Biconservative surfaces

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Harmonic and biharmonic maps

Let (M^m, g) and (N^n, h) be two Riemannian manifolds
$C^\infty(M,N)$ the set of all smooth maps between M and N

Energy functional

$$E(\varphi) = E_1(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 \, v_g$$

Euler-Lagrange equation

$$\begin{aligned}\tau(\varphi) &= \tau_1(\varphi) = \mathrm{trace}_g \nabla d\varphi \\ &= 0\end{aligned}$$

Critical points of E:
harmonic maps

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Bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \, v_g$$

Euler-Lagrange equation

$$\begin{aligned} \tau_2(\varphi) &= -\Delta^\varphi \tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi \\ &= 0 \end{aligned}$$

Critical points of E_2:
biharmonic maps

The biharmonic equation (G.Y. Jiang, 1986)

$$\tau_2(\varphi) = -\Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0,$$

where

$$\Delta^{\varphi} = -\operatorname{trace}_g\left(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla}\right)$$

is the rough Laplacian on sections of $\varphi^{-1}TN$ and

$$R^N(X,Y)Z = \nabla^N_X\nabla^N_Y Z - \nabla^N_Y\nabla^N_X Z - \nabla^N_{[X,Y]}Z.$$

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- is a fourth-order non-linear elliptic equation;
- any harmonic map is biharmonic;
- a non-harmonic biharmonic map is called proper biharmonic;
- a submanifold $\varphi : M \to N$ of a Riemannian manifold N is called biharmonic if the map $\varphi : M \to N$ is biharmonic (φ is a harmonic map if and only if M is minimal).

Stress-energy tensor

- D. Hilbert, 1924, described a symmetric 2-covariant tensor S, associated to a variational problem, conservative at critical points, i.e., S satisfies $\operatorname{div} S = 0$ at these points, and called it the stress-energy tensor.

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- P. Baird and J. Eells, 1981; A. Sanini, 1983, used the tensor

$$S = \frac{1}{2}|d\varphi|^2 g - \varphi^* h$$

that satisfies

$$\operatorname{div} S = -\langle \tau(\varphi), d\varphi \rangle$$

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  Obviously

  $$\varphi : M \to N \text{ is any isometric immersion} \Rightarrow \tau(\varphi) \text{ is normal} \Rightarrow \operatorname{div} S = 0.$$

Stress-energy tensor; variational meaning

Assume that M is compact

$$\mathscr{G} = \{g : g \text{ is a Riemannian metric on M}\}$$

$$T_g\mathscr{G} = C(\odot^2 T^* M).$$

For a deformation $\{g_t\}$ of g we denote $\omega = \frac{d}{dt}\big|_{t=0} g_t \in T_g\mathscr{G}$.

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Fix $\varphi : M \to (N,h)$ and define

$$\mathscr{F} : \mathscr{G} \to \mathbb{R}, \quad \mathscr{F}(g) = E(\varphi)$$

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Theorem (Baird-Eells, Sanini)

Let $\varphi : M \to (N,h)$ and assume that M is compact, then

$$\frac{d}{dt}\Big|_{t=0} \mathscr{F}(g_t) = \frac{1}{2}\int_M \langle \omega, e(\varphi)g - \varphi^*h \rangle \; v_g,$$

so g is a critical point of $\mathscr{F}$ if and only if the stress-energy tensor $S = S_1 = e(\varphi)g - \varphi^*h$ vanishes.

Stress-bienergy tensor

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$$S_2(X,Y) = \frac{1}{2}|\tau(\varphi)|^2\langle X,Y\rangle + \langle d\varphi, \nabla\tau(\varphi)\rangle\langle X,Y\rangle - \langle d\varphi(X), \nabla_Y\tau(\varphi)\rangle - \langle d\varphi(Y), \nabla_X\tau(\varphi)\rangle,$$

that satisfies

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that satisfies

$$\operatorname{div} S_2 = -\langle \tau_2(\varphi), d\varphi\rangle.$$

If $\varphi : M \to N$ is an isometric immersion, then $(\operatorname{div} S_2)^\sharp = -\tau_2(\varphi)^\top$.

Stress-bienergy tensor; variational meaning

If $\varphi : M \to (N, h)$ is a fixed map, then E_2 can be thought as a functional on the set of all Riemannian metrics on M. This new functional's critical points are Riemannian metrics determined by $S_2 = 0$.

