I *is the identity tensor field of type* $(1,1)$*.*

Rough Laplacian $\Delta^R S_2$

Proposition (N.-B., 2016)

Let (M^2, g) *be a surface and let* T *be a symmetric tensor field of type* $(1,1)$*. We assume that* $\operatorname{div} T = 0$*. Then*

$$\Delta^R T = -2KT + tKI + (\Delta t)I + \nabla \operatorname{grad} t, \tag{2}$$

where $\Delta^R T = -\operatorname{trace}\left(\nabla^2 T\right)$*,* $t = \operatorname{trace} T$ *and* I *is the identity tensor field of type* $(1,1)$*.*

Corollary

If $\varphi : M^2 \to N^n$ *is a biconservative surface, then*

$$\Delta^R S_2 = -2KS_2 + \nabla \operatorname{grad}\left(|\tau(\varphi)|^2\right) + \left(K|\tau(\varphi)|^2 + \Delta|\tau(\varphi)|^2\right) I, \tag{3}$$

where K *is the Gaussian curvature of* M*.*

Rough Laplacian $\Delta^R S_2$

Proposition (N.-B., 2016)

Let $\varphi : M^2 \to N^n$ be a biconservative surface and we assume that M is compact. Then

$$\int_M \left(|\nabla S_2|^2 + 2K \left(|S_2|^2 - \frac{|\tau(\varphi)|^4}{2} \right) \right) v_g = \int_M \left| \operatorname{grad} \left(|\tau(\varphi)|^2 \right) \right|^2 v_g,$$

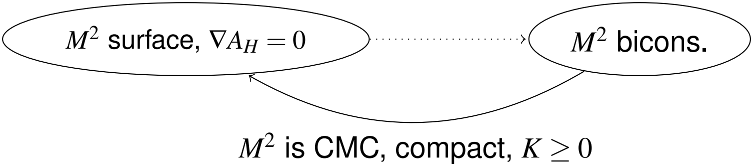
or, equivalent,

$$\int_M \left(|\nabla A_H|^2 + 2K \left(|A_H|^2 - 2h^4 \right) \right) v_g = \frac{5}{2} \int_M \left| \operatorname{grad} \left(h^2 \right) \right|^2 v_g.$$

The converse result

Theorem (N.-B., 2016)

Let $\varphi : M^2 \to N^n$ be a CMC biconservative surface and we assume that M is compact. If $K \geq 0$, then $\nabla A_H = 0$ and M is flat or M is pseudoumbilical.



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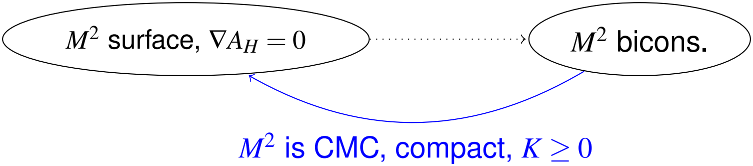
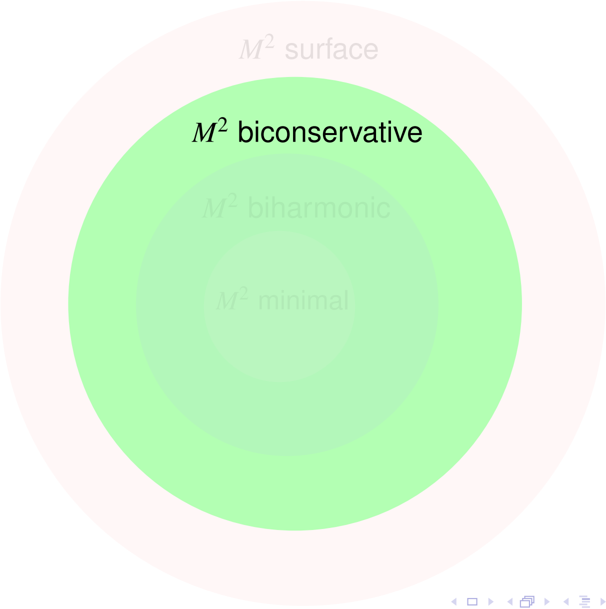


