Biharmonic tori in Euclidean spheres

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Harmonic and biharmonic maps

Let (M^m, g) and (N^n, h) be two Riemannian manifolds
$C^\infty(M,N)$ the set of all smooth maps between M and N

Energy functional

$$E(\varphi) = E_1(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 \, v_g$$

Euler-Lagrange equation

$$\begin{aligned} \tau(\varphi) &= \tau_1(\varphi) = \operatorname{trace}_g \nabla d\varphi \\ &= 0 \end{aligned}$$

Critical points of E:
harmonic maps

Bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \, v_g$$

Euler-Lagrange equation

$$\begin{aligned} \tau_2(\varphi) &= -\Delta^\varphi \tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi \\ &= 0 \end{aligned}$$

Critical points of E_2:
biharmonic maps

The biharmonic equation (G.Y. Jiang - 1986)

$$\tau_2(\varphi) = -\Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0,$$

where

$$\Delta^{\varphi} = -\operatorname{trace}_g \left(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla}\right)$$

is the rough Laplacian on sections of $\varphi^{-1}TN$ and

$$R^N(X,Y)Z = \nabla^N_X\nabla^N_Y Z - \nabla^N_Y\nabla^N_X Z - \nabla^N_{[X,Y]}Z.$$

- is a fourth-order non-linear elliptic equation;
- any harmonic map is biharmonic;
- a non-harmonic biharmonic map is called proper biharmonic;
- a submanifold $\varphi : M \to N$ of a Riemannian manifold N is called biharmonic if the map $\varphi : M \to N$ is biharmonic (φ is a harmonic map if and only if M is minimal).

Biharmonic maps; non-existence results

- If M compact and $\mathrm{Riem}^N \leq 0$ then biharmonic = harmonic (G.Y. Jiang - 1986).
- A biharmonic Riemannian immersion is minimal (harmonic) if $|\tau(\varphi)|^2$ is constant, i.e. it is CMC, and $\mathrm{Riem}^N \leq 0$ (C.O. - 2002).
- $\Rightarrow$ study biharmonic maps into spheres.

The biharmonic equation in spheres

Theorem (B.Y. Chen - 1984; C.O. - 2002)

A submanifold M^m in a Euclidean sphere $\mathbb{S}^n$ of radius 1 is biharmonic if and only if

$$\begin{cases} \Delta^\perp H + \operatorname{trace} B(A_H(\cdot),\cdot) - mH = 0 \\ 4 \operatorname{trace} A_{\nabla^\perp_{(\cdot)} H}(\cdot) + m \operatorname{grad}(|H|^2) = 0, \end{cases}$$

where $\Delta^\perp$ is the Laplacian in the normal bundle, B is the second fundamental form of the submanifold and H is the mean curvature vector field.

Main examples in spheres

- The extrinsec product

$$\mathbb{S}^{m_1}\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^{m_2}\left(\frac{1}{\sqrt{2}}\right) \to \mathbb{S}^{m_1+m_2+1}, \quad m_1 \neq m_2$$

  is proper biharmonic (G.Y. Jiang).

- 45^{th}-parallel:

$$\mathbb{S}^{m}\left(\frac{1}{\sqrt{2}}\right) \to \mathbb{S}^{m+1}$$

  is proper biharmonic.

Main examples in spheres

Theorem (R. Caddeo, S. Montaldo, C.O. - 2002)

Let $\psi : M^m \to \mathbb{S}^n(a)$ *be a minimal submanifold,* $a \in (0,1)$*. Then* $\varphi = \mathbf{i} \circ \psi : M \to \mathbb{S}^{n+1}$ *is proper biharmonic if and only if* $a = \frac{1}{\sqrt{2}}$*, where* $\mathbf{i} : \mathbb{S}^n(a) \to \mathbb{S}^{n+1}$ *is the canonical inclusion.*

Theorem (R. Caddeo, S. Montaldo, C.O. - 2002)

