SEMINAR NR. 11, REZOLVĂRI
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CALCUL OPERAŢIONAL

7. TRANSFORMATA LAPLACE

7.1. Definiţii. Determinare de tranformată Laplace (funcţie imagine) pentru o funcţie original dată, respectiv de funcţie original pentru o funcţie imagine dată

Transformata Laplace se utilizează în studiul proceselor aproape continue, în rezolvarea unor ecuaţii şi sisteme de ecuaţii diferenţiale, integro-diferenţiale.

Definiţia 7.1.1. O funcţie $f : \mathbb{R} \to \mathbb{R}$ (sau $f : \mathbb{R} \to \mathbb{C}$) se numeşte *funcţie original* dacă:

(i) $f(t) = 0, \forall t \in]-\infty, 0[$;

(ii) pe orice interval mărginit, f are cel mult un număr finit de puncte de discontinuitate de speţa întâi (există limite laterale finite);

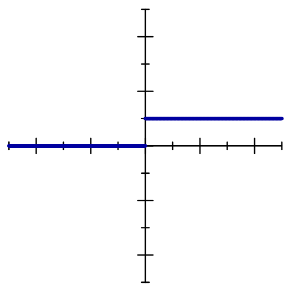
(iii) f are o creştere de tip exponenţial, adică $\exists M \in \mathbb{R}$, $M > 0$, $\exists \sigma \in \mathbb{R}$, astfel încât
$|f(t)| \leq Me^{\sigma t}, \forall t \in]0, +\infty[$.

Numărul real $\sigma_f = \inf \left\{ \sigma \in \mathbb{R}; |f(t)| \leq Me^{\sigma t}, \forall t \in]0, +\infty[\right\}$ se numeşte *abscisă de convergenţă* sau *indice de creştere.*

Se notează cu $\mathcal{O}$ mulţimea tuturor funcţiilor original.

Exemplul 1. a) Funcţia $\eta : \mathbb{R} \to \mathbb{R}$ (sau $\eta : \mathbb{R} \to \mathbb{C}$), $\eta(t) = \begin{cases} 0, & \text{dacă } t < 0 \\ 1, & \text{dacă } t \geq 0. \end{cases}$

se numeşte f*uncţia treapta unitate* (*funcţia Heaviside*).



$\eta \in \mathcal{O}$ cu abscisa de convergenţă $\sigma_\eta = 0$.

b) Dacă f verifică (ii) şi (iii) din Definiţia 1 atunci funcţia $f \cdot \eta$ este funcţie original.

c) Dacă $f : \mathbb{R} \to \mathbb{R}$ este o funcţie elementară, atunci $f \cdot \eta$ îndeplineşte (i) şi (ii). Dacă există $\beta \geq 0$ astfel încât $\lim\limits_{t \to +\infty} f(t) e^{-\beta t} = 0$, atunci este îndeplinită şi condiţia (iii).

d) Fie $f : \mathbb{R} \to \mathbb{R}, f(t) = e^{\alpha t} (P(t) \cos \beta t + Q(t) \sin \beta t)$,
unde $\alpha, \beta \in \mathbb{R}$, iar P, Q sunt polinoame. Atunci funcţia $f \cdot \eta$ este funcţie original.

e) Dacă $f : \mathbb{R} \to \mathbb{R}$ (sau $f : \mathbb{R} \to \mathbb{C}$) este o funcţie continuă şi mărginită (sau îndeplineşte (*ii*) şi este mărginită), atunci $f \cdot \eta$ este funcţie original cu abscisa de convergenţă $\sigma_f = 0$.

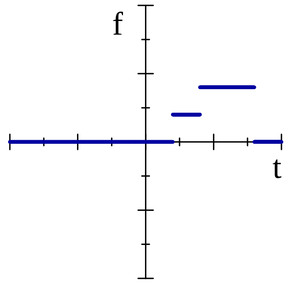
Exerciţiul 1. Să se studieze dacă următoarele funcţii sunt funcţii original

a) $f : \mathbb{R} \to \mathbb{R}, f(t) = \begin{cases} 0, & \text{dacă } t < 2, \\ 2, & \text{dacă } t \in [2, 4], \\ 4, & \text{dacă } t \in]4, 8], \\ 0, & \text{dacă } t > 8. \end{cases}$; **b)** $f : \mathbb{R} \to \mathbb{R}, f(t) = \begin{cases} 0, & \text{dacă } t < 0 \\ \dfrac{1}{t+1}, & \text{dacă } t \geq 0. \end{cases}$

c) $f:\mathbb{R}\to\mathbb{R}, f(t)=\cos t^2\cdot\eta(t)$; **d)** $f:\mathbb{R}\to\mathbb{R}, f(t)=e^{t^2}\cdot\eta(t)$; **e)** $f:\mathbb{R}\to\mathbb{R}, f(t)=e^{-t^2}\cdot\eta(t)$.

Rezolvare. a) Fie $f:\mathbb{R}\to\mathbb{R}, f(t)=\begin{cases}0, & \text{dacă } t<2,\\ 2, & \text{dacă } t\in[2,4],\\ 4, & \text{dacă } t\in\,]4,8],\\ 0, & \text{dacă } t>8.\end{cases}$

Se reprezintă grafic funcţia anterioară.



Se observă că:

(i) $f(t)=0, \forall t\in\,]-\infty,0[$;

(ii) pe orice interval mărginit, f are cel mult un număr finit de puncte de discontinuitate, iar în aceste puncte de discontinuitate există limite laterale finite. Într-adevăr, $t_1=2, t_2=4, t_3=8$ sunt singurele puncte de discontinuitate ale funcţiei f şi

$$\begin{cases} l_s(2)=0 \text{ şi } l_d(2)=2;\\ l_s(4)=2 \text{ şi } l_d(4)=4;\\ l_s(8)=4 \text{ şi } l_d(8)=0.\end{cases}$$

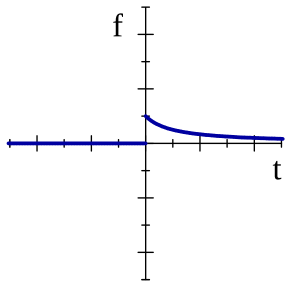
(iii) f are o creştere de tip exponenţial, adică $\exists M=4>0$, $\exists\sigma=0\in\mathbb{R}$, astfel încât

$|f(t)|\le 4e^{0t}, \forall t\in\,]0,+\infty[$.

Atunci $f\in\mathcal{O}$, cu abscisa de convergenţă $\sigma_f=0$.

b) Fie $f:\mathbb{R}\to\mathbb{R}, f(t)=\begin{cases}0, & \text{dacă } t<0\\ \dfrac{1}{t+1}, & \text{dacă } t\ge 0.\end{cases}$

Se reprezintă grafic funcţia anterioară.



Se observă că:

(i) $f(t)=0, \forall t\in\,]-\infty,0[$;

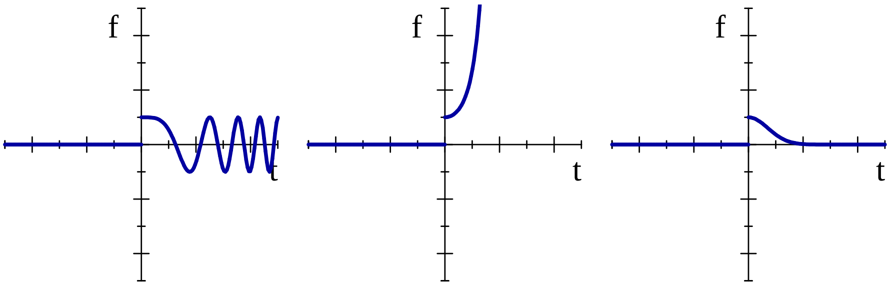
(ii) pe orice interval mărginit, f are cel mult un număr finit de puncte de discontinuitate, iar în aceste puncte de discontinuitate există limite laterale finite. Într-adevăr, $t_1=0$ este singurul punct de discontinuitate al funcţiei f şi

$l_s(0)=0$ şi $l_d(0)=\lim\limits_{t\to 0, t>0}\dfrac{1}{t+1}=1.$

(iii) f are o creştere de tip exponenţial, adică $\exists M=1>0$, $\exists\sigma=0\in\mathbb{R}$, astfel încât

$|f(t)|\le 1e^{0t}, \forall t\in\,]0,+\infty[$.

Atunci $f \in \mathcal{O}$, cu abscisa de convergenţă $\sigma_f = 0$.
c) Analog cu **b)**$\Rightarrow f \in \mathcal{O}$; **d)** Nu se verifică (iii)$\Rightarrow f \notin \mathcal{O}$; **e)** Analog cu **b)**$\Rightarrow f \in \mathcal{O}$.



Definiţia 2. Fie $f \in \mathcal{O}$ o funcţie original cu abscisa de convergenţă σ_f. Fie $D = \{s \in \mathbb{C}; \operatorname{Re} s > \sigma_f\}$. Funcţia

$$\boxed{F : D \to \mathbb{C}, F(s) = \int_0^{+\infty} e^{-st} f(t)\, dt}$$

se numeşte *transformata Laplace a funcţiei* f sau *imaginea funcţiei* f *prin transformata Laplace*. Se notează

$\boxed{F(s) = \mathcal{L}\{f(t)\}(s).}$, chiar $\boxed{f(t) \longleftrightarrow F(s)}$.