Our second objective



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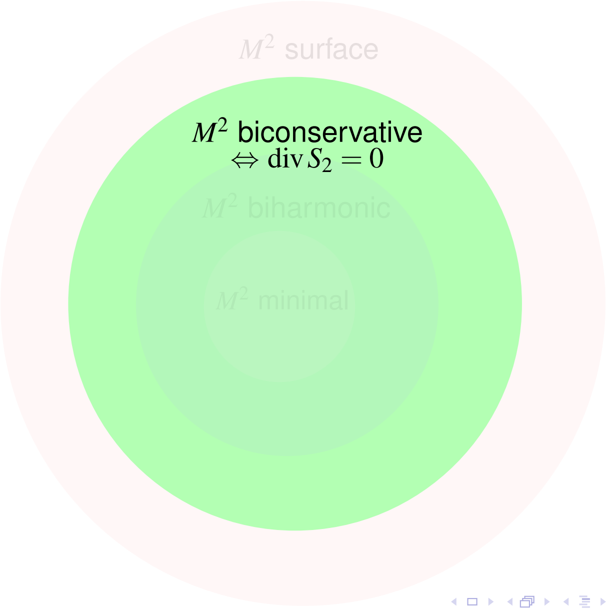

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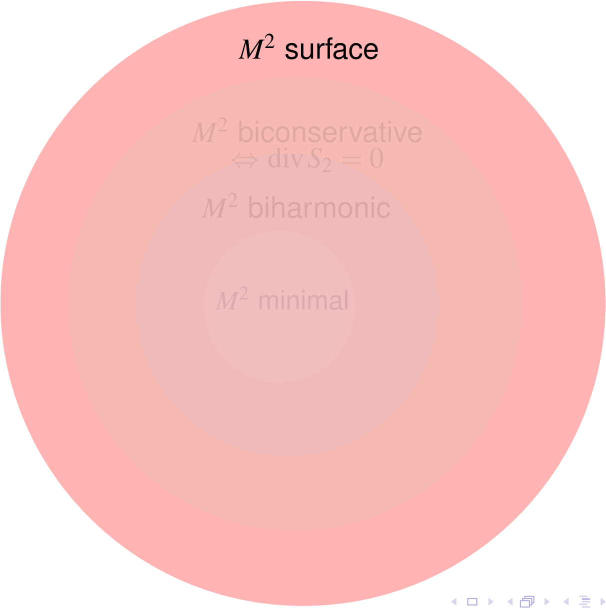

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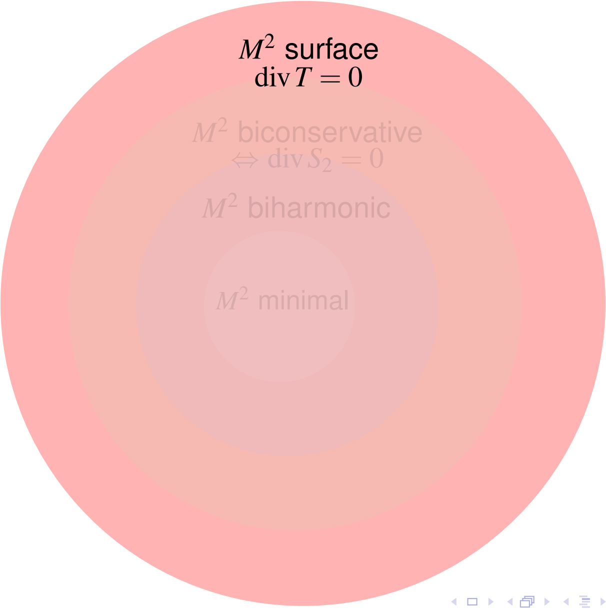