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$$\mathscr{F}_2 : \mathscr{G} \to \mathbb{R}, \quad \mathscr{F}_2(g) = E_2(\varphi)$$

Theorem (Loubeau-Montaldo-O., 2008)

We have

$$\frac{d}{dt}\Big|_{t=0} \mathscr{F}_2(g_t) = -\frac{1}{2}\int_M \langle \omega, S_2 \rangle \, v_g,$$

so g is a critical point of $\mathscr{F}_2$ if and only if $S_2 = 0$.

Definition

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A submanifold $\varphi : M \to N$ *of a Riemannian manifold* N *is called a* ***biconservative submanifold*** *if* $\operatorname{div} S_2 = 0$*, i.e.* $\tau_2(\varphi)^\top = 0$.

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Theorem (Loubeau-Montaldo-O., 2008)

A submanifold M^m *in a Riemannian manifold* N^n*, with second fundamental form* B*, mean curvature vector field* H*, and shape operator* A*, is biharmonic if and only if*

$$\frac{m}{2}\nabla|H|^2 + 2\operatorname{trace} A_{\nabla^\perp_{\cdot} H}(\cdot) + 2\operatorname{trace}(R^N(\cdot,H)\cdot)^\top = 0$$

and

$$\Delta^\perp H + \operatorname{trace} B(\cdot, A_H \cdot) + \operatorname{trace}(R^N(\cdot,H)\cdot)^\perp = 0,$$

where $\Delta^\perp$ *is the Laplacian in the normal bundle and* R^N *is the curvature tensor of* N*.*

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Using the Codazzi equation

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Proposition

A submanifold $\varphi : M \to N$ is biconservative if and only if

$$-\frac{m}{2}\nabla|H|^2 + 2\,\mathrm{trace}\,\nabla A_H = 0.$$

Some immediate properties

Proposition

Let $\varphi : M \to N$ *be a submanifold. If* $\nabla A_H = 0$ *then* M *is biconservative.*

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Proposition

Let $\varphi : M \to N$ be a submanifold. Assume that N is a space form, i.e. it has constant sectional curvature, and $\nabla^{\perp} H = 0$, i.e. M is PMC. Then M is biconservative.

Some immediate properties

Theorem

Let $\varphi : M^m \to N^n$ be a biconservative submanifold. Assume that M is pseudo-umbilical, i.e. $A_H = |H|^2 I$, and $m \neq 4$. Then M is CMC, i.e. $|H|$ is constant.

The case of biconservative hypersurfaces

Theorem

If M^m is a hypersurface in a Riemannian manifold N^{m+1}, then M is biharmonic if and only if

$$2A(\nabla f) + f\nabla f - 2f(\mathrm{Ricci}^N(\eta))^\top = 0,$$

(or $2A(\nabla f) - f\nabla f + 2f\,\mathrm{trace}\,\nabla A = 0$) and

$$\Delta f + f|A|^2 - f\,\mathrm{Ricci}^N(\eta,\eta) = 0,$$

where η is a unit normal vector field of M in N and $f = \mathrm{trace}\,A$ is the mean curvature function.

Biconservative surfaces in 3-dimensional space forms

Corollary

A surface M^2 in a space form $N^3(c)$ is biconservative if and only if

$$A(\nabla f) = -\frac{f}{2}\nabla f.$$

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A surface M^2 in a space form $N^3(c)$ is biconservative if and only if

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Remark

Any CMC surface in $N^3(c)$ is biconservative.

Biconservative surfaces in 3-dimensional space forms

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For simplicity, we will first restrict ourselves to biconservative surfaces with $\nabla f \neq 0$ at *any point* and $f > 0$ *everywhere* (local results).

Biconservative surfaces in 3-dimensional space forms

In $N^3(c)$ it is interesting to study biconservative surfaces that are not CMC, i.e. $\nabla f \neq 0$.

For simplicity, we will first restrict ourselves to biconservative surfaces with $\nabla f \neq 0$ at *any point* and $f > 0$ *everywhere* (local results).

Then, we will study complete biconservative surfaces with $\nabla f \neq 0$ at any point of an open dense subset of M^2 and $f > 0$ everywhere (global results).