Observaţia 2 (**Teorema fundamentală a transformatei Laplace**). Dacă $f \in \mathcal{O}$ este o funcţie original cu abscisa de convergenţă σ_f, atunci integrala improprie pe interval nemărginit, cu parametrul $s \in D$, din Definiţia 2, este absolut convergentă şi uniform convergentă pe D, deci transformata Laplace f este bine definită. Mai mult, este *continuă* şi *mărginită*, cu

$$\lim_{|s| \to \infty} F(s) = 0.$$

Exerciţiul 2. Utilizând definiţia, să se determine transformata Laplace pentru:

a) $f : \mathbb{R} \to \mathbb{R}, f(t) = \begin{cases} 0, & \text{dacă } t < 0, \\ t^2, & \text{dacă } 0 \le t \le 1, \\ 0, & \text{dacă } t > 1. \end{cases}$; **b)**○ $f : \mathbb{R} \to \mathbb{R}, f(t) = \begin{cases} 0, & \text{dacă } t < 1 \\ (t-1)^n, & \text{dacă } t \ge 1, \end{cases}$

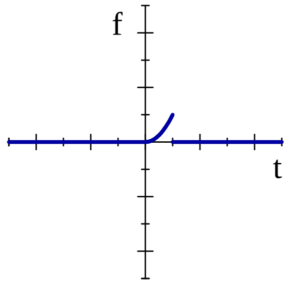
unde $n \in \mathbb{N}^*$ este dat.

c) $\eta : \mathbb{R} \to \mathbb{R}, \eta(t) = \begin{cases} 0, & \text{dacă } t < 0 \\ 1, & \text{dacă } t \ge 0, \end{cases}$; **d)** $\eta_a : \mathbb{R} \to \mathbb{R}, \eta_a(t) = \eta(t-a) = \begin{cases} 0, & \text{dacă } t < a \\ 1, & \text{dacă } t \ge a, \end{cases}$

unde $a > 0$ este dat.

e) $f : \mathbb{R} \to \mathbb{R}, f(t) = e^{\lambda t} \cdot \eta(t)$, unde $\lambda \in \mathbb{R}$;.($f : \mathbb{R} \to \mathbb{C}$, pentru $\lambda \in \mathbb{C}$);
f) $f : \mathbb{R} \to \mathbb{R}, f(t) = (\operatorname{ch} \omega t) \cdot \eta(t)$, unde $\omega \in \mathbb{R}, \omega > 0$; ($f : \mathbb{R} \to \mathbb{C}$, pentru $\omega \in \mathbb{C}^*$);
g) $f : \mathbb{R} \to \mathbb{R}, f(t) = (\operatorname{sh} \omega t) \cdot \eta(t)$, unde $\omega \in \mathbb{R}, \omega > 0$; ($f : \mathbb{R} \to \mathbb{C}$, pentru $\omega \in \mathbb{C}^*$);
h) $f : \mathbb{R} \to \mathbb{R}, f(t) = (\cos \omega t) \cdot \eta(t)$, unde $\omega \in \mathbb{R}, \omega > 0$; ($f : \mathbb{R} \to \mathbb{C}$, pentru $\omega \in \mathbb{C}^*$);
i) $f : \mathbb{R} \to \mathbb{R}, f(t) = (\sin \omega t) \cdot \eta(t)$, unde $\omega \in \mathbb{R}, \omega > 0$; ($f : \mathbb{R} \to \mathbb{C}$, pentru $\omega \in \mathbb{C}^*$);
j) $f : \mathbb{R} \to \mathbb{R}, f(t) = t^n \cdot \eta(t)$.

Rezolvare. a) Fie $f : \mathbb{R} \to \mathbb{R}, f(t) = \begin{cases} 0, & \text{dacă } t < 0, \\ t^2, & \text{dacă } 0 \le t \le 1, \\ 0, & \text{dacă } t > 1. \end{cases}$



•Se observă că $f \in \mathcal{O}$, cu abscisa de convergenţă $\sigma_f = 0$.
•$D = \{s \in \mathbb{C}; \operatorname{Re} s > 0\}$. Se calculează

$$F(s) = \int_0^{+\infty} e^{-st} f(t)\, dt = \int_0^1 e^{-st} t^2 dt.$$

$$\mathcal{I}(t;c) = \int e^{-st} t^2 dt \underset{\text{de integrare}}{\overset{t \text{ este variabilă}}{=}} \frac{1}{-s} \int t^2 \frac{d}{dt}\left(e^{-st}\right) dt = \frac{1}{-s}\left(t^2 e^{-st} - \int 2te^{-st} dt\right) =$$
$$= \frac{1}{-s}\left(t^2 e^{-st} - \frac{1}{-s}\int 2t \frac{d}{dt}\left(e^{-st}\right) dt\right) = \frac{1}{-s} t^2 e^{-st} - \left(\frac{1}{-s}\right)^2 \left(2te^{-st} - \int 2e^{-st} dt\right) =$$
$$= \frac{1}{-s} t^2 e^{-st} - \left(\frac{1}{-s}\right)^2 \left(2te^{-st} - 2\frac{e^{-st}}{-s}\right) + c = \frac{1}{s^3}\left(-s^2 t^2 e^{-st} - 2ste^{-st} - 2e^{-st}\right) + c,$$

pentru $\forall t \in [0,1], \forall c \in \mathbb{R}, \forall s \in D$. Atunci

$$\int_0^1 e^{-st} t^2 dt = \mathcal{I}(t;0)\big|_{t=0}^{t=1} = \frac{1}{s^3}\left(-s^2 e^{-s} - 2se^{-s} - 2e^{-s} + 2\right).$$

Deci $F : D \to \mathbb{C}, F(s) = \dfrac{1}{s^3}\left(-s^2 e^{-s} - 2se^{-s} - 2e^{-s} + 2\right)$.

b) Fie $n \in \mathbb{N}^*$ fixat. Fie $f : \mathbb{R} \to \mathbb{R}, f(t) = \begin{cases} 0, & \text{dacă } t < 1 \\ (t-1)^n & \text{dacă } t \geq 1. \end{cases}$

•Se observă că $f \in \mathcal{O}$.
•$D = \{s \in \mathbb{C}; \operatorname{Re} s > \sigma_f\}$.
Se calculează

$$F(s) = \int_0^{+\infty} e^{-st} f(t)\, dt = \int_1^{+\infty} e^{-st} (t-1)^n dt.$$

$$\mathcal{I}_0(t;c) = \int e^{-st} dt = \frac{e^{-st}}{-s} + c.$$

$$\mathcal{I}_1(t;c) = \int e^{-st}(t-1)\, dt = \frac{1}{-s}\int (t-1)\frac{d}{dt}\left(e^{-st}\right) dt =$$
$$= \frac{1}{-s}\left((t-1)e^{-st} - \int e^{-st} dt\right) = \frac{1}{-s}\left((t-1)e^{-st} - \frac{e^{-st}}{-s}\right) + c.$$

...

$$\mathcal{I}_n(t;c) = \int e^{-st}(t-1)^n dt \underset{\text{de integrare}}{\overset{t \text{ este variabilă}}{=}} \frac{1}{-s}\int (t-1)^n \frac{d}{dt}\left(e^{-st}\right) dt =$$
$$= \frac{1}{-s}\left((t-1)^n e^{-st} - \int n(t-1)^{n-1} e^{-st} dt\right) = \frac{1}{-s}(t-1)^n e^{-st} - \frac{1}{-s} n\mathcal{I}_{n-1}(t;c),$$

$\forall t \in [1, +\infty[, \forall c \in \mathbb{R}, \forall s \in D, \forall n \in \mathbb{N}^*$.

Se obţine

$$\mathcal{I}_n(t;c) = \left(\frac{1}{-s}\right)^1 (t-1)^n e^{-st} - \left(\frac{1}{-s}\right)^2 n(t-1)^{n-1} e^{-st} + \left(\frac{1}{-s}\right)^3 n(n-1)(n-2)(t-1)^{n-2} e^{-st} +$$
$$+ ...$$

Atunci

$$F(s)=\int_0^{+\infty}e^{-st}f(t)\,dt=\int_1^{+\infty}e^{-st}(t-1)^n\,dt=\mathcal{I}_n(t;0)\Big|_{t=1}^{t\to+\infty}=0-\left(\frac{-1}{s}\right)^{n+1}n!e^{-s},$$
$\forall t\in[0,1], \forall s\in D.$

Deci, pentru $n\in\mathbb{N}^*, F:D\to\mathbb{C},\ F(s)=\left(\frac{-1}{s}\right)^n n!e^{-s}.$

c) Fie $\eta:\mathbb{R}\to\mathbb{R}, \eta(t)=\begin{cases}0, & \text{dacă } t<0\\ 1, & \text{dacă } t\geq 0\end{cases}$ -a se vedea Curs:

$\boxed{\mathcal{L}\{\eta(t)\}(s)=\frac{1}{s}}$ sau $\boxed{f(t)=\eta(t)\Rightarrow F(s)=\frac{1}{s}.}$

d) Se obţine, cu definiţia, pentru $a>0$ dat

$\boxed{\mathcal{L}\{\eta(t-a)\}(s)=\frac{e^{-as}}{s}}$ sau $\boxed{f(t)=\eta(t-a)\Rightarrow F(s)=\frac{e^{-as}}{s}.}$

e) Fie $f:\mathbb{R}\to\mathbb{R}, f(t)=e^{\lambda t}\cdot\eta(t)$, unde $\lambda\in\mathbb{R}$ ($f:\mathbb{R}\to\mathbb{C}$, pentru $\lambda\in\mathbb{C}$). A se vedea Curs:

$\boxed{\mathcal{L}\left\{e^{\lambda t}\cdot\eta(t)\right\}(s)=\frac{1}{s-\lambda}}$ sau $\boxed{f(t)=e^{\lambda t}\cdot\eta(t)\Rightarrow F(s)=\frac{1}{s-\lambda}.}$

f), **g)** analog cu **e)**. Se obţine

$\boxed{\mathcal{L}\{(\operatorname{ch}\omega t)\cdot\eta(t)\}(s)=\frac{s}{s^2-\omega^2};\mathcal{L}\{(\operatorname{sh}\omega t)\cdot\eta(t)\}(s)=\frac{\omega}{s^2-\omega^2}.}$

h) Fie $f:\mathbb{R}\to\mathbb{R}, f(t)=(\cos\omega t)\cdot\eta(t)$.