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1. $$\operatorname{div} T = \operatorname{grad} t - \langle Z_{12}, X_2\rangle X_1 + \langle Z_{12}, X_1\rangle X_2,$$

   where $t = \operatorname{trace} T$, $\{X_1, X_2\}$ is a locally orthonormal frame in M and $Z_{12} = (\nabla_{X_1} T)(X_2) - (\nabla_{X_2} T)(X_1)$;

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2. $\langle T(\partial_z), \partial_z\rangle$ *is holomorphic is and only if* $\operatorname{grad} t = 2 \operatorname{div} T$;
3. $\langle T(\partial_z), \partial_z\rangle$ *is holomorphic if and only if*
   $$\operatorname{grad} t = 2\left(\langle Z_{12}, X_2\rangle X_1 - \langle Z_{12}, X_1\rangle X_2\right).$$

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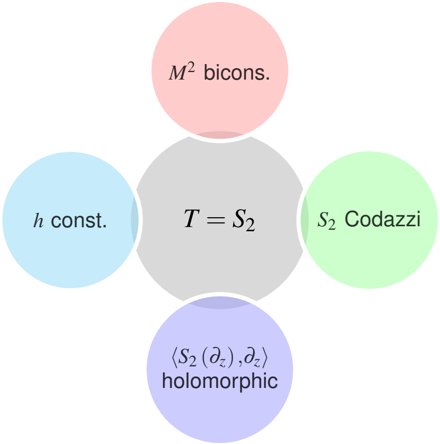
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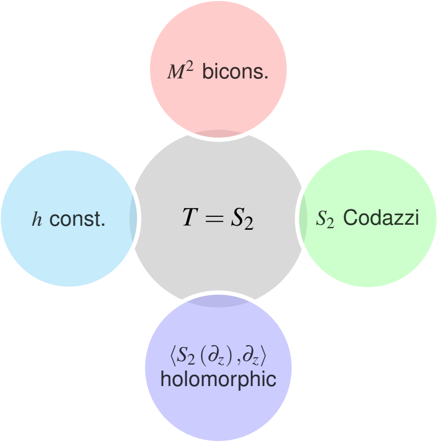
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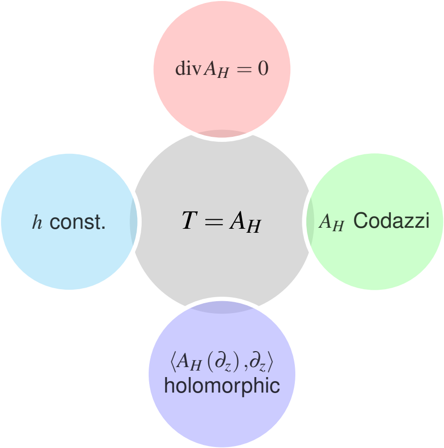
$\mathrm{div} A_H = 0$
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A_H Codazzi
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We recall that for a surface

$$S_2 = -2h^2 I + 4A_H,$$

therefore, in general,

$$S_2 \text{ is Codazzi } \not\Leftrightarrow A_H \text{ is Codazzi}$$
$$\operatorname{div} S_2 = 0 \not\Leftrightarrow \operatorname{div} A_H = 0.$$

$\operatorname{div} S_2 = -2 \operatorname{grad}\left(h^2\right) + 4 \operatorname{div} A_H.$

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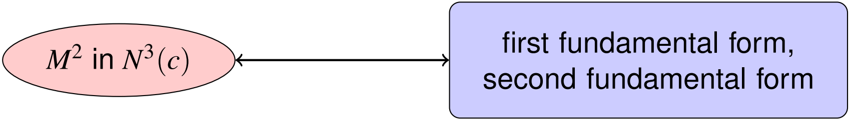
Proposition