Let $\psi : M^m \to \mathbb{S}^n\left(\frac{1}{\sqrt{2}}\right)$ *be a submanifold. Then* $\varphi = \mathbf{i} \circ \psi : M \to \mathbb{S}^{n+1}$ *is proper biharmonic if and only if* ψ *is minimal.*

Main examples in spheres

Theorem (R. Caddeo, S. Montaldo, C.O. - 2002)

Let n_1, n_2 be two positive integers such that $n_1 + n_2 = n - 1$, and let M_1 be a submanifold in $\mathbb{S}^{n_1}\left(\frac{1}{\sqrt{2}}\right)$ of dimension m_1, with $0 \leq m_1 \leq n_1$, and let M_2 be a submanifold in $\mathbb{S}^{n_2}\left(\frac{1}{\sqrt{2}}\right)$ of dimension m_2, with $0 \leq m_2 \leq n_2$. Then $M_1 \times M_2$ is proper biharmonic in $\mathbb{S}^n$ if and only if

$$\begin{cases} m_1 \neq m_2 \quad \text{or} \quad |\tau(\psi_1)| > 0 \\ \tau_2(\psi_1) + 2(m_2 - m_1)\tau(\psi_1) = 0 \\ \tau_2(\psi_2) - 2(m_2 - m_1)\tau(\psi_2) = 0 \\ |\tau(\psi_1)| = |\tau(\psi_2)| = constant, \end{cases}$$

where $\psi_1 : M_1 \to \mathbb{S}^{n_1}\left(\frac{1}{\sqrt{2}}\right)$ and $\psi_2 : M_2 \to \mathbb{S}^{n_2}\left(\frac{1}{\sqrt{2}}\right)$ are the Riemannian immersions.

Main examples in spheres

Corollary

If $\tau(\psi_1) = \tau(\psi_2) = 0$, then $M_1^{m_1} \times M_2^{m_2}$ is proper biharmonic in $\mathbb{S}^n$ if and only if $m_1 \neq m_2$.

Constraint on $|H|$

Theorem (C.O. - 2002)

Let $\varphi : M^m \to \mathbb{S}^n$ *be a CMC proper biharmonic immersion. Then*

(i) $|H| \in (0,1]$;

(ii) $|H| = 1$ *if and only if* φ *induces a minimal immersion* ψ *of* M^m *in the small hypersphere* $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{2}}\right) \subset \mathbb{S}^n$.

Interesting case: $|H| \in (0,1)$.

Type decomposition (B.Y. Chen)

A Riemannian immersion $\phi : M^m \to \mathbb{R}^{n+1}$ is called of finite type if it can be expressed as a finite sum of $\mathbb{R}^{n+1}$-valued eigenmaps of the Laplacian Δ of (M, g), i.e.

$$\phi = \phi_0 + \phi_{t_1} + \cdots + \phi_{t_k}, \qquad (1)$$

where $\phi_0 \in \mathbb{R}^{n+1}$ is a constant vector and

$$\Delta\phi_{t_i} = \lambda_{t_i}\phi_{t_i}, \quad i = 1, \ldots, k.$$

If, in particular, all eigenvalues λ_{t_i} are mutually distinct, the submanifold is said to be of k-type and (1) is called the spectral decomposition of ϕ.

Types for biharmonic

Theorem (A. Balmuş, S. Montaldo, C.O. - 2008; E. Loubeau, C.O. - 2016)

Let $\varphi : M^m \to \mathbb{S}^n$ be a proper biharmonic immersion. Denote by $\phi = \mathbf{i} \circ \varphi : M \to \mathbb{R}^{n+1}$ the immersion of M in $\mathbb{R}^{n+1}$, where $\mathbf{i} : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is the canonical inclusion map. Then