Biconservative surfaces in $\mathbb{R}^3$

Theorem (Hasanis-Vlachos, 1995)

Let S^2 be a biconservative surface in $\mathbb{R}^3$ with $\nabla f(p) \neq 0$ and $f(p) > 0$ for any $p \in M$. Then, locally, S^2 is a surface of revolution, and the curvature $k = k(u)$ of the profile curve $\sigma = \sigma(u)$, $|\sigma'(u)| = 1$, is a positive solution of the following ODE

$$k''k = \frac{7}{4}\left(k'\right)^2 - 4k^4.$$

Biconservative surfaces in $\mathbb{R}^3$

Theorem (Caddeo-Montaldo-O.-Piu, 2014)

Let S^2 be a biconservative surface of revolution in $\mathbb{R}^3$ with non constant mean curvature. Then, locally, the surface can be parametrized by

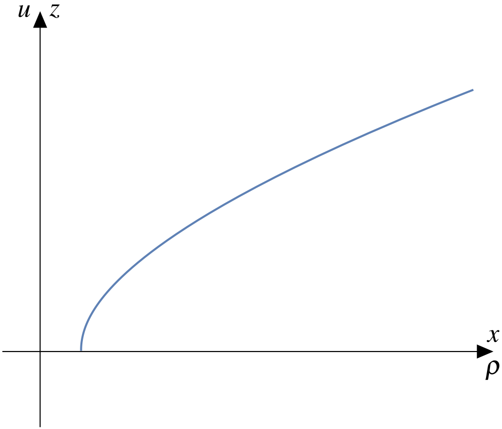
$$X_C(\rho, v) = (\rho \cos v, \rho \sin v, u_C(\rho)),$$

where

$$u_C(\rho) = \frac{3}{2C}\left(\rho^{1/3}\sqrt{C\rho^{2/3}-1} + \frac{1}{\sqrt{C}}\ln\left(\sqrt{C}\rho^{1/3} + \sqrt{C\rho^{2/3}-1}\right)\right)$$

with C a positive constant and $\rho \in (C^{-3/2}, \infty)$.

Biconservative surfaces in $\mathbb{R}^3$



Biconservative surfaces in $\mathbb{R}^3$

The boundary of S_C, i.e. $\bar{S}_C \setminus S_C$, is the circle

$$\left(C^{-3/2}\cos v, C^{-3/2}\sin v, 0\right)$$

which lies in the xOy plane (a plane perpendicular to the rotation axis Oz).

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At a boundary point, the tangent plane to the closure $\overline{S}_C$ of S_{C_1} is parallel to Oz. Moreover, along the boundary, the mean curvature function is constant $f_C = \frac{2}{3C^{-3/2}}$ and $\nabla f_C = 0$.

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Thus, we can expect to "glue" along the boundary two biconservative surfaces of type S_C corresponding to the same C and symmetric to each other, at the level of C^∞ smoothness.

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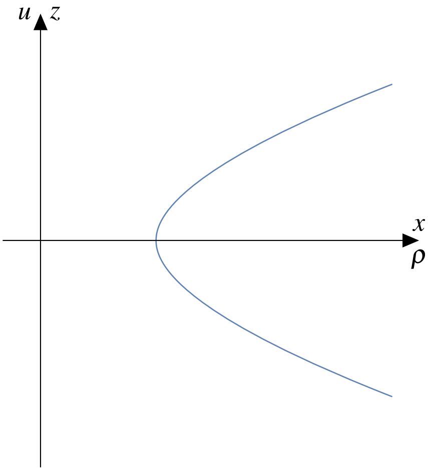
In fact, we will prove that we can glue two biconservative surfaces S_C and $S_{C'}$, at the level of C^∞ smoothness, only along the boundary.

Biconservative surfaces in $\mathbb{R}^3$

Proposition

If we consider the symmetry of $\operatorname{Graf} u_C$, $\rho \in (C^{-3/2}, \infty)$, *with respect to the* $O\rho (= Ox)$ *axis, we get a smooth, complete, biconservative surface* $\tilde{S}_C$ *in* $\mathbb{R}^3$. *Moreover, its mean curvature function* f *is positive and* ∇f *is different from zero at any point of an open dense subset of* $\tilde{S}_C$.

Biconservative surfaces in $\mathbb{R}^3$



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The homothety $(x,y,z) \to C(x,y,z)$ *renders* $\tilde{S}_1$ *onto* $\tilde{S}_{C^{-\frac{2}{3}}}$.