Let $\varphi : M^2 \to N^n$ *be a biconservative surface. We assume that* $\nabla^{\perp} H = 0$*. Then* A_H *is Codazzi and* $\left(R^N(X,Y)H\right)^T = 0$ *for any* $X, Y \in C(TM)$*.*

Our third objective



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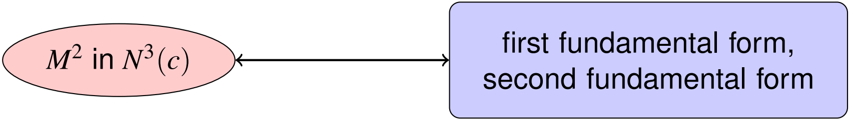

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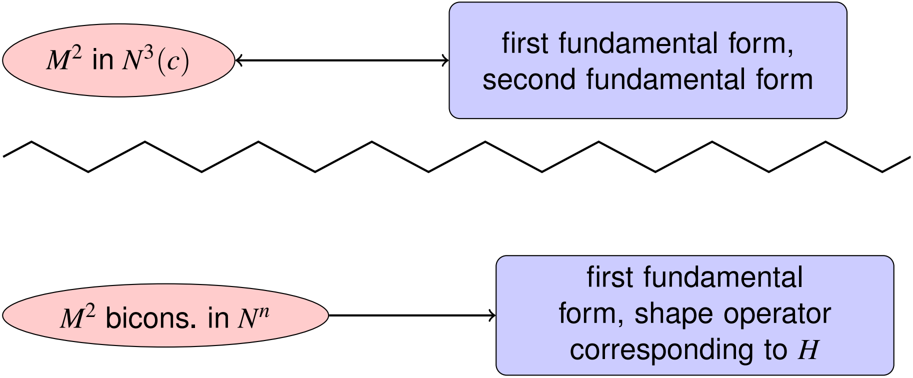

Theorem (N.-B., 2016)

Let $\varphi : M^2 \to N^n$ be a CMC biconservative surface. We denote by λ_1 and λ_2 the principal curvatures of A_H, with $\lambda_1 \geq \lambda_2$ and consider $\mu = \lambda_1 - \lambda_2$. We assume that M does not have pseudoumbilical points, i.e., for any point $p \in M$, we have $\mu(p) > 0$. Then, around any point p there exists a local chart $(U; x, y)$ which is both isothermal and a line of curvature coordinate system for A_H. Moreover, on U, we have

$$\langle \cdot, \cdot \rangle = \frac{1}{\mu} \langle \cdot, \cdot \rangle_0,$$

and A_H is given by

$$\langle A_H(\cdot), \cdot \rangle = \left(\frac{h^2}{\mu} + \frac{1}{2} \right) dx^2 + \left(\frac{h^2}{\mu} - \frac{1}{2} \right) dy^2,$$

where $\langle \cdot, \cdot \rangle_0$ is the Euclidian metric from $\mathbb{R}^2$.

Corollary

Let $\varphi : M^2 \to N^3$ be a CMC biconservative surface. In the hypotheses of previous theorem, we have, on U:

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where μ is solution of the equation

$$\mu \overset{0}{\Delta} \mu + \left| \overset{0}{\operatorname{grad}} \mu \right|_0^2 + 2\mu \left(K^N + h^2 - \frac{\mu^2}{4h^2} \right) = 0,$$

where $\overset{0}{\Delta}$ şi $\overset{0}{\operatorname{grad}}$ are the Laplacian and, the gradient, respectively, with respect to $\langle \cdot, \cdot \rangle_0$, and K^N is the sectional curvature of N along M.