(i) *The immersion ϕ is of 1-type if and only if $|H| = 1$. In this case, $\phi = \phi_0 + \phi_{t_1}$, with $\Delta\phi_{t_1} = 2m\phi_{t_1}$, ϕ_0 is a constant vector. Moreover, $\langle \phi_0, \phi_{t_1} \rangle = 0$ at any point, $|\phi_0| = |\phi_{t_1}| = \frac{1}{\sqrt{2}}$ and $\psi_{t_1} : M \to \mathbb{S}^{n-1}\left(\frac{1}{\sqrt{2}}\right)$ is a minimal immersion.*

Types for biharmonic

Theorem (Continued)

(ii) *The immersion ϕ is of 2-type if and only if $|H|$ is constant, $|H| \in (0,1)$. In this case $\phi = \phi_{t_1} + \phi_{t_2}$, with $\Delta\phi_{t_1} = m(1-|H|)\phi_{t_1}$, $\Delta\phi_{t_2} = m(1+|H|)\phi_{t_2}$ and*

$$\phi_{t_1} = \frac{1}{2}\phi + \frac{1}{2|H|}H, \quad \phi_{t_2} = \frac{1}{2}\phi - \frac{1}{2|H|}H.$$

Moreover, $\langle \phi_{t_1}, \phi_{t_2} \rangle = 0$, $|\phi_{t_1}| = |\phi_{t_2}| = \frac{1}{\sqrt{2}}$ and

$$\psi_{t_i} : (M,g) \to \mathbb{S}^n\left(\frac{1}{\sqrt{2}}\right), \quad i = 1,2,$$

are harmonic maps with constant density energy.

Type decomposition $\Rightarrow$ uniqueness

Theorem (E. Loubeau, C.O. - 2016)

Let $\varphi_1, \varphi_2 : M^m \to \mathbb{S}^n$ be two CMC proper biharmonic immersions. If φ_1 and φ_2 agree on an open subset of M, then they agree everywhere.

Definition of biconservative submanifolds

A submanifold $\varphi : M \to N$ is called biconservative if $\tau_2(\varphi)^\top = 0$.

Properties of biharmonic (biconservative) surfaces in spheres

Theorem (E. Loubeau, C.O. - 2014)

Let $\varphi : M^2 \to N^n$ be a CMC proper biharmonic (biconservative) surface. If M is compact and $K^M \geq 0$, then $\nabla A_H = 0$ and M is flat or pseudo-umbilical.

Properties of biharmonic (biconservative) surfaces in spheres

Hopf differential

- Let $\varphi : M^2 \to N^n$ be a proper biharmonic (biconservative) surface with mean curvature vector field H.
- Let z be a complex coordinate on M^2.
- Then the function $\langle B(\partial_z, \partial_z), H\rangle$ is holomorphic if and only if the norm of H is constant (CMC).

Properties of biharmonic (biconservative) surfaces in spheres

- Let $\varphi : M^2 \to N^n$ be a CMC proper biharmonic (biconservative) surface. If M^2 is a topological sphere, then it is pseudo-umbilical.
- Let $\varphi : M^2 \to N^n$ be a CMC proper biharmonic (biconservative) surface. If M is not pseudo-umbilical, then its pseudo-umbilical points are isolated.
- Let $\varphi : M^2 \to N^n$ be a CMC proper biharmonic (biconservative) surface. Assume M is compact and has no pseudo-umbilical point, then it is topologically a torus.

Constant Gaussian curvature

- Boruvka spheres and their uniqueness (Bryant);
- Bryant results for minimal surfaces in spheres which are flat or have negative constant Gaussian curvature;
- Miyata results for 2-type immersions.

Positive constant curvature

Theorem (E. Loubeau, C.O. - 2016)

A Riemannian immersion φ is a CMC proper biharmonic map from a surface with constant positive Gaussian curvature in $\mathbb{S}^n$ if and only if n is odd and it is the diagonal map

$$\phi = \phi_{t_1} + \phi_{t_2} = (\alpha\psi_1, \beta\psi_2),$$

where

$$\psi_1 : \mathbb{S}^2(r) \to \mathbb{S}^{2n_1}(r_1)$$
$$\psi_2 : \mathbb{S}^2(r) \to \mathbb{S}^{2n_2}(r_2)$$

are Boruvka minimal immersions and the parameters α, β, r_1, r_2, r are given by

Positive constant curvature

Theorem (Continued)

$$\alpha^2 = \frac{q_1}{q_1+q_2} \quad \textit{and} \quad \beta^2 = \frac{q_2}{q_1+q_2}$$

$$r_1 = \sqrt{\frac{q_1+q_2}{2q_1}} \quad \textit{and} \quad r_2 = \sqrt{\frac{q_1+q_2}{2q_2}}$$

$$r = \frac{1}{2}\sqrt{q_1+q_2},$$

with $q_1 = n_1(n_1+1)$ *and* $q_2 = n_2(n_2+1)$*, and* $n_1 \neq n_2$*. Moreover,* $|H|^2 = \frac{(q_1-q_2)^2}{(q_1+q_2)^2}$ *and* φ *is pseudo-umbilical.*

Negative constant curvature

There can be no CMC proper biharmonic surface in a sphere with K^M constant and $K^M < 0$.

The flat case

Local description: Let D be a small disk about the origin in $\mathbb{R}^2$ and $\varphi : D \to \mathbb{S}^n$ a CMC proper biharmonic immersion with $|H| \in (0,1)$. Then

(i) n is odd, $n \geq 5$;

(ii) φ extends uniquely to a CMC proper biharmonic immersion of $\mathbb{R}^2$ in $\mathbb{S}^n$;

(iii) $\phi = \mathbf{i} \circ \varphi : \mathbb{R}^2 \to \mathbb{R}^{n+1}$

The flat case

$$\phi(z) = \frac{1}{\sqrt{2}} \sum_{k=1}^{m} \sqrt{R_k} \left(e^{\frac{\sqrt{\lambda_1}}{2}(\mu_k z - \bar{\mu}_k \bar{z})} Z_k + e^{\frac{\sqrt{\lambda_1}}{2}(-\mu_k z + \bar{\mu}_k \bar{z})} \bar{Z}_k \right) + \frac{1}{\sqrt{2}} \sum_{j=1}^{m'} \sqrt{R'_j} \left(e^{\frac{\sqrt{\lambda_2}}{2}(\eta_j z - \bar{\eta}_j \bar{z})} W_j + e^{\frac{\sqrt{\lambda_2}}{2}(-\eta_j z + \bar{\eta}_j \bar{z})} \overline{W_j} \right), \qquad (2)$$

where

The flat case

- $Z_k = \frac{1}{2}\left(E_{2k-1} - \mathrm{i}E_{2k}\right)$, $k = 1, \ldots, m$, $\mathrm{i}^2 = -1$,
- $W_j = \frac{1}{2}\left(E_{2(m+j)-1} - \mathrm{i}E_{2(m+j)}\right)$, $j = 1, \ldots, m'$,
- $\{E_1, \ldots, E_{2m+2m'}\}$ is an orthonormal basis of $\mathbb{R}^{n+1}$, $n = 2m + 2m' - 1$,
- $\lambda_1 = 2(1 - |H|)$, $\lambda_2 = 2(1 + |H|)$, $|H|$ constant, $|H| \in (0, 1)$,
- $\sum_k R_k = 1$, $\sum_j R'_j = 1$, $R_k > 0$, $R'_j > 0$,
- $(1 - |H|)\sum_k \mu_k^2 R_k + (1 + |H|)\sum_j \eta_j^2 R'_j = 0$,
- $\{\pm\mu_k\}_{k=1}^{m}$ are $2m$ complex numbers of norm 1,
- $\{\pm\eta_j\}_{j=1}^{m'}$ are $2m'$ complex numbers of norm 1.

The flat case

$$(1-|H|)\sum_k \mu_k^2 R_k + (1+|H|)\sum_j \eta_j^2 R'_j = 0. \quad (3)$$

Symmetries of the solutions

The flat case

Theorem

Let $\varphi : \mathbb{R}^2 \to \mathbb{S}^n$ *be a CMC proper biharmonic immersion with* $|H| \in (0,1)$. *Then it is a diagonal map*

$$\phi = \phi_{t_1} + \phi_{t_2} = (\phi_1, \phi_2),$$

where $\phi_1 : \mathbb{R}^2 \to \mathbb{S}^{2m-1}\left(\frac{1}{\sqrt{2}}\right)$ *and* $\phi_2 : \mathbb{R}^2 \to \mathbb{S}^{2m'-1}\left(\frac{1}{\sqrt{2}}\right)$ *are harmonic maps with constant density energy,* $n = 2m + 2m' - 1$.

Theorem

Let $\varphi : \mathbb{R}^2 \to \mathbb{S}^n$ *be a CMC proper biharmonic immersion with* $|H| \in (0,1)$. *Then it is pseudo-umbilical if and only if*

$$\phi_1^* \langle , \rangle = \frac{1 - |H|}{2} \langle , \rangle \quad \text{and} \quad \phi_2^* \langle , \rangle = \frac{1 + |H|}{2} \langle , \rangle.$$

The flat case

AIMS

- Solve completely (3).
- Quotient the biharmonic immersion (2) to a cylinder.
- Quotient the biharmonic immersion (2) to a torus.

$n = 5$; Structure Theorem

- Take $h \in (0, 1)$.
- Then there is a one-parameter family of CMC proper biharmonic surfaces $\varphi_{h,\rho} = \varphi_\rho : \mathbb{R}^2 \to \mathbb{S}^5$ with mean curvature $|H| = h$,
- $\rho \in \left[0, \frac{1}{2} \arccos \frac{h-1}{1+h}\right]$,

$n = 5$; Structure Theorem

$\phi_\rho = \mathbf{i} \circ \varphi_\rho : \mathbb{R}^2 \to \mathbb{R}^6$ can be written as

$$\phi_\rho(z) = \frac{1}{\sqrt{2}} \left(e^{\frac{\sqrt{\lambda_1}}{2}(z-\bar{z})} Z_1 + e^{\frac{\sqrt{\lambda_1}}{2}(-z+\bar{z})} \bar{Z}_1 \right)$$
$$+ \frac{1}{\sqrt{2}} \sum_{j=1}^{2} \left(\sqrt{R'_j} e^{\frac{\sqrt{\lambda_2}}{2}(\eta_j z - \bar{\eta}_j \bar{z})} W_j + \sqrt{R'_j} e^{\frac{\sqrt{\lambda_2}}{2}(-\eta_j z + \bar{\eta}_j \bar{z})} \bar{W}_j \right),$$

where

$n=5$; Structure Theorem

- $Z_1 = \frac{1}{2}\left(E_1 - \mathrm{i}E_2\right)$,
- $W_j = \frac{1}{2}\left(E_{2(1+j)-1} - \mathrm{i}E_{2(1+j)}\right),\ j=1,2$,
- $\{E_1,\ldots,E_6\}$ is an orthonormal basis of $\mathbb{R}^6$,
- $\lambda_1 = 2(1-h)$, $\lambda_2 = 2(1+h)$,
- R_1', R_2', $\eta_1 = e^{\mathrm{i}\rho}$ and $\eta_2 = e^{\mathrm{i}\tilde{\rho}}$ are given by:

$n=5$; Structure Theorem

$$R_1' = \frac{1-\left(\frac{h-1}{h+1}\right)^2}{2\left(1+\frac{h-1}{h+1}\cos 2\rho\right)},$$

$$R_2' = 1-\frac{1-\left(\frac{h-1}{h+1}\right)^2}{2\left(1+\frac{h-1}{h+1}\cos 2\rho\right)},$$

$$\tilde{\rho} = \arctan\left(-\frac{1}{h\tan\rho}\right),$$

if $\rho \in \left(0, \frac{1}{2}\arccos\frac{h-1}{1+h}\right]$,

and

$n = 5$; Structure Theorem

$$R_1' = \frac{h}{1+h},\ R_2' = \frac{1}{1+h},\ \tilde{\rho} = -\frac{\pi}{2},$$

if $\rho = 0$.

$n = 5$; Structure Theorem

Conversely, assume that $\varphi : \mathbb{R}^2 \to \mathbb{S}^5$ is a CMC proper biharmonic surface with mean curvature $|H| = h \in (0,1)$. Then, up to isometries of $\mathbb{R}^2$ and $\mathbb{R}^6$, $\phi = \mathbf{i} \circ \varphi : \mathbb{R}^2 \to \mathbb{R}^6$ is one of the above maps.

Existence of biharmonic planes

Let $h \in (0,1)$, then there exist proper biharmonic immersions of $\mathbb{R}^2$ in $\mathbb{S}^5$ with CMC equal to h.

Existence of biharmonic cylinders

Let $h \in (0, 1)$, then there exist proper biharmonic cylinders in $\mathbb{S}^5$ with CMC equal to h.

Biharmonic tori in $\mathbb{S}^5$

Theorem

The CMC proper biharmonic immersion $\varphi_{h,\rho}:\mathbb{R}^2\to\mathbb{S}^5$, $\rho\in\left[0,\frac{1}{2}\arccos\frac{h-1}{1+h}\right]$, *quotients to a torus if and only if either*

(i) $\rho=0$ *and*

$$h=\frac{1-b}{1+b},$$

where $b=\left(\frac{r}{t}\right)^2$, $r,t\in\mathbb{N}^*$, *with* $r<t$ *and* $(r,t)=1$*; or*

(ii) $\rho\in\left(0,\frac{1}{2}\arccos\frac{h-1}{1+h}\right]$ *is a constant depending on* a *and* b *and*

$$h=\frac{1-(a-b)^2}{1+(a-b)^2+2(a+b)},$$

where $a=\left(\frac{p}{q}\right)^2$ *and* $b=\left(\frac{r}{t}\right)^2$, *with* $p,q,r,t\in\mathbb{N}^*$, *such that* $0\leq b-a<1$.

Biharmonic tori in $\mathbb{S}^5$

Theorem (Continued)

Moreover, in this case, the corresponding lattice $\Lambda_{\psi_{h,\rho}}$ is given by

$$\begin{aligned}\Lambda_{\psi_{h,0}} =& \{l\cdot r\cdot v_2 + k\cdot v_1 : k,l\in\mathbb{Z}\}\\ =& \{l\cdot t\cdot \tilde{v}_2 + k\cdot v_1 : k,l\in\mathbb{Z}\},\end{aligned}$$

where

$$v_1 = (-\pi\sqrt{1+b},0)$$

$$v_2 = \left(0,\pi\sqrt{\frac{1+b}{b}}\right)$$

and

$$\tilde{v}_2 = (0,\pi\sqrt{1+b});$$

or

Biharmonic tori in $\mathbb{S}^5$

Theorem (Continued)

$$\begin{aligned}\Lambda_{\psi_{h,\rho}} &= \Big\{ m\cdot v_2 + n\cdot v_1 : m,n\in\mathbb{Z} \text{ s.t. } m\frac{q}{p} - n\frac{qr}{pt}\in\mathbb{Z}\Big\}\\ &= \Big\{ m\cdot \tilde{v}_2 + k\cdot \tilde{v}_1 : m,k\in\mathbb{Z} \text{ s.t. } m\frac{t}{r} - k\frac{pt}{qr}\in\mathbb{Z}\Big\},\end{aligned}$$

where

$$\begin{aligned} v_1 &= \Big(\frac{\pi\sqrt{(a-b)^2+a+b}}{\sqrt{a}}, 0\Big)\\ v_2 &= \Big(-\frac{\pi\sqrt{b}(1-a+b)}{\sqrt{a((a-b)^2+a+b)}}, \frac{\pi\sqrt{1+(a-b)^2+2(a+b)}}{\sqrt{(a-b)^2+a+b}}\Big)\\ \tilde{v}_1 &= \Big(\frac{\pi\sqrt{(a-b)^2+a+b}}{\sqrt{b}}, 0\Big)\\ \tilde{v}_2 &= \Big(-\frac{\pi\sqrt{a}(1+a-b)}{\sqrt{b((a-b)^2+a+b)}}, \frac{\pi\sqrt{1+(a-b)^2+2(a+b)}}{\sqrt{(a-b)^2+a+b}}\Big).\end{aligned}$$

Biharmonic tori in $\mathbb{S}^5$

Theorem

Let $h \in (0,1)$. Then there exists a CMC proper biharmonic immersion from some torus T^2 in $\mathbb{S}^5$ with mean curvature h if and only if either

(i)

$$h = \frac{1-b}{1+b},$$

where $b = \left(\frac{r}{t}\right)^2$, $r,t \in \mathbb{N}^*$, with $r < t$; or

(ii)

$$h = \frac{1-(a-b)^2}{1+(a-b)^2+2(a+b)},$$

where $a = \left(\frac{p}{q}\right)^2$ and $b = \left(\frac{r}{t}\right)^2$, with $p,q,r,t \in \mathbb{N}^*$, such that $0 \leq b-a < 1$.

Flat tori biharmonically immersed in $\mathbb{S}^n$

AIM

- Given a flat torus T^2, does it admit a biharmonic immersion in some $\mathbb{S}^n$?

A class of CMC biharmonic rectangular tori

Theorem (D. Fetcu, E. Loubeau, C.O. - 2016)

Consider a rectangular lattice $\Lambda = \{(2\pi k, 2\pi l\theta) : k, l \in \mathbb{Z}\}$ with a side of length 2π and the other of length $2\pi\theta$ and the torus $T^2 = \mathbb{R}^2/\Lambda$, where $\theta \in \mathbb{R}^*_+$. Then T^2 admits a proper biharmonic immersion in $\mathbb{S}^n$ with constant mean curvature $h \in (0,1)$ if and only if

$$\theta^2 = \frac{q_1^2 + q_2^2}{2} \quad \text{and} \quad n \in \{5, 7\},$$

where $q_1, q_2 \in \mathbb{N}$ and $q_1 < q_2$. In this case

$$h = \frac{q_2^2 - q_1^2}{2(q_1^2 + q_2^2)},$$

with $q_1 \geq 0$ when $n = 5$ and $q_1 > 0$ when $n = 7$.

A class of CMC biharmonic rectangular tori

Theorem (Continued)

Moreover, the CMC proper biharmonic immersion from T^2 in $\mathbb{S}^n$ corresponds to the map $\phi : \mathbb{R}^2 \to \mathbb{R}^{n+1}$ determined by one of the following sets of data:

- *when $n = 5$*

$$R_1 = 1, \quad R_1' = \frac{1}{2}\left(1 - \sqrt{\frac{1-2h}{1+2h}}\right), \quad R_2' = \frac{1}{2}\left(1 + \sqrt{\frac{1-2h}{1+2h}}\right)$$

$$\mu_1 = \sqrt{\frac{1-2h}{2(1-h)}} + \frac{\mathrm{i}}{\sqrt{2(1-h)}}, \quad \eta_1 = \sqrt{\frac{1+2h}{2(1+h)}} + \frac{\mathrm{i}}{\sqrt{2(1+h)}}, \quad \eta_2 = \bar{\eta}_1,$$

where $h \in (0, \frac{1}{2}]$;

A class of CMC biharmonic rectangular tori

Theorem (Continued)

- *when* $n = 7$

$$R_{1,2} = \frac{1}{2}\left(1 \pm \frac{\omega}{\sqrt{1-2h}}\right), \quad R'_{1,2} = \frac{1}{2}\left(1 \mp \frac{\omega}{\sqrt{1+2h}}\right),$$

$$\mu_1 = \sqrt{\frac{1-2h}{2(1-h)}} + \frac{\mathrm{i}}{\sqrt{2(1-h)}}, \mu_2 = \bar{\mu}_1, \quad \eta_1 = \sqrt{\frac{1+2h}{2(1+h)}} + \frac{\mathrm{i}}{\sqrt{2(1+h)}}, \eta_2 = \bar{\eta}_1,$$

where $\omega \in (-\sqrt{1-2h}, \sqrt{1-2h})$ *and* $h \in (0, \frac{1}{2})$.

Remark

The proof relies on explicitly finding all admissible CMC proper biharmonic immersions of T^2 *in* $\mathbb{S}^n$.

A class of CMC biharmonic rectangular tori

Remark

For $0 < q_1^2 < q_2^2$ the same torus can be immersed in $\mathbb{S}^5$ and $\mathbb{S}^7$ as a CMC proper biharmonic surface, with the same constant mean curvature.

Remark

The same rectangular torus can be immersed in $\mathbb{S}^5$ (or $\mathbb{S}^7$) as a CMC proper biharmonic surface in different ways with different mean curvatures.

Remark

A rectangular torus with both sides of length less than $\frac{1}{\sqrt{2}}$ cannot be immersed in a sphere $\mathbb{S}^n$ as a CMC proper biharmonic surface.

CMC biharmonic square tori in $\mathbb{S}^n$

Theorem

Consider a square lattice $\Lambda = \{(2\pi ka, 2\pi la) : k,l \in \mathbb{Z}\}$ with the side of length $2\pi a$ and the torus $T^2 = \mathbb{R}^2/\Lambda$, where $a \in \mathbb{R}^*_+$. Then we have

(i) *T^2 admits a proper biharmonic immersion in $\mathbb{S}^n$, $n \equiv 3 \pmod 4$, with constant mean curvature $h \in (0,1)$ if and only if*

$$4a^2 = p_1^2 + q_1^2 + p_2^2 + q_2^2, \quad h = \frac{p_2^2 + q_2^2 - p_1^2 - q_1^2}{p_1^2 + q_1^2 + p_2^2 + q_2^2},$$

and

$$7 \le n \le r_2(p_1^2 + q_1^2) + r_2(p_2^2 + q_2^2) - 1,$$

where $r_2(p)$ is the number of representations of $p \in \mathbb{N}$ as the sum of two squares of integers and $p_1, q_1, p_2, q_2 \in \mathbb{N}$ such that $0 < p_1^2 + q_1^2 < p_2^2 + q_2^2$.

(ii) *If $4a^2 = p^2 + q^2$, where $p, q \in \mathbb{N}$ such that $0 < p < q$, then T^2 admits a CMC proper biharmonic immersion in $\mathbb{S}^n$ with $h = \frac{q^2 - p^2}{p^2 + q^2}$ for any odd n, $5 \le n \le r_2(p^2) + r_2(q^2) - 1$.*

CMC biharmonic square tori in $\mathbb{S}^n$

Remark

One obtains that $a \geq \frac{\sqrt{3}}{2}$. Moreover, if $a = \frac{\sqrt{3}}{2}$, the corresponding square torus can be immersed only in $\mathbb{S}^7$, in a unique manner.

Remark

While any positive integer can be written as a sum of four squares (not necessarily satisfying the condition in the theorem), a positive integer can be written as a sum of two squares if and only if each of its prime factors of the form $4p-1$ occurs with an even power in its prime factorization.

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As positive integers p and q can be chosen such that $r_2(p^2) + r_2(q^2)$ is arbitrarily large, we have the following

Theorem

For any sphere $\mathbb{S}^n$, with n odd, $n \geq 5$, there exists a square torus that can be immersed in $\mathbb{S}^n$ as a CMC proper biharmonic surface.

Thank You!