•Se observă că $f\in\mathcal{O}$, cu abscisa de convergenţă $\sigma_f=0$

•$D=\{s\in\mathbb{C};\operatorname{Re}s>0\}$. Se calculează

$$F(s)=\int_0^{+\infty}e^{-st}f(t)\,dt=\int_0^{+\infty}e^{-st}\cos(\omega t)\,dt.$$

$$\mathcal{I}(t;c)=\int e^{-st}\cos(\omega t)\,dt\overset{t \text{ este variabilă}}{\underset{\text{de integrare}}{=}}\frac{1}{s^2+\omega}e^{-st}(\omega\sin(\omega t)-s\cos(\omega t))+c,$$
$\forall t\in[0,+\infty[, \forall c\in\mathbb{R}, \forall s\in D.$ Atunci

$$\int_0^{+\infty}e^{-st}\cos\omega t dt=\mathcal{I}(t;0)\Big|_{t=0}^{t\to+\infty}=0-\frac{-s}{s^2+\omega^2}.$$

Deci $F:D\to\mathbb{C}, F(s)=\frac{s}{s^2+\omega^2}$. Se scrie

$\boxed{\mathcal{L}\{(\cos\omega t)\cdot\eta(t)\}(s)=\frac{s}{s^2+\omega^2}}$ sau $\boxed{f(t)=(\cos\omega t)\cdot\eta(t)\Rightarrow F(s)=\frac{s}{s^2+\omega^2}.}$

i) analog cu **h)**. Se obţine

$\boxed{\mathcal{L}\{(\sin\omega t)\cdot\eta(t)\}(s)=\frac{\omega}{s^2+\omega^2}}$ sau $\boxed{f(t)=(\sin\omega t)\cdot\eta(t)\Rightarrow F(s)=\frac{\omega}{s^2+\omega^2}.}$

Teorema 1 (**de liniaritate**). *Transformata Laplace este o funcţie liniară, adică* $\forall f,g\in\mathcal{O},\forall\alpha,\beta\in\mathbb{C}$:

$\boxed{\mathcal{L}\{\alpha f(t)+\beta g(t)\}(s)=\alpha\mathcal{L}\{f(t)\}(s)+\beta\mathcal{L}\{g(t)\}(s),}\ \forall s\in\mathbb{C}$ *cu* $\operatorname{Re}s>\max\{\sigma_f,\sigma_g\}$.

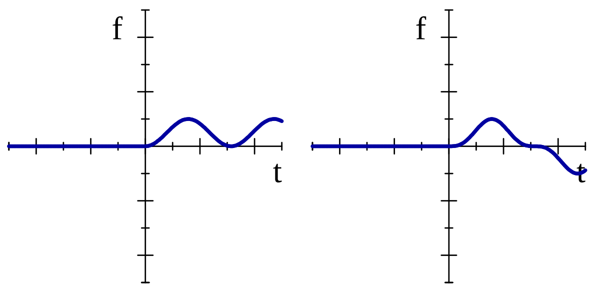
Observaţia 2. Transformatele Laplace pentru

$f(t)=(\operatorname{ch}\omega t)\cdot\eta(t); f(t)=(\operatorname{sh}\omega t)\cdot\eta(t); f(t)=(\cos\omega t)\cdot\eta(t); f(t)=(\sin\omega t)\cdot\eta(t)$

se pot determina folosind Teorema 1 şi $\mathcal{L}\left\{e^{\lambda t}\cdot\eta(t)\right\}(s)=\frac{1}{s-\lambda}$. A se vedea Curs.

Exerciţiul 3. Utilizând Teorema 1, de liniaritate, să se determine transformata Laplace pentru :

a) $f:\mathbb{R}\to\mathbb{R}, f(t)=(\sin t)^2\cdot\eta(t)$; **b)** $f:\mathbb{R}\to\mathbb{R}, f(t)=(\sin t)^3\cdot\eta(t)$;



c) $f:\mathbb{R}\to\mathbb{R}, f(t)=\dfrac{e^{at}-e^{bt}}{2}\cdot\eta(t).$

Rezolvare. **a)** $\mathcal{L}\left\{(\sin t)^2\cdot\eta(t)\right\}(s)=\mathcal{L}\left\{\dfrac{1-\cos 2t}{2}\cdot\eta(t)\right\}(s)=$

$$\overset{T1,\text{ de lin}}{=}\frac{1}{2}\mathcal{L}\{\eta(t)\}(s)-\frac{1}{2}\mathcal{L}\{(\cos(2t))\cdot\eta(t)\}(s)=\frac{1}{2}\cdot\frac{1}{s}-\frac{1}{2}\cdot\frac{s}{s^2+2^2}=\frac{1}{2}\cdot\frac{4}{s^2+2^2}.$$

b) $\mathcal{L}\left\{(\sin t)^3\cdot\eta(t)\right\}(s)=\mathcal{L}\left\{\dfrac{3\sin t-\sin(3t)}{4}\cdot\eta(t)\right\}(s)=$

$$\overset{T1,\text{ de lin}}{=}\frac{3}{4}\mathcal{L}\{(\sin t)\,\eta(t)\}(s)-\frac{1}{4}\mathcal{L}\{(\sin(3t))\cdot\eta(t)\}(s)=\frac{3}{4}\cdot\frac{1}{s^2+1^2}-\frac{1}{4}\cdot\frac{3}{s^2+3^2}.$$

c) $\mathcal{L}\left\{\dfrac{e^{at}-e^{bt}}{2}\cdot\eta(t)\right\}(s)=\mathcal{L}\left\{\dfrac{1-\cos 2t}{2}\cdot\eta(t)\right\}(s)=$

$$\overset{T1,\text{ de lin}}{=}\frac{1}{2}\mathcal{L}\left\{e^{at}\eta(t)\right\}(s)-\frac{1}{2}\mathcal{L}\left\{e^{bt}\eta(t)\right\}(s)=\frac{1}{2}\cdot\frac{1}{s-a}-\frac{1}{2}\cdot\frac{1}{s-b}.$$

Teorema 2 (**a asemănării, comportarea la omotetie**). *Fie* $f\in\mathcal{O}$ *cu* σ_f *abscisa de convergenţă şi* $F(s)=\mathcal{L}\{f(t)\}(s)$. *Atunci,* $\forall\omega\in\mathbb{R},\omega>0$

$$\boxed{\mathcal{L}\{f(\omega t)\}(s)=\frac{1}{\omega}F\left(\frac{s}{\omega}\right),}\ \forall s\in\mathbb{C}\ \textit{cu}\ \operatorname{Re}s>\sigma_f.$$

Observaţia 3. Transformatele Laplace pentru
a) $f(t)=(\operatorname{ch}\omega t)\cdot\eta(t)$; **b)** $f(t)=(\operatorname{sh}\omega t)\cdot\eta(t)$;
c) $f(t)=(\cos\omega t)\cdot\eta(t)$; **d)** $f(t)=(\sin\omega t)\cdot\eta(t)$
se pot determina folosind Teorema 2 şi transformatele funcţiilor f cu $\omega=1$. A se vedea Curs.

Teorema 3 (**a deplasării, frequency-shift**). *Fie* $f\in\mathcal{O}$ *cu* σ_f *abscisa de convergenţă şi* $F(s)=\mathcal{L}\{f(t)\}(s)$. *Atunci,* $\forall\lambda\in\mathbb{C}$,

$$\boxed{\mathcal{L}\left\{e^{-\lambda t}f(t)\right\}(s)=F(s+\lambda),}\ \forall s\in\mathbb{C}\ \textit{cu}\ \operatorname{Re}s>\sigma_f-\operatorname{Re}\lambda.$$

Exerciţiul 4. Utilizând teorema deplasării , să se determine transformata Laplace pentru :
a) $f:\mathbb{R}\to\mathbb{R}, f(t)=e^{-\lambda t}(\operatorname{ch}(\omega t))\cdot\eta(t), \lambda\in\mathbb{C},\omega\in\mathbb{R},\omega>0$;
b) $f:\mathbb{R}\to\mathbb{R}, f(t)=e^{-\lambda t}(\operatorname{sh}(\omega t))\cdot\eta(t), \lambda\in\mathbb{C},\omega\in\mathbb{R},\omega>0$;
c) $f:\mathbb{R}\to\mathbb{R}, f(t)=e^{-\lambda t}(\cos(\omega t))\cdot\eta(t), \lambda\in\mathbb{C},\omega\in\mathbb{R},\omega>0$;
d) $f:\mathbb{R}\to\mathbb{R}, f(t)=e^{-\lambda t}(\sin(\omega t))\cdot\eta(t), \lambda\in\mathbb{C},\omega\in\mathbb{R},\omega>0$;
(rezultatele sunt valabile şi pentru $f:\mathbb{R}\to\mathbb{C}$, dacă $\omega\in\mathbb{R}^\star$);
e) $f:\mathbb{R}\to\mathbb{R}, f(t)=e^{-2t}(\sin(3t))\cdot\eta(t)$;
f) $f:\mathbb{R}\to\mathbb{R}, f(t)=e^{-\lambda t}t^n\cdot\eta(t), \lambda\in\mathbb{C}, n\in\mathbb{N}^*$.