Theorem (N.-B., 2016)

Let $\varphi : M^2 \to N^n$ be a biconservative surface. We denote by λ_1 and λ_2 the principal curvatures corresponding to $A^{\varphi}_{H^{\varphi}}$. Assume that λ_1 and λ_2 are constants and $\lambda_1 > \lambda_2$. We have:

a) *locally, there exists $\psi : M^2 \to N^3(c)$ an isoparametric surface such that $A^{\varphi}_{H^{\varphi}}$ is the second fundamental form of ψ for some unit normal vector field, where*

$$c = \frac{\mu^2}{4} - \left(h^{\varphi}\right)^2;$$

moreover $h^{\psi} = \left(h^{\varphi}\right)^2$;

b) *locally, there exists $\psi : M^2 \to N^3(c)$ an isoparametric surface such that S^{φ}_2 is the second fundamental form of ψ for some unit normal vector field, where*

$$c = 4\left(\mu^2 - \left(h^{\varphi}\right)^2\right);$$

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a) *locally, there exists $\psi : M^2 \to N^3(c)$ an umbilical surface such that $A^{\varphi}_{H^{\varphi}}$ is the second fundamental form of ψ for some unit normal vector field, where*

$$c = -\left(h^{\varphi}\right)^4;$$

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b) *locally, there exists $\psi : M^2 \to N^3(c)$ an umbilical surface such that S^{φ}_2 is the second fundamental form of ψ for some unit normal vector field, where*

$$c = -4\left(h^{\varphi}\right)^4;$$

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Bibliography

A. Balmuş,
Biharmonic maps and submanifolds, Th.D. Thesis, University of Cagliari, Italy, Geometry Balkan Press, 2007.

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Biharmonic maps and submanifolds, Th.D. Thesis, University of Cagliari, Italy, Geometry Balkan Press, 2007.

W. M. Boothby ,
An Introduction to Differentiable Manifolds and Riemannian Geometry, Academic Press, Inc., Second Edition, 1986.

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W. M. Boothby ,
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R. Caddeo, S. Montaldo, C. Oniciuc, P. Piu,
Surfaces in three-dimensional space forms with divergence-free stress-bienergy tensor, Ann. Mat. Pura Appl. (4) 193 (2014), 529–550.

Bibliography

M. P. do Carmo,
Differential Geometry of Curves and Surfaces,
Prentice-Hall, Inc., Englewood Cliffs, New Jersey,
1976.

Bibliography

- M. P. do Carmo,
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- M. P. do Carmo,
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Bibliography

M. P. do Carmo,
Differential Geometry of Curves and Surfaces,
Prentice-Hall, Inc., Englewood Cliffs, New Jersey,
1976.

M. P. do Carmo,
Riemannian Geometry, Birkhauser Boston, 1992.

D. Fetcu, S. N., C. Oniciuc,
On biconservative surfaces in 3-dimensional space forms, Comm. Anal. Geom, acc.;

E. Loubeau, S. Montaldo, C. Oniciuc,
The stress-energy tensor for biharmonic maps, Math. Z. 259 (2008), 503–524.

Bibliography

E. Loubeau, S. Montaldo, C. Oniciuc,
The stress-energy tensor for biharmonic maps, Math. Z. 259 (2008), 503–524.

E. Loubeau, C. Oniciuc,
Biharmonic surfaces of constant mean curvature, Pacific J. Math. 271 (2014), 213–230.

Bibliography

E. Loubeau, S. Montaldo, C. Oniciuc,
The stress-energy tensor for biharmonic maps, Math. Z. 259 (2008), 503–524.

E. Loubeau, C. Oniciuc,
Biharmonic surfaces of constant mean curvature, Pacific J. Math. 271 (2014), 213–230.

S. N.,
Complete biconservative surfaces in $\mathbb{R}^3$ *and* $\mathbb{S}^3$, J. Geom. Phys. 110,(2016), 130-153.

S. N.-B.,
On biconservative surfaces, work in progress.

Bibliography

S. N.-B.,
On biconservative surfaces, work in progress.

C. Oniciuc,
O introducere în teoria aplicaţiilor armonice, Casa Editorială Demiurg, Iaşi, 2007.

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