Differential Geometry and its Applications
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Global Properties of Biconservative Surfaces

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Biconservative submanifolds and general properties

Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between two Riemannian manifolds. **Biharmonic maps** are critical points of the bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \, v_g,$$

where $\tau(\varphi) = \tau_1(\varphi) = \operatorname{trace}_g \nabla d\varphi$; thus they are solutions of Euler-Lagrange equation associated to E_2 (also called **biharmonic equation**):

$$\tau_2(\varphi) = -\Delta^\varphi \tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0.$$

Here, $\Delta^\varphi = -\operatorname{trace}_g(\nabla^\varphi\nabla^\varphi - \nabla^\varphi_\nabla)$ is the rough Laplacian on sections of $\varphi^{-1}TN$ and $R^N(X,Y)Z = \nabla^N_X\nabla^N_Y Z - \nabla^N_Y\nabla^N_X Z - \nabla^N_{[X,Y]}Z$ is the curvature on N.

Definition 1 *A submanifold M^m in N^n, i.e., $\varphi : M^m \to N^n$ Riemannian immersion, is called **biconservative** if $\tau_2(\varphi)^\top = 0$.*

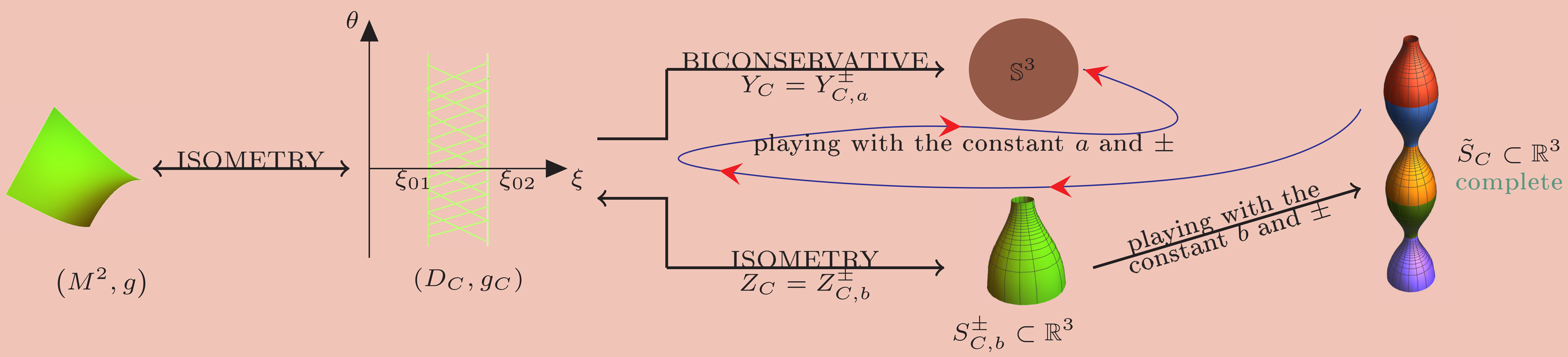
Theorem 1 ([3]) *A submanifold M^m in a Riemannian manifold N^n, with second fundamental form B, mean curvature vector field H, and shape operator A, is biconservative if and only if*

$$\frac{m}{2}\nabla |H|^2 + 2\operatorname{trace} A_{\nabla^\perp_\cdot H}(\cdot) + 2\operatorname{trace}(R^N(\cdot, H)\cdot)^\top = 0,$$

where $\Delta^\perp$ is the Laplacian in the normal bundle.

Remark 1 *A submanifold M^m in N^n is biconservative if and only if $\operatorname{div} S_2^\varphi = 0$, where S_2^φ is the **stress-bienergy tensor** associated to the bienergy functional.*

Biconservative surfaces in $\mathbb{S}^3$ - Local and Global Results



Local results

[1] If S^2 is a biconservative surface in $\mathbb{S}^3$ with $(\nabla f)(p) \neq 0$ and $f(p) > 0$ at any point $p \in S^2$, then, locally, $S^2 \subset \mathbb{R}^4$ can be parametrized by

$$Y_C(k,v) = \left(\sqrt{1 - \frac{16}{9C}k^{-3/2}}\cos\mu(k), \sqrt{1 - \frac{16}{9C}k^{-3/2}}\sin\mu(k), \frac{4\cos v}{3\sqrt{C}k^{3/4}}, \frac{4\sin v}{3\sqrt{C}k^{3/4}}\right),$$

where $(k,v) \in (k_{01}, k_{02}) \times \mathbb{R}$, k_{01} and k_{02} are the solutions of $-\frac{16}{9}k^2 - 16k^4 + Ck^{7/2} = 0$, and $\mu(k) = \pm\int_{k_0}^k E(\tau)\, d\tau + c_0$ with $c_0 \in \mathbb{R}$ and $k_0 = (\frac{3}{64}C)^2 \in (k_{01}, k_{02})$.

Global results

[2] A Riemannian surface (M^2, g) with Gaussian curvature K satisfying $(\nabla K)(p) \neq 0$ and $1 - K(p) > 0$ at any point $p \in M^2$ can be locally isometrically embedded in $\mathbb{S}^3$ as a biconservative surface with $(\nabla f)(p) \neq 0$ and $f(p) > 0$ at any point $p \in M^2$ if and only if (M^2, g) is isometric to (D_C, g_C). Moreover, the biconservative embedding is unique.

- $(D_C, g_C) = \left((\xi_{01}, \xi_{02}) \times \mathbb{R}, \frac{3}{\xi^2(-\xi^{8/3} + 3C\xi^2 - 3)}d\xi^2 + \frac{1}{\xi^2}d\theta^2\right)$, $C \in \left(\frac{4}{3^{3/2}}, \infty\right)$ and ξ_{01} and ξ_{02} are the positive vanishing points of $-\xi^{8/3} + 3C\xi^2 - 3$, with $0 < \xi_{01} < \xi_{02}$.
- $Y_C = Y^\pm_{C,a} : (D_C, g_C) \to \mathbb{S}^3$ is biconservative,

$$Y_C(\xi,\theta) = \left(\sqrt{1 - \frac{1}{C\xi^2}}\cos\zeta(\xi), \sqrt{1 - \frac{1}{C\xi^2}}\sin\zeta(\xi), \frac{\cos(\sqrt{C}\theta)}{\sqrt{C}\xi}, \frac{\sin(\sqrt{C}\theta)}{\sqrt{C}\xi}\right),$$

$\zeta(\xi) = \pm\int_{\xi_{00}}^\xi E(\tau)\, d\tau + a$, with $a \in \mathbb{R}$ and $\xi_{00} = \left(\frac{9}{4}C\right)^{3/2}$;
- (D_C, g_C) is isometric to a surface of revolution in $\mathbb{R}^3$, $Z_C(\xi,\theta) = \left(f(\xi)\cos\left(\sqrt{C}\theta\right), f(\xi)\sin\left(\sqrt{C}\theta\right), h(\xi)\right)$, $f(\xi) = \frac{1}{\sqrt{C}\xi}$, $h(\xi) = \pm\int_{\xi_{00}}^\xi E(\tau)\, d\tau + b$, and $b \in \mathbb{R}$.

Biconservative surfaces in $\mathbb{R}^3$ - Local and Global results

Local results

[1] If S^2 is a biconservative surface of revolution in $\mathbb{R}^3$ with $(\nabla f)(p) \neq 0$ and $f(p) > 0$ at any point $p \in S^2$ then, locally, the surface can be parametrized by

$$X_C(\rho, v) = (\rho\cos v, \rho\sin v, u_C(\rho)),$$

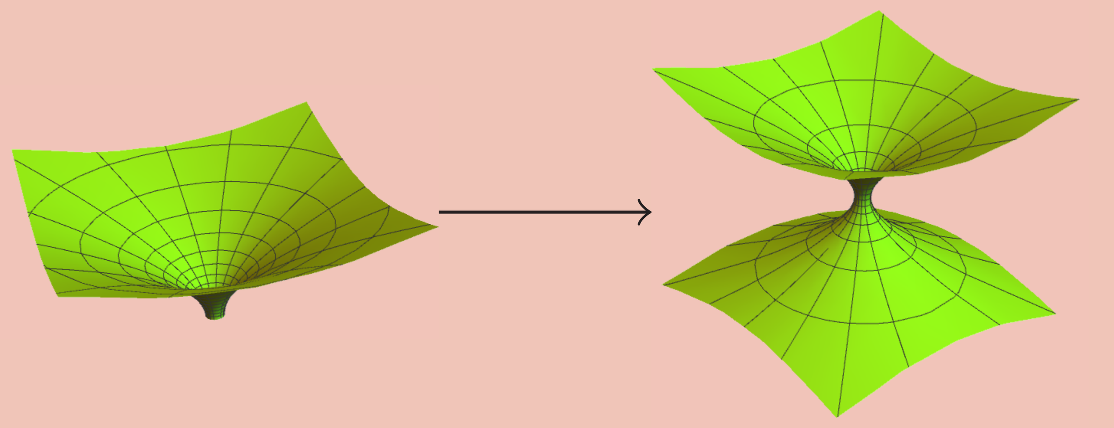
where

$$u_C(\rho) = \frac{3}{2C}\left(\rho^{1/3}\sqrt{C\rho^{2/3} - 1} + \frac{1}{\sqrt{C}} \cdot \ln\left(\sqrt{C}\rho^{1/3} + \sqrt{C\rho^{2/3} - 1}\right)\right)$$

with $C \in \mathbb{R}^*_+$ and $\rho \in (C^{-3/2}, \infty)$.

Global results

[2] If we consider the symmetry of Graf u_C, $\rho \in (C^{-3/2}, \infty)$, with respect to the $O\rho$ axis, we get a smooth, **complete**, biconservative surface $\tilde{S}_C$ in $\mathbb{R}^3$. Moreover, its mean curvature function f is positive and ∇f is different from zero at any point of an open dense subset of $\tilde{S}_C$.



References

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