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Remark

The profile curve can be parametrized as:

$$\sigma_C(\theta) = C^{-\frac{3}{2}} \left(\theta^{\frac{3}{2}}, \frac{3}{2} \left(\sqrt{\theta^2 - \theta} + \ln\left(\sqrt{\theta} + \sqrt{\theta - 1} \right) \right) \right), \quad \theta > 1.$$

Biharmonic surfaces in $\mathbb{R}^3$

Theorem (Chen, 1991)

There is no proper biharmonic surface in $\mathbb{R}^3$.

Biconservative surfaces in $\mathbb{S}^3$

Theorem (Caddeo-Montaldo-O.-Piu, 2014)

Let S^2 be a biconservative surface in $\mathbb{S}^3$ with $\nabla f(p) \neq 0$ and $f(p) > 0$ at any point $p \in M$. Then, locally, $S^2 \subset \mathbb{R}^4$ can be parametrized by

$$Y_C(u,v) = \sigma(u) + \frac{4}{3\sqrt{C}k(u)^{3/4}}\left(C_1(\cos v - 1) + C_2 \sin v\right),$$

where $C \in \left(\frac{64}{3^{5/4}}, \infty\right)$ is a positive constant; $C_1, C_2 \in \mathbb{R}^4$ are two constant orthonormal vectors; $\sigma(u)$ is a curve parametrized by arclength that satisfies

$$\langle \sigma(u), C_1 \rangle = \frac{4}{3\sqrt{C}k(u)^{3/4}}, \quad \langle \sigma(u), C_2 \rangle = 0,$$

Biconservative surfaces in $\mathbb{S}^3$

Theorem (continued)

and, as a curve in $\mathbb{S}^2$, its curvature $k = k(u)$ is a positive non constant solution of the following ODE

$$k''k = \frac{7}{4}\left(k'\right)^2 + \frac{4}{3}k^2 - 4k^4$$

such that

$$\left(k'\right)^2 = -\frac{16}{9}k^2 - 16k^4 + Ck^{7/2}.$$

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Remark

The constant C determines uniquely the curvature k, up to a translation of u, and then k, C_1 and C_2 determines uniquely the curve σ.

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If $C_1 = e_3$ and $C_2 = e_4$, we get

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$$Y_C(k,v) = \left(\sqrt{1 - \frac{16}{9C} k^{-3/2}} \cos \mu(k), \quad \sqrt{1 - \frac{16}{9C} k^{-3/2}} \sin \mu(k), \quad \frac{4\cos v}{3\sqrt{C} k^{3/4}}, \quad \frac{4 \sin v}{3\sqrt{C} k^{3/4}} \right)$$

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where $(k,v) \in (k_{01}, k_{02}) \times \mathbb{R}$, k_{01} and k_{02} are the solutions of

$$-\frac{16}{9}k^2 - 16k^4 + Ck^{7/2} = 0$$

and

$$\mu(k) = \pm \int_{k_0}^{k} \frac{108\sqrt{\frac{\tau^2}{-16+9C\tau^{3/2}}}}{\sqrt{\tau^{1/2}\frac{\left(-16+9C\tau^{3/2}\right)\left(9C\tau^{3/2}-16\left(1+9\tau^2\right)\right)}{C}}}d\tau + c_0$$

with c_0 a real constant and $k_0 = \left(\frac{3}{64}C\right)^2 \in (k_{01}, k_{02})$.

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Thus, we can expect to glue along the boundary two biconservative surfaces of type S_C, corresponding to the same C.

But this process is not as clear as in $\mathbb{R}^3$ since we should repeat it infinitely many times.

Biconservative surfaces in $\mathbb{S}^3$

AIM:

Biconservative surfaces in $\mathbb{S}^3$

AIM:

Construct complete biconservative surfaces in $\mathbb{S}^3$ with $\nabla f \neq 0$ at any point of an open dense subset!

Biharmonic surfaces in $\mathbb{S}^3$

Theorem (Caddeo-Mondaldo-O., 2001)

M^2 is a proper biharmonic surface in $\mathbb{S}^3$ if and only if $\varphi(M)$ is an open part of the small hypersphere $\mathbb{S}^2\left(\frac{1}{\sqrt{2}}\right)$.

The (local) characterization theorem

Theorem (Fetcu-Nistor-O., 2015)

Let (M^2, g) be a Riemannian surface and $c \in \mathbb{R}$ a constant. Then M can be locally isometrically embedded in a space form $N^3(c)$ as a biconservative surface with positive mean curvature function having the gradient different from zero at any point $p \in M$ if and only if the Gaussian curvature K satisfies $(\nabla K)(p) \neq 0$, $c - K(p) > 0$, and its level curves are circles in M with curvature $\kappa = (3|\nabla K|)/(8(c-K))$.

Rigidity

Proposition

Let (M^2, g) be a simply connected Riemannian surface and $c \in \mathbb{R}$ a constant. If M admits two biconservative isometric immersions in $N^3(c)$ such that their mean curvature functions are positive with gradients different from zero at any point $p \in M$, then the two immersions differ by an isometry of $N^3(c)$.

Remark

Not as in the case of minimal immersions.

The characterization theorem

The level curves of K are circles with constant curvature

$$\kappa = \frac{3|\nabla K|}{8(c-K)}$$

if and only if $X_2X_1K = 0$ and $\nabla_{X_2}X_2 = -\frac{3X_1K}{8(c-K)}X_1$, where $X_1 = \frac{\nabla K}{|\nabla K|}$ and $X_2 \in C(TM)$ are two vector fields on M such that $\{X_1(p), X_2(p)\}$ is a positively oriented orthonormal basis at any point $p \in M$.

The characterization theorem

Proposition (Nistor-2016)

Let (M^2, g) be a Riemannian surface with Gaussian curvature K satisfying $(\nabla K)(p) \neq 0$ and $c - K(p) > 0$ at any point $p \in M$, where $c \in \mathbb{R}$ is a constant. Then, the level curves of K are circles in M with curvature $\kappa = (3|\nabla K|)/(8(c-K))$ if and only if the Riemannian metric g can be locally written as $g = e^{2\varphi(u)}(du^2 + dv^2)$, where φ satisfies the equation

$$8ce^{2\varphi(u)}\varphi'(u) + 2\varphi'(u)\varphi''(u) + 3\varphi'''(u) = 0$$

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$$K'(u) = e^{-2\varphi(u)}(2\varphi'(u)\varphi''(u) - \varphi'''(u)) \neq 0$$

and

$$c - K(u) = c + e^{-2\varphi(u)}\varphi''(u) > 0,$$

for any u in some open interval I.

The $\mathbb{R}^3$ case

Proposition

Let (M^2, g) be a Riemannian surface with Gaussian curvature K satisfying $(\nabla K)(p) \neq 0$ and $K(p) < 0$ at any point $p \in M$. Then, the level curves of K are circles in M with curvature $\kappa = -(3|\nabla K|)/(8K)$ if and only if the Riemannian metric g can be locally written as

$$g_C = C(\cosh u)^6 (du^2 + dv^2),$$

where $C \in \mathbb{R}$ is a positive constant.

The $\mathbb{S}^3$ case

Proposition

Let (M^2, g) be a Riemannian surface with Gaussian curvature K satisfying $(\nabla K)(p) \neq 0$ and $1 - K(p) > 0$ at any point $p \in M$. Then, the level curves of K are circles in M with curvature $\kappa = (3|\nabla K|)/(8(1-K))$ if and only if the Riemannian metric g can be locally written as $g = e^{2\varphi(u)}(du^2 + dv^2)$, where $u = u(\varphi)$ satisfies

$$u = u(\varphi) = \pm \int_{\varphi_0}^{\varphi} \frac{d\tau}{\sqrt{\frac{b}{3}e^{-\frac{2}{3}\tau} - e^{2\tau} + a}} + c, \qquad \varphi \in I,$$

where $a, b, c \in \mathbb{R}$, $a > 0$, $b < 0$, and $\frac{b}{3}e^{-\frac{2}{3}\varphi} - e^{2\varphi} + a > 0$, for every $\varphi \in I$, where I is some open interval.

Complete biconservative surfaces; the $\mathbb{R}^3$ case

Theorem

Let $\left(\mathbb{R}^2, g_C = C(\cosh u)^6(du^2+dv^2)\right)$ *be a Riemannian surface, where* C *is a positive constant. Then we have:*

(a) *the metric on* $\mathbb{R}^2$ *is complete;*

(b) $K_C(u,v) = K_C(u) = -\frac{3}{C(\cosh u)^8} < 0$, $K_C'(u) = \frac{24}{C}\frac{\sinh u}{(\cosh u)^9}$, *and therefore* $\nabla K_C \neq 0$ *at any point of* $\mathbb{R}^2 \setminus Ov$*;*

Complete biconservative surfaces; the $\mathbb{R}^3$ case

Theorem (continued)

(c) *the immersion* $X_C : \left(\mathbb{R}^2, C(\cosh u)^6(du^2+dv^2)\right) \to \mathbb{R}^3$ *given by*

$$X_C(u,v) = \left(\sigma_C^1(u)\cos 3v, \sigma_C^1(u)\sin 3v, \sigma_C^2(u)\right)$$

is biconservative in $\mathbb{R}^3$*, where*

$$\sigma_C^1(u) = \frac{C^{1/2}}{3}(\cosh u)^3, \qquad u \in \mathbb{R}$$

and

$$\sigma_C^2(u) = \frac{C^{1/2}}{2}\left(\frac{1}{2}\sinh 2u + u\right), \qquad u \in \mathbb{R}.$$

Complete biconservative surfaces; the $\mathbb{S}^3$ case

Proposition

Let (M^2,g) be a Riemannian surface with $g=e^{2\varphi(u)}(du^2+dv^2)$, where $u=u(\varphi)$ satisfies

$$u=u(\varphi)=\pm\int_{\varphi_0}^{\varphi}\frac{d\tau}{\sqrt{\frac{b}{3}e^{-2\tau/3}-e^{2\tau}+a}}+c, \qquad \varphi\in I,$$

where $a,b,c\in\mathbb{R}$, $a>0$, $b<0$, and $\frac{b}{3}e^{-2\varphi/3}-e^{2\varphi}+a>0$, for every $\varphi\in I$, with I some open interval. Then (M^2,g) is isometric to

$$\left(D_C, g_C=\frac{3}{\xi^2(-\xi^{8/3}+3C\xi^2-3)}d\xi^2+\frac{1}{\xi^2}d\theta^2\right),$$

where $D_C=(\xi_{01},\xi_{02})\times\mathbb{R}$, $C\in\left(\frac{4}{3^{3/2}},\infty\right)$ is a positive constant, and ξ_{01} and ξ_{02} are the positive vanishing points of $-\xi^{8/3}+3C\xi^2-3$, with $0<\xi_{01}<\xi_{02}$.

Complete biconservative surfaces; the $\mathbb{S}^3$ case

Theorem

Consider (D_C, g_C). *Then, we have*

(a) $1 - K(\xi,\theta) = \frac{1}{9}\xi^{8/3} > 0$, $K'(\xi) = -\frac{8}{27}\xi^{5/3}$ *and* $\nabla K \neq 0$ *at any point of* D_C*;*

(b) *the immersion* $Y_C : (D_C, g_C) \to \mathbb{S}^3$ *given by*

$$Y_C(\xi,\theta) = \left(\sqrt{1 - \frac{1}{C\xi^2}}\cos\zeta(\xi), \sqrt{1 - \frac{1}{C\xi^2}}\sin\zeta(\xi), \frac{\cos(\sqrt{C}\theta)}{\sqrt{C}\xi}, \frac{\sin(\sqrt{C}\theta)}{\sqrt{C}\xi}\right),$$

is biconservative in $\mathbb{S}^3$*, where*

$$\zeta(\xi) = \pm\int_{\xi_{00}}^{\xi} \frac{\sqrt{C}\tau^{4/3}}{(-1+C\tau^2)\sqrt{-\tau^{8/3}+3C\tau^2-3}}\,d\tau + c,$$

with c *a real constant and* $\xi_{00} = \left(\frac{9}{4}C\right)^{3/2}$*;* $Y_C = Y^{\pm}_{C,c}$.

Complete biconservative surfaces; the $\mathbb{S}^3$ case

Theorem

Let us consider (D_C, g_C)*, where* $D_C = (\xi_{01}, \xi_{02}) \times \mathbb{R}$ *and* $C \in \left(\frac{4}{3^{3/2}}, \infty\right)$*. Then* (D_C, g_C) *is the universal cover of the surface of revolution* $S^{\pm}_{C,C^*,a}$ *in* $\mathbb{R}^3$ *given by*

$$Z_{C,C^*}(\xi,\theta) = \left(f(\xi)\cos\frac{\theta}{C^*}, f(\xi)\sin\frac{\theta}{C^*}, h(\xi)\right), \tag{1}$$

where $f(\xi) = \frac{C^*}{\xi}$*,*

$$h(\xi) = \pm\int_{\xi_{00}}^{\xi}\sqrt{\frac{3\tau^2 - (C^*)^2\left(-\tau^{8/3} + 3C\tau^2 - 3\right)}{\tau^4\left(-\tau^{8/3} + 3C\tau^2 - 3\right)}}\,d\tau + a, \tag{2}$$

$C^* \in \left(0, \left(C - \frac{4}{3^{3/2}}\right)^{-1/2}\right)$ *is a positive constant and* a *is a real constant;* $Z_{C,C^*} = Z^{\pm}_{C,C^*,a}$*.*

Complete biconservative surfaces; the $\mathbb{S}^3$ case

We have now a family of surfaces of revolution $S^{\pm}_{C,C^*,a}$ in $\mathbb{R}^3$, where C is fixed.

Complete biconservative surfaces; the $\mathbb{S}^3$ case

We have now a family of surfaces of revolution $S^{\pm}_{C,C^*,a}$ in $\mathbb{R}^3$, where C is fixed.

We fix now C^* and, alternating the sign and with appropriate choices of the constant a, we can construct a complete surface of revolution S_{C,C^*} which on an open dense subset is locally isometric to (D_C, g_C)

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$$\tilde{S}_{C,C^*} = \bigcup_{k\in\mathbb{Z}} \overline{S^{\pm}_{C,C^*,a_k}}.$$

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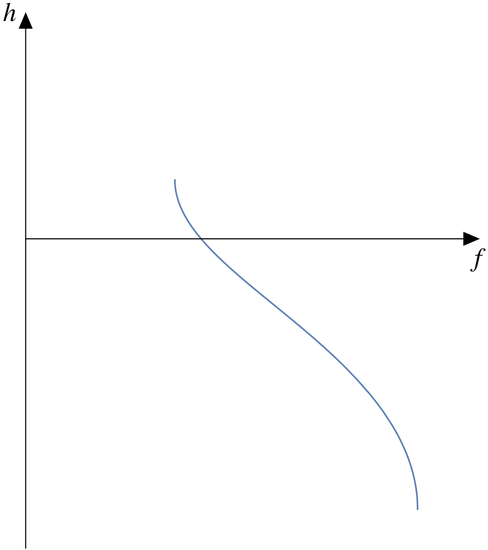
The profile curve of $\tilde{S}_{C,C^*}$ is the graph of a function $f \circ F$ depending on h and defined on the whole Oz (or Oh); $F : \mathbb{R} \to [\xi_{01}, \xi_{02}]$.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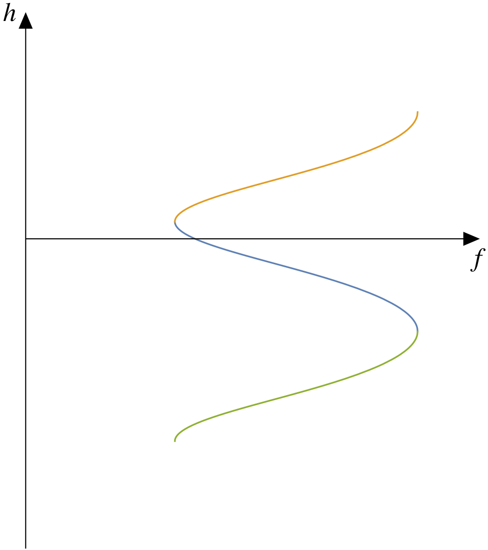
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Complete biconservative surfaces; the $\mathbb{S}^3$ case

Proposition

The universal cover of the surface of revolution S_{C,C^*} *is* $\mathbb{R}^2$ *endowed with the metric* g_{C,C^*}*. It is complete,* $1-K>0$ *on* $\mathbb{R}^2$ *and, on an open dense subset, it is locally isometric to* (D_C, g_C) *and* $\nabla K \neq 0$ *at any point. Moreover any two* $\left(\mathbb{R}^2, g_{C,C_1^*}\right)$ *and* $\left(\mathbb{R}^2, g_{C,C_2^*}\right)$ *are isometric.*

Complete biconservative surfaces; the $\mathbb{S}^3$ case

$$S^{\pm}_{C,C^*,a_k} \simeq (D_C, g_C) \xrightarrow{Y^{\pm}_{C,c_k}} \mathbb{S}^3, \quad k \in \mathbb{Z}.$$

Complete biconservative surfaces; the $\mathbb{S}^3$ case

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Using the function F and modifying c_k in a certain way we obtain the biconservative immersion

$$\tilde{S}_{C,C^*} \longrightarrow \mathbb{S}^3.$$

Biconservative surfaces in $\mathbb{H}^3$

Let us consider the following model for the hyperbolic space

$$\mathbb{H}^3 = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{L}^4 : x_1^2 + x_2^2 + x_3^2 - x_4^2 = -1,\, x_4 > 0\right\},$$

where $\mathbb{L}^4$ is the 4-dimensional Lorentz-Minkowski space. Then we have the following description of biconservative surfaces in $\mathbb{H}^3$.

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where $\mathbb{L}^4$ is the 4-dimensional Lorentz-Minkowski space. Then we have the following description of biconservative surfaces in $\mathbb{H}^3$.

Theorem (Caddeo-Montaldo-O.-Piu, 2014)

Let S^2 be a biconservative surface in $\mathbb{H}^3$ with $\nabla f(p) \neq 0$ and $f(p) > 0$ at any point $p \in M$. Define

$$W = \frac{9|\nabla f|^2}{16f^2} + \frac{9}{4}f^2 - 1.$$

Then, locally, $S^2 \subset \mathbb{L}^4$ can be parametrized by:

Biconservative surfaces in $\mathbb{H}^3$

Theorem (continued)

(a) *if* $W > 0$

$$X_C(u,v) = \sigma(u) + \frac{4}{3\sqrt{C}k(u)^{3/4}}\left(C_1(\cos v - 1) + C_2 \sin v\right),$$

where C *is a positive constant;* $C_1, C_2 \in \mathbb{L}^4$ *are two constant vectors such that* $\langle C_i, C_j \rangle = \delta_{ij}$*;* $\sigma(u)$ *is a curve parametrized by arclength that satisfies*

$$\langle \sigma(u), C_1 \rangle = \frac{4}{3\sqrt{C}k(u)^{3/4}}, \quad \langle \sigma(u), C_2 \rangle = 0,$$

so $\sigma(u)$ *is a curve lying in the totally geodesic* $\mathbb{H}^2 = \mathbb{H}^3 \cap \mathbb{A}$ *(*$\mathbb{A}$ *the linear hyperspace of* $\mathbb{L}^4$ *defined by* $\langle \mathbf{r}, C_2 \rangle = 0$*), while its curvature* $k = k(u)$ *is a positive non constant solution of the following ODE*

$$k''k = \frac{7}{4}\left(k'\right)^2 - \frac{4}{3}k^2 - 4k^4.$$

Biconservative surfaces in $\mathbb{H}^3$

Theorem (continued)

(b) *if* $W<0$

$$X_C(u,v)=\sigma(u)+\frac{4}{3\sqrt{-C}k(u)^{3/4}}\left(C_1(e^v-1)+C_2(e^{-v}-1)\right),$$

where C *is a negative constant;* $C_1,C_2\in\mathbb{L}^4$ *are two constant vectors such that* $\langle C_i,C_i\rangle=0$, $\langle C_1,C_2\rangle=-1$*;* $\sigma(u)$ *is a curve parametrized by arclength that satisfies*

$$\langle\sigma(u),C_1\rangle=\langle\sigma(u),C_2\rangle=-\frac{2\sqrt{2}}{3\sqrt{-C}k(u)^{3/4}},$$

so $\sigma=\sigma(u)$ *is a curve lying in the totally geodesic* $\mathbb{H}^2=\mathbb{H}^3\cap\mathbb{A}$ *(*$\mathbb{A}$ *the linear hyperspace of* $\mathbb{L}^4$ *orthogonal to* C_1-C_2*), while its curvature* $k=k(u)$ *is a positive non constant solution of the same ODE as in the case* (a)*.*

Biconservative surfaces in $\mathbb{H}^3$

Theorem (continued)

(c) *if $W = 0$, solved by Yu Fu.*

Biharmonic surfaces in $\mathbb{H}^3$

Theorem (Caddeo-Mondaldo-O., 2001)

There is no proper biharmonic surface in $\mathbb{H}^3$.

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