Rezolvare (schiţă).
a) $\widetilde{f}(t)=(\operatorname{ch}(\omega t))\cdot\eta(t)\Rightarrow\widetilde{F}(s)=\mathcal{L}\{(\operatorname{ch}(\omega t))\cdot\eta(t)\}(s)=\dfrac{s}{s^2-\omega^2}\Rightarrow$

$$\mathcal{L}\left\{e^{-\lambda t}(\operatorname{ch}(\omega t))\cdot\eta(t)\right\}(s)=\widetilde{F}(s+\lambda)=\frac{s+\lambda}{(s+\lambda)^2-\omega^2}.$$

b) $\widetilde{f}(t) = (\operatorname{sh}(\omega t)) \cdot \eta(t) \Rightarrow \widetilde{F}(s) = \mathcal{L}\{(\operatorname{sh}(\omega t)) \cdot \eta(t)\}(s) = \dfrac{\omega}{s^2 - \omega^2} \Rightarrow$

$\mathcal{L}\left\{e^{-\lambda t}(\operatorname{sh}(\omega t)) \cdot \eta(t)\right\}(s) = \widetilde{F}(s + \lambda) = \dfrac{\omega}{(s+\lambda)^2 - \omega^2}.$

c) $\widetilde{f}(t) = (\cos(\omega t)) \cdot \eta(t) \Rightarrow \widetilde{F}(s) = \mathcal{L}\{(\cos(\omega t)) \cdot \eta(t)\}(s) = \dfrac{s}{s^2 + \omega^2} \Rightarrow$

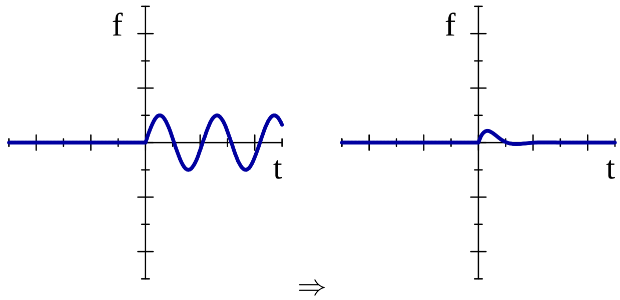
$\mathcal{L}\left\{e^{-\lambda t}(\cos(\omega t)) \cdot \eta(t)\right\}(s) = \widetilde{F}(s + \lambda) = \dfrac{s+\lambda}{(s+\lambda)^2 + \omega^2}.$

d) $\widetilde{f}(t) = (\sin(\omega t)) \cdot \eta(t) \Rightarrow \widetilde{F}(s) = \mathcal{L}\{(\sin(\omega t)) \cdot \eta(t)\}(s) = \dfrac{\omega}{s^2 + \omega^2} \Rightarrow$

$\mathcal{L}\left\{e^{-\lambda t}(\sin(\omega t)) \cdot \eta(t)\right\}(s) = \widetilde{F}(s + \lambda) = \dfrac{\omega}{(s+\lambda)^2 + \omega^2}.$

e) $\widetilde{f}(t) = (\sin(3t)) \cdot \eta(t) \Rightarrow \widetilde{F}(s) = \mathcal{L}\{(\sin(3t)) \cdot \eta(t)\}(s) = \dfrac{3}{s^2 + 3^2} \Rightarrow$

$\mathcal{L}\left\{e^{-2t}(\sin(3t)) \cdot \eta(t)\right\}(s) = \widetilde{F}(s + 2) = \dfrac{3}{(s+2)^2 + 3^2}.$



Teorema 4 (**a întârzierii argumentului, comportarea la translaţie, positive time-shift**.). *Fie* $f \in \mathcal{O}$ *cu* σ_f *abscisa de convergenţă şi* $F(s) = \mathcal{L}\{f(t)\}(s)$. *Atunci,* $\forall t_0 \in \mathbb{R}, t_0 > 0$,

$\boxed{\mathcal{L}\{f(t - t_0)\}(s) = e^{-t_0 s} F(s),}\ \forall s \in \mathbb{C}$ *cu* $\operatorname{Re} s > \sigma_f$.

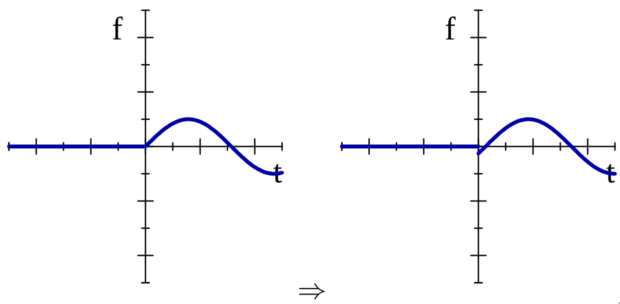
Exerciţiul 5. Utilizând teorema întârzierii argumentului, să se determine transformata Laplace pentru :

a) $f : \mathbb{R} \to \mathbb{R}, f(t) = \left(\sin\left(t - \frac{\pi}{12}\right)\right) \cdot \eta(t)$; **b)** $f : \mathbb{R} \to \mathbb{R}, f(t) = e^{7t-14} \cdot \eta(t)$.

Rezolvare.

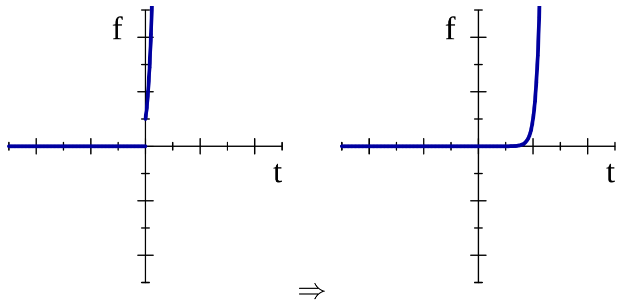
a) $\widetilde{f}(t) = (\sin t) \cdot \eta(t) \Rightarrow \widetilde{F}(s) = \mathcal{L}\{(\sin t) \cdot \eta(t)\}(s) = \dfrac{1}{s^2 + 1} \Rightarrow$

$\mathcal{L}\left\{\left(\sin\left(t - \frac{\pi}{12}\right)\right) \cdot \eta(t)\right\}(s) = e^{-t_0 s}\widetilde{F}(s) = e^{-\frac{\pi}{12}s}\dfrac{1}{s^2+1}.$



b) $\widetilde{f}(t) = e^{7t} \cdot \eta(t) \Rightarrow \widetilde{F}(s) = \mathcal{L}\left\{e^{7t} \cdot \eta(t)\right\}(s) = \dfrac{1}{s-7} \Rightarrow$

$\mathcal{L}\left\{e^{7(t-2)} \cdot \eta(t)\right\}(s) = e^{-t_0 s}\widetilde{F}(s) = e^{-2s}\dfrac{1}{s-7}.$



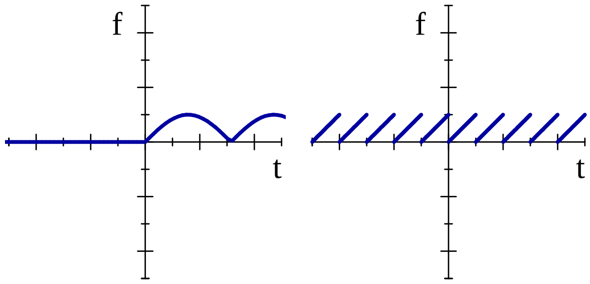
$\Rightarrow$

Teorema 5 (de periodicitate). *Fie* $f \in \mathcal{O}$ *cu* σ_f *abscisa de convergenţă şi* $F(s) = \mathcal{L}\{f(t)\}(s)$. *Dacă* f *este funcţie periodică pentru* $t \geq 0$, *de perioadă* $T > 0$, *atunci*

$$\boxed{\mathcal{L}\{f(t)\}(s) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt,} \forall s \in \mathbb{C} \text{ cu } \operatorname{Re} s > \sigma_f.$$

Exerciţiul 6. Utilizând teorema de periodicitate, să se determine transformata Laplace pentru :

a) $f : \mathbb{R} \to \mathbb{R}, f(t) = \begin{cases} 0, & \text{dacă } t < 0 \\ |\sin t|, & \text{dacă } t \geq 0. \end{cases}$; **b)** $f : \mathbb{R} \to \mathbb{R}, f(t) = \{t\}$.



Rezolvare. **a)** Se observă că f este funcţie original şi periodică pentru $t \geq 0$, cu $T = \pi$.

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{1-e^{-s\pi}}\int_0^\pi e^{-st} f(t)\,dt = \frac{1}{1-e^{-s\pi}}\int_0^\pi e^{-st}\sin t dt.$$

Dar $\int e^{-st}\sin t dt = \frac{1}{s^2+1}e^{-st}(-\cos t - s\sin t) + c \Rightarrow$

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{1-e^{-s\pi}}\left(\frac{1}{s^2+1}e^{-s\pi}\cdot 1 - \frac{1}{s^2+1}1(-1)\right) = \frac{1}{s^2+1}\frac{1+e^{-s\pi}}{1-e^{-s\pi}}.$$

○**b)** Se observă (a se vedea Curs) că f este funcţie original şi periodică pentru $t \geq 0$, cu $T = 1$.

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{1-e^{-s\cdot 1}}\int_0^1 e^{-st} t dt = \frac{1}{1-e^{s\pi}}\int_0^1 e^{-st} t dt$$

$$= \frac{1}{1-e^{-s}}\left(-\frac{1}{s}te^{-st}\Big|_{t=0}^{t=1} - \frac{1}{s^2}e^{-st}\Big|_{t=0}^{t=1}\right) = \frac{1}{1-e^{-s}}\left(-\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2}\right).$$

Teorema 6 (de derivare a originalului, i**maginea derivatei originalului).**

a) *Fie* f *o funcţie continuă pentru* $t > 0$ *astfel încât* $f \in \mathcal{O}$ *cu* σ_f *abscisa de convergenţă şi* $F(s) = \mathcal{L}\{f(t)\}(s)$. *Se presupune că* $\exists f'$ *peste tot, eventual cu excepţia originii, şi* $f' \in \mathcal{O}$ *cu* $\sigma_{f'}$ *abscisa de convergenţă. Se notează* $f(0_+) = \lim_{t\to 0, t>0} f(t)$. *Atunci*

$$\boxed{\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0_+),} \forall s \in \mathbb{C} \text{ cu } \operatorname{Re} s > \max\{\sigma_f, \sigma_{f'}\}.$$

b) *Fie* f *o funcţie continuă pentru* $t > 0$ *astfel încât* $f \in \mathcal{O}$ *cu* σ_f *abscisa de convergenţă şi* $F(s) = \mathcal{L}\{f(t)\}(s)$. *Se presupune că* f *este de* n *ori derivabilă,* $f', ..., f^{(n)} \in \mathcal{O}$ *cu* $\sigma_{f'}, ..., \sigma_{f^{(n)}}$ *abscise de convergenţă şi cu* $f', ..., f^{(n-1)}$ *continue pentru* $t > 0$. *Se notează* $f(0_+) = \lim_{t\to 0, t>0} f(t)$,...,

$f^{(n-1)}(0_+) = \lim\limits_{t\to 0, t>0} f^{(n-1)}(t)$. *Atunci,* $\forall n \in \mathbb{N}^*$,

$$\boxed{\mathcal{L}\left\{f^{(n)}(t)\right\}(s) = s^n F(s) - \left[s^{n-1} f(0_+) + s^{n-2} f'(0_+) + ... + s f^{(n-2)}(0_+) + f^{(n-1)}(0_+)\right],}$$

$\forall s \in \mathbb{C}$ *cu* $\operatorname{Re} s > \max\left\{\sigma_f, \sigma_{f'}, ..., \sigma_{f^{(n)}}\right\}$.

○**Exerciţiul 7**. Fie $f : \mathbb{R} \to \mathbb{R}, f(t) = \begin{cases} 0, & \text{dacă } t < 0 \\ 2t, & \text{dacă } 0 \le t \le 1 \\ 0, & \text{dacă } t > 1 \end{cases}$

a) Să se calculeze $\mathcal{L}\{f(t)\}$;
b) Să se calculeze $\mathcal{L}\{f'(t)\}$;
c) Se poate aplica formula $\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0+)$? Justificare.
Rezolvare. A se vedea Curs.

Teorema 7 (de derivare a imaginii). *Fie* $f \in \mathcal{O}$ *cu* σ_f *abscisa de convergenţă şi* $F(s) = \mathcal{L}\{f(t)\}(s)$. *Atunci*
a) $\boxed{\mathcal{L}\{(-t) f(t)\}(s) = F'(s),}$ $\forall s \in \mathbb{C}$ *cu* $\operatorname{Re} s > \sigma_f$.
b) $\forall n \in \mathbb{N}^*$, $\boxed{\mathcal{L}\{(-t)^n f(t)\}(s) = F^{(n)}(s),}$ $\forall s \in \mathbb{C}$ *cu* $\operatorname{Re} s > \sigma_f$.

Exerciţiul 8. Utilizând teorema de derivare a imaginii, să se determine transformata Laplace pentru :
a) $f : \mathbb{R} \to \mathbb{R}, f(t) = -te^{\lambda t} \cdot \eta(t), \lambda \in \mathbb{C}$;**b)** $f : \mathbb{R} \to \mathbb{R}, f(t) = t^n e^{\lambda t} \cdot \eta(t), n \in \mathbb{N}^*$;
c) $f : \mathbb{R} \to \mathbb{R}, f(t) = t(\operatorname{ch}(\omega t)) \cdot \eta(t)$;**d)** $f : \mathbb{R} \to \mathbb{R}, f(t) = t(\operatorname{sh}(\omega t)) \cdot \eta(t)$;
e) $f : \mathbb{R} \to \mathbb{R}, f(t) = t(\cos(\omega t)) \cdot \eta(t)$;**f)** $f : \mathbb{R} \to \mathbb{R}, f(t) = t(\sin(\omega t)) \cdot \eta(t)$;
g) $f : \mathbb{R} \to \mathbb{R}, f(t) = t^2(\sin 2t)(\cos 3t) \cdot \eta(t)$; **h)** $f : \mathbb{R} \to \mathbb{R}, f(t) = t^2(\cos 3t) \cdot \eta(t)$;
i) $f : \mathbb{R} \to \mathbb{R}, f(t) = t(\sin 4t) \cdot \eta(t)$; **j)** $f : \mathbb{R} \to \mathbb{R}, f(t) = t^2 e^{2t}(\sin t) \cdot \eta(t)$;
k) $f : \mathbb{R} \to \mathbb{R}, f(t) = te^t(\cos t) \cdot \eta(t)$.
Rezolvare. a), b) A se vedea Curs

$\boxed{\mathcal{L}\left\{t^n e^{\lambda t} \cdot \eta(t)\right\}(s) = \dfrac{n!}{(s-\lambda)^{n+1}}.}$ În particular, $\boxed{\mathcal{L}\{t^n \cdot \eta(t)\}(s) = \dfrac{n!}{s^{n+1}}.}$

c) $\widetilde{f}(t) = \operatorname{ch}(\omega t) \cdot \eta(t) \Rightarrow \widetilde{F}(s) = \mathcal{L}\{\operatorname{ch}(\omega t) \cdot \eta(t)\}(s) = \dfrac{s}{s^2 - \omega^2} \Rightarrow$

$$\mathcal{L}\{t(\operatorname{ch}(\omega t)) \cdot \eta(t)\}(s) \overset{T1,\text{de lin}}{=} (-1)\mathcal{L}\{(-t)\operatorname{ch}(\omega t) \cdot \eta(t)\}(s) =$$
$$= (-1)\widetilde{F}'(s) = (-1)\frac{d}{ds}\left(\frac{s}{s^2-\omega^2}\right) = (-1)\frac{1\left(s^2-\omega^2\right) - s \cdot 2s}{\left(s^2-\omega^2\right)^2} = \frac{s^2+\omega^2}{\left(s^2-\omega^2\right)^2}.$$

$$\boxed{\mathcal{L}\{t(\operatorname{ch}(\omega t)) \cdot \eta(t)\}(s) = \frac{s^2+\omega^2}{\left(s^2-\omega^2\right)^2}; \mathcal{L}\{t(\operatorname{sh}(\omega t)) \cdot \eta(t)\}(s) = \frac{2s\omega}{\left(s^2-\omega^2\right)^2}}$$

$$\boxed{\mathcal{L}\{t(\cos(\omega t)) \cdot \eta(t)\}(s) = \frac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}; \mathcal{L}\{t(\sin(\omega t)) \cdot \eta(t)\}(s) = \frac{2s\omega}{\left(s^2+\omega^2\right)^2}.}$$

Se poate arăta şi că

$$\boxed{\mathcal{L}\{t^n(\operatorname{ch}(\omega t)) \cdot \eta(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\frac{s}{s^2-\omega^2}\right); \mathcal{L}\{t^n(\operatorname{sh}(\omega t)) \cdot \eta(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\frac{\omega}{s^2-\omega^2}\right).}$$

$$\boxed{\mathcal{L}\{t(\cos(\omega t)) \cdot \eta(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\frac{s}{s^2+\omega^2}\right); \mathcal{L}\{t^n(\sin(\omega t)) \cdot \eta(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\frac{\omega}{s^2+\omega^2}\right).}$$

d), e), f) analog cu **c)**

g) Deoarece $\widetilde{f}(t)=\sin(\omega t)\cdot\eta(t)\Rightarrow\widetilde{F}(s)=\mathcal{L}\{\sin(\omega t)\cdot\eta(t)\}(s)=\dfrac{\omega}{s^2+\omega^2}$, atunci

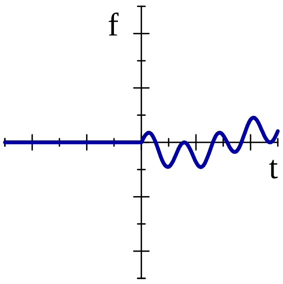
$$\mathcal{L}\left\{t^2(\sin 2t)(\cos 3t)\cdot\eta(t)\right\}(s)=\mathcal{L}\left\{t^2\frac{\sin 5t+\sin(-t)}{2}\cdot\eta(t)\right\}(s)=$$

$$\overset{T1,\text{de lin}}{=}\frac{1}{2}\mathcal{L}\left\{(-t)^2(\sin 5t)\cdot\eta(t)\right\}(s)-\frac{1}{2}\mathcal{L}\left\{(-t)^2(\sin t)\cdot\eta(t)\right\}(s)=$$

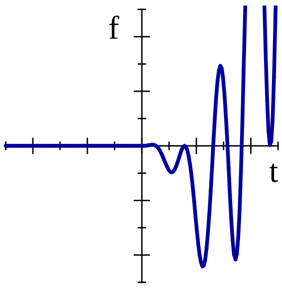
$$\overset{T7,\text{de derivare a imaginii}}{=}\frac{1}{2}\left(\frac{5}{s^2+5^2}\right)''_s-\frac{1}{2}\left(\frac{1}{s^2+1^2}\right)''_s=$$

$$=\frac{1}{2}\left(\frac{-5\cdot 2s}{(s^2+5^2)^2}\right)'-\frac{1}{2}\left(\frac{-1\cdot 2s}{(s^2+1^2)^2}\right)'=$$

$$=-5\left(\frac{(s^2+5^2)-s\cdot 2\cdot 2s}{(s^2+5^2)^3}\right)+\left(\frac{(s^2+1^2)-s\cdot 2\cdot 2s}{(s^2+1^2)^3}\right)=-5\frac{-3s^2+5^2}{(s^2+5^2)^3}+\frac{-3s^2+1^2}{(s^2+1^2)^3}.$$

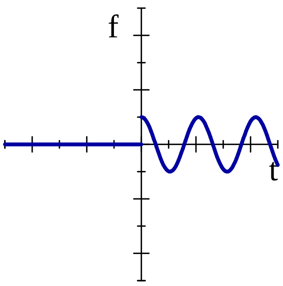


$\Rightarrow$

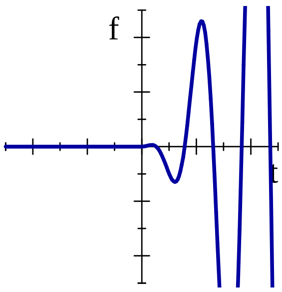


Analog

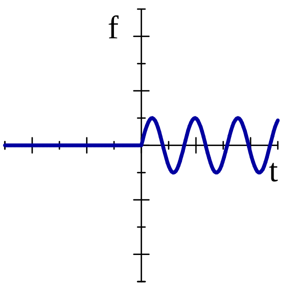
$$\mathcal{L}\left\{t^2(\cos 3t)\cdot\eta(t)\right\}(s)\overset{T7,\text{de derivare a imaginii}}{=}\left(\frac{s}{s^2+3^2}\right)''=\frac{2s^3-54s}{(s^2+9)^3}.$$



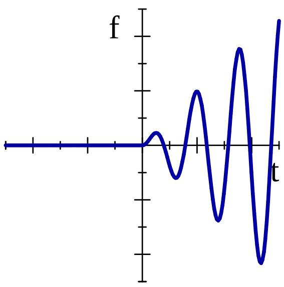
$\Rightarrow$



$$\mathcal{L}\left\{t(\sin 4t)\cdot\eta(t)\right\}(s)\overset{T7,\text{de derivare a imaginii}}{=}-\left(\frac{4}{s^2+4^2}\right)'=\frac{8s}{(s^2+16)^2}.$$



$\Rightarrow$



$$\mathcal{L}\left\{t^2e^{2t}(\sin t)\cdot\eta(t)\right\}(s)\overset{T7,\text{de derivare a imaginii}}{=}\left(\frac{1}{(s-2)^2+1}\right)''=2\frac{(s-2)^2-1}{((s-2)^2+1)^3}.$$

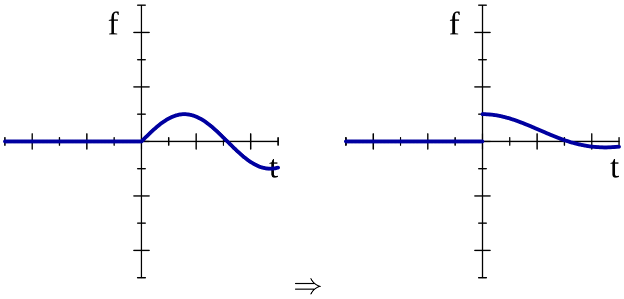
$$\mathcal{L}\left\{te^{t}(\cos t)\cdot\eta(t)\right\}(s)\overset{T7,\text{de derivare a imaginii}}{=}-\left(\frac{s-1}{(s-1)^2+1}\right)'=\frac{s^2-2s}{\left((s-1)^2+1\right)^2}.$$

Teorema 8 (de integrare a imaginii). *Fie $f\in\mathcal{O}$ cu σ_f abscisa de convergenţă şi $F(s)=\mathcal{L}\{f(t)\}(s)$. Fie $g:\mathbb{R}\to\mathbb{R}$ (sau $g:\mathbb{R}\to\mathbb{C}$) definită prin $g(t)=\dfrac{f(t)}{t},\forall t\in\mathbb{R}^*$. Dacă $g\in\mathcal{O}$ cu σ_g abscisa de convergenţă, atunci*

$$\boxed{\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s)=\int_s^{+\infty}F(p)\,dp,}\ \forall s\in\mathbb{C}\ cu\ \operatorname{Re}s>\max\{\sigma_f,\sigma_g\}.$$

Exerciţiul 9. Utilizând teorema de integrare a imaginii, să se determine transformata Laplace pentru :

a) $f:\mathbb{R}\to\mathbb{R},f(t)=\dfrac{\sin t}{t}\cdot\eta(t)$.



$\Rightarrow$

Rezolvare. Deoarece $\widetilde{f}(t)=\sin t\cdot\eta(t)\Rightarrow\widetilde{F}(s)=\mathcal{L}\{\sin t\cdot\eta(t)\}(s)=\dfrac{1}{s^2+1}$,atunci

$$\mathcal{L}\left\{\frac{\sin t}{t}\cdot\eta(t)\right\}(s)\overset{T8}{=}\int_s^{+\infty}\widetilde{F}(p)\,dp=\int_s^{+\infty}\frac{1}{p^2+1}dp=\operatorname{arctg}p\Big|_{p=s}^{p\to+\infty}=\tfrac{\pi}{2}-\operatorname{arctg}s.$$

b) $f:\mathbb{R}\to\mathbb{R},f(t)=\dfrac{\cos at-\cos bt}{t}\cdot\eta(t),\forall a,b\in\mathbb{R}$

Rezolvare. Deoarece

$$\widetilde{f}(t)=(\cos at-\cos bt)\cdot\eta(t)\Rightarrow\widetilde{F}(s)=\mathcal{L}\{(\cos at-\cos bt)\cdot\eta(t)\}(s)=\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2},$$

atunci

$$\mathcal{L}\left\{\frac{\cos at-\cos bt}{t}\cdot\eta(t)\right\}(s)\overset{T8}{=}\int_s^{+\infty}\widetilde{F}(p)\,dp=\int_s^{+\infty}\left(\frac{p}{p^2+a^2}-\frac{p}{p^2+b^2}\right)dp$$
$$=\tfrac{1}{2}\ln\frac{p^2+a^2}{p^2+b^2}\Big|_{p=s}^{p\to+\infty}=-\tfrac{1}{2}\ln\frac{s^2+a^2}{s^2+b^2}.$$

Teorema 9 (de integrare a originalului, imaginea originalului integrat). *Fie $f\in\mathcal{O}$ cu σ_f abscisa de convergenţă şi $F(s)=\mathcal{L}\{f(t)\}(s)$. Atunci, fără a scrie "$\cdot\eta(t)$",*

a) $\boxed{\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s)=\frac{1}{s}F(s),}\ \forall s\in\mathbb{C}\ cu\ \operatorname{Re}s>\sigma_f.$

b) $\forall n\in\mathbb{N}^*,\boxed{\mathcal{L}\left\{\underbrace{\int_0^t\left(\int_0^{\tau_1}\underbrace{\left(...\int_0^{\tau_{n-1}}f(\tau_n)\,d\tau_n\right)}\,d\tau_2\right)}\,d\tau_1\right\}(s)=\frac{1}{s^n}F(s),}\forall s\in\mathbb{C}\ cu\ \operatorname{Re}s>\sigma_f.$

Exerciţiul 10. Utilizând teorema de integrare a originalului, să se determine transformata Laplace pentru :

a) $f:\mathbb{R}\to\mathbb{R},f(t)=\left(\int_0^t\frac{\sin\tau}{\tau}d\tau\right)\cdot\eta(t);$

Rezolvare. Deoarece $\widetilde{f}(t)=\dfrac{\sin\tau}{\tau}\cdot\eta(t)\Rightarrow\widetilde{F}(s)=\mathcal{L}\left\{\dfrac{\sin\tau}{\tau}\cdot\eta(t)\right\}(s)=\frac{\pi}{2}-\operatorname{arctg}s$, atunci

$$\mathcal{L}\left\{\left(\int_0^t\frac{\sin\tau}{\tau}d\tau\right)\cdot\eta(t)\right\}(s)\overset{T9}{=}\frac{1}{s}\widetilde{F}(s)=\frac{1}{s}\left(\tfrac{\pi}{2}-\operatorname{arctg}s\right);$$

b) $f:\mathbb{R}\to\mathbb{R}, f(t)=\left(\displaystyle\int_0^t\tau^4e^{-2\tau}d\tau\right)\cdot\eta(t)$;

Rezolvare. Deoarece $\widetilde{f}(t)=t^4e^{-2t}\cdot\eta(t)\Rightarrow\widetilde{F}(s)=\mathcal{L}\left\{t^4e^{-2t}\cdot\eta(t)\right\}(s)=\dfrac{4!}{(s+2)^5}$, atunci

$$\mathcal{L}\left\{\left(\int_0^t\tau^4e^{-2\tau}d\tau\right)\cdot\eta(t)\right\}(s)\overset{T9}{=}\frac{1}{s}\widetilde{F}(s)=\frac{1}{s}\frac{4!}{(s+2)^5};$$

c) $f:\mathbb{R}\to\mathbb{R}, f(t)=\left(\displaystyle\int_0^t e^{3\tau}\sin 2\tau d\tau\right)\cdot\eta(t)$;

Rezolvare. Deoarece $\widetilde{f}(t)=e^{3t}\sin 2t\cdot\eta(t)\Rightarrow\widetilde{F}(s)=\mathcal{L}\left\{e^{3t}\sin 2t\cdot\eta(t)\right\}(s)=\dfrac{2}{(s-3)^2+2^2}$, atunci

$$\mathcal{L}\left\{\left(\int_0^t e^{3\tau}\sin 2\tau d\tau\right)\cdot\eta(t)\right\}(s)\overset{T9}{=}\frac{1}{s}\widetilde{F}(s)=\frac{1}{s}\frac{2}{(s-3)^2+2^2};$$

Definiţia 4. Fie $f,g\in\mathcal{O}$ funcţii original. Funcţia $f*g:\mathbb{R}\to\mathbb{R}$ (sau $f*g:\mathbb{R}\to\mathbb{C}$), definită prin

$$\boxed{(f*g)(t)=\begin{cases}0, & \text{dacă } t<0\\ \displaystyle\int_0^t f(\tau)g(t-\tau)d\tau, & \text{dacă } t\geq 0\end{cases}}$$

se numeşte *produs de convoluţie a funcţiilor original* f şi g, şi se notează $f*g$.

Teorema 10 (Borel, a imaginii produsului de convoluţie a două funcţii original). Fie $f\in\mathcal{O}$ o funcţie original cu abscisa de convergenţă σ_f şi $F(s)=\mathcal{L}\{f(t)\}(s)$, respectiv $g\in\mathcal{O}$ o funcţie original cu abscisa de convergenţă σ_g şi $G(s)=\mathcal{L}\{g(t)\}(s)$. Atunci $f*g\in\mathcal{O}$ şi

$\boxed{\mathcal{L}\{f*g\}(s)=F(s)\cdot G(s),}$ pentru $\forall s\in\mathbb{C}$ cu $\operatorname{Re}s>\max\left\{s_0^f,s_0^g\right\}$.

Se scrie, prin convenţie, şi

$$\boxed{\mathcal{L}\left\{\left(\int_0^t f(\tau)g(t-\tau)d\tau\right)\cdot\eta(t)\right\}(s)=\mathcal{L}\{f(t)\}(s)\cdot\mathcal{L}\{g(t)\}(s).}$$

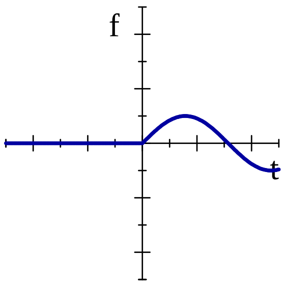
Exerciţiul 11. **a)** Fie $f:\mathbb{R}\to\mathbb{R}, f(t)=\sin t\cdot\eta(t)$. Să se determine $f*f$. Să se determine $\mathcal{L}\{f*f\}$.

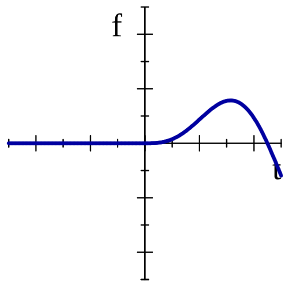
Rezolvare. •Pentru $t\geq 0$,

$$(f*f)(t)=\int_0^t f(\tau)f(t-\tau)d\tau=\int_0^t\sin\tau\cdot\sin(t-\tau)d\tau=$$

$$=\int_0^t\frac{\cos(\tau-(t-\tau))-\cos(\tau+(t-\tau))}{2}d\tau=\frac{1}{2}\int_0^t(\cos(2\tau-t)-\cos t)d\tau\overset{\tau\text{ var. de integrare}}{=}$$

$$=\frac{1}{2}\left(\tfrac{\sin(2\tau-t)}{2}-(\cos t)\cdot\tau\right)\Big|_{\tau=0}^{\tau=t}=\frac{1}{2}\left(\tfrac{\sin t}{2}-(\cos t)\cdot t-\tfrac{-\sin t}{2}+0\right)=\frac{1}{2}(\sin t-t\cos t).$$





$\Rightarrow$

•Folosind Teorema Borel, se obţine imediat:

$\mathcal{L}\left\{(f*f)(t)\right\}(s)=\mathcal{L}\left\{f(t)\right\}(s)\cdot\mathcal{L}\left\{f(t)\right\}(s)=\dfrac{1}{s^2+1}\cdot\dfrac{1}{s^2+1}$.**b)** Fie $f:\mathbb{R}\to\mathbb{R}, f(t)=t\cdot\eta(t)$.

Fie $f:\mathbb{R}\to\mathbb{R}, f(t)=\sin t\cdot\eta(t)$.

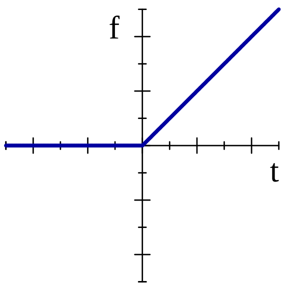
Să se determine $f*g$; Să se determine $\mathcal{L}\left\{f*g\right\}$.

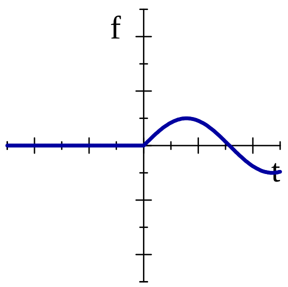
Rezolvare. •Pentru $t\geq 0$,

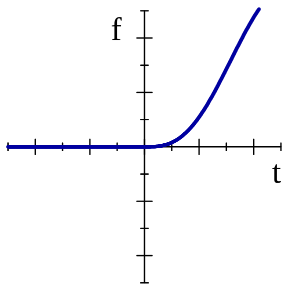
$$(f*g)(t)=\int_0^t f(\tau)\,g(t-\tau)\,d\tau=\int_0^t \tau\cdot\sin(t-\tau)\,d\tau=$$

$$=\int_0^t \tau\cdot\tfrac{d}{d\tau}\left(\cos(t-\tau)\right)d\tau \overset{\tau \text{ var. de integrare}}{=} \left(\tau\cdot\cos(t-\tau)\right)\Big|_{\tau=0}^{\tau=t}-\int_0^t 1\cdot\cos(t-\tau)\,d\tau=$$

$$=(t\cdot\cos 0-0)-\left(-\sin(t-\tau)\right)\Big|_{\tau=0}^{\tau=t}=t-\sin t$$







şi $\Rightarrow$

•Folosind Tabelul pentru produsul de convoluţie calculat:

$\mathcal{L}\left\{(f*g)(t)\right\}(s)=\mathcal{L}\left\{(t-\sin t)\cdot\eta(t)\right\}(s)=\dfrac{1}{s^2}-\dfrac{1}{s^2+1}$

•Folosind Teorema Borel se obţine imediat:

$\mathcal{L}\left\{(f*g)(t)\right\}(s)=\mathcal{L}\left\{f(t)\right\}(s)\cdot\mathcal{L}\left\{g(t)\right\}(s)=\dfrac{1}{s^2}\cdot\dfrac{1}{s^2+1}$

TRANSFORMATA LAPLACE

	$f(t)$-funcţia original	$F(s)=\mathcal{L}\{f(t)\}(s)$-funcţia imagine
1.	$\delta(t)$-distribuţia Dirac (impuls unitar)	1
2.	$\eta(t)=\begin{cases}0, & \text{dacă } t<0\\ 1, & \text{dacă } t\geq 0,\end{cases}$	$\frac{1}{s}, \operatorname{Re} s>0$
3.	$e^{\lambda t}\cdot\eta(t), \lambda\in\mathbb{C}$	$\frac{1}{s-\lambda}, \operatorname{Re} s>\operatorname{Re}\lambda$
4.	$\operatorname{ch}(\omega t)\cdot\eta(t), \omega\in\mathbb{R}, \omega>0$	$\frac{s}{s^2-\omega^2}, \operatorname{Re} s>0$
5.	$\operatorname{sh}(\omega t)\cdot\eta(t), \omega\in\mathbb{R}, \omega>0$	$\frac{\omega}{s^2-\omega^2}, \operatorname{Re} s>0$
6.	$\cos(\omega t)\cdot\eta(t), \omega\in\mathbb{R}, \omega>0$	$\frac{s}{s^2+\omega^2}, \operatorname{Re} s>0$
7.	$\sin(\omega t)\cdot\eta(t), \omega\in\mathbb{R}, \omega>0$	$\frac{\omega}{s^2+\omega^2}, \operatorname{Re} s>0$
8.	$e^{-\lambda t}\cdot\operatorname{ch}(\omega t)\cdot\eta(t), \lambda\in\mathbb{C}$	$\frac{s+\lambda}{(s+\lambda)^2-\omega^2}, \operatorname{Re} s>-\operatorname{Re}\lambda$
9.	$e^{-\lambda t}\cdot\operatorname{sh}(\omega t)\cdot\eta(t), \lambda\in\mathbb{C}$	$\frac{\omega}{(s+\lambda)^2-\omega^2}, \operatorname{Re} s>-\operatorname{Re}\lambda$
10.	$e^{-\lambda t}\cdot\cos(\omega t)\cdot\eta(t), \lambda\in\mathbb{C}$	$\frac{s+\lambda}{(s+\lambda)^2+\omega^2}, \operatorname{Re} s>-\operatorname{Re}\lambda$
11.	$e^{-\lambda t}\cdot\sin(\omega t)\cdot\eta(t), \lambda\in\mathbb{C}$	$\frac{\omega}{(s+\lambda)^2+\omega^2}, \operatorname{Re} s>-\operatorname{Re}\lambda$
12.	$t^n\cdot e^{\lambda t}\cdot\eta(t), \lambda\in\mathbb{C}, n\in\mathbb{N}^*$	$\frac{n!}{(s-\lambda)^{n+1}}, \operatorname{Re} s>\operatorname{Re}\lambda$
13.	$t^n\cdot\eta(t), n\in\mathbb{N}^*$	$\frac{n!}{s^{n+1}}, \operatorname{Re} s>0$
14.	$t\cdot\operatorname{ch}(\omega t)\cdot\eta(t), \omega\in\mathbb{R}, \omega>0$	$\frac{s^2+\omega^2}{(s^2-\omega^2)^2}, \operatorname{Re} s>0$
15.	$t\cdot\operatorname{sh}(\omega t)\cdot\eta(t), \omega\in\mathbb{R}, \omega>0$	$\frac{2s\omega}{(s^2-\omega^2)^2}, \operatorname{Re} s>0$
16.	$t\cdot\cos(\omega t)\cdot\eta(t), \omega\in\mathbb{R}, \omega>0$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}, \operatorname{Re} s>0$
17.	$t\cdot\sin(\omega t)\cdot\eta(t), \omega\in\mathbb{R}, \omega>0$	$\frac{2s\omega}{(s^2+\omega^2)^2}, \operatorname{Re} s>0$
18.	$\widetilde{f}(t-t_0)$	$e^{-t_0 s}\widetilde{F}(s)$
19.	$\begin{cases}0, & t<0\\ \int_0^t \widetilde{f}(\tau)\,\widetilde{g}(t-\tau)\,d\tau, & t\geq 0\end{cases}$	$\widetilde{F}(s)\cdot\widetilde{G}(s)$

Exerciţiul 12. Să se determine funcţiile original pentru :
a) $F(s)=\dfrac{1}{s+3}$; **b)** $F(s)=\dfrac{1}{2s+3}$; **c)** $F(s)=\dfrac{3s+2}{s^2+16}$;**d)** $F(s)=\dfrac{6s+7}{s^2-25}$; **e)** $F(s)=\dfrac{e^{-s}}{s+2}$;
f) $F(s)=\dfrac{1}{s^2+4s+13}$;**g)** $F(s)=\dfrac{1}{s(s+1)(s^2+1)}$; **h)** $F(s)=\dfrac{1}{s^2-3s+2}$;
i) $F(s)=\dfrac{s}{s^4+5s^2+4}$; **j)** $F(s)=\dfrac{1}{s^4-s^2}$; **k)** $F(s)=\dfrac{s^3}{s^4-1}$;
l) $F(s)=\dfrac{s^2+1}{(s+1)^3}$; **m)** $F(s)=\dfrac{2s-1}{s^4-1}$; **n)** $F(s)=\dfrac{s^2+1}{s^4+s^3+s+1}$;
o) $F(s)=\dfrac{s^3+1}{s^2(s^2+1)}$; **p)** $F(s)=\dfrac{1}{(s^2+1)^2}$.

Rezolvare. Se caută $f(t)=?$ funcţie original astfel încât $F(s)=\mathcal{L}\{f(t)\}(s)$.

a) $F(s)=\dfrac{1}{s+3}\overset{\text{tabel}}{\Rightarrow} f(t)=e^{-3t}\cdot\eta(t)$.

b) $F(s)=\dfrac{1}{2s+3}=\dfrac{1}{2}\dfrac{1}{s+\frac{3}{2}}\overset{\text{tabel}}{=}\dfrac{1}{2}\mathcal{L}\left\{e^{-\frac{3}{2}t}\cdot\eta(t)\right\}(s)\overset{T1,\text{lin}}{=}\mathcal{L}\left\{\dfrac{1}{2}e^{-\frac{3}{2}t}\cdot\eta(t)\right\}(s)$

$\Rightarrow f(t)=\dfrac{1}{2}e^{-\frac{3}{2}t}\cdot\eta(t)$.

c) $F(s)=\dfrac{3s+2}{s^2+16}=3\dfrac{s}{s^2+4^2}+2\dfrac{1}{4}\dfrac{4}{s^2+4^2}\overset{\text{tabel}}{=}$

$=3\mathcal{L}\{(\cos 4t)\cdot\eta(t)\}(s)+\dfrac{1}{2}\mathcal{L}\{(\sin 4t)\cdot\eta(t)\}(s)\overset{T1,\text{lin}}{=}\mathcal{L}\left\{3(\cos 4t)\cdot\eta(t)+\dfrac{1}{2}(\sin 4t)\cdot\eta(t)\right\}(s)$

$\Rightarrow f(t)=\left(3\cos 4t+\dfrac{1}{2}\sin 4t\right)\cdot\eta(t)$.

d) $F(s)=\dfrac{6s+7}{s^2-25}=6\dfrac{s}{s^2-5^2}+7\dfrac{1}{5}\dfrac{5}{s^2-5^2}\overset{\text{tabel}}{=}$

$=6\mathcal{L}\{(\operatorname{ch}5t)\cdot\eta(t)\}(s)+\dfrac{7}{5}\mathcal{L}\{(\operatorname{sh}5t)\cdot\eta(t)\}(s)\overset{T1,\text{lin}}{=}\mathcal{L}\left\{6(\operatorname{ch}5t)\cdot\eta(t)+\dfrac{7}{5}(\operatorname{sh}5t)\cdot\eta(t)\right\}(s)$

$\Rightarrow f(t)=\left(6\operatorname{ch}5t+\dfrac{7}{5}\operatorname{sh}5t\right)\cdot\eta(t)$.

e) $F(s)=\dfrac{e^{-s}}{s+2}$;

$\widetilde{F}(s)=\dfrac{1}{s+2}\overset{\text{tabel}}{\Rightarrow}\widetilde{f}(t)=e^{-2t}\cdot\eta(t)$. Atunci

$F(s)=e^{-1\cdot s}\dfrac{1}{s+2}=e^{-1\cdot s}\widetilde{F}(s)\underset{\text{sau }T4\text{, a întârzierii}}{\overset{\text{tabel}}{=}}\mathcal{L}\left\{\widetilde{f}(t-1)\right\}(s)=\mathcal{L}\left\{e^{-2(t-1)}\cdot\eta(t-1)\right\}(s)$

$\Rightarrow f(t)=e^{-2(t-1)}\cdot\eta(t-1)$.

f) $F(s)=\dfrac{1}{s^2+4s+13}$;

$F(s)=\dfrac{1}{(s+2)^2+3^2}=\frac{1}{3}\dfrac{3}{(s+2)^2+3^2}\underset{\text{sau }T3\text{, a deplasării}}{\overset{\text{tabel}}{=}}\frac{1}{3}\mathcal{L}\left\{e^{-2t}\sin(3t)\cdot\eta(t)\right\}(s)$.

g) $F(s)=\dfrac{1}{s(s+1)(s^2+1)}$;

Se descompune F în fracţii din tabel (chiar simple) $\Rightarrow$

$F(s)=\dfrac{1}{s}-\dfrac{1}{2}\dfrac{1}{s+1}-\dfrac{1}{2}\dfrac{s}{s^2+1}-\dfrac{1}{2}\dfrac{1}{s^2+1}\overset{\text{tabel}}{\Rightarrow}$

$F(s)=\mathcal{L}\{\eta(t)\}(s)-\dfrac{1}{2}\mathcal{L}\left\{e^{-t}\cdot\eta(t)\right\}(s)-\dfrac{1}{2}\mathcal{L}\{(\cos t)\cdot\eta(t)\}(s)-\dfrac{1}{2}\mathcal{L}\{(\sin t)\cdot\eta(t)\}(s)=$

$$\overset{T1,\text{lin}}{=} \mathcal{L}\left\{\eta(t) - \frac{1}{2}e^{-t}\cdot\eta(t) - \frac{1}{2}(\cos t)\cdot\eta(t) - \frac{1}{2}(\sin t)\cdot\eta(t)\right\}(s)$$
$$\Rightarrow f(t) = \left(1 - \frac{1}{2}e^{-t} - \frac{1}{2}\cos t - \frac{1}{2}\sin t\right)\cdot\eta(t).$$

h) $F(s) = \dfrac{1}{s^2-3s+2}$;

<u>modul 1.</u> Se descompune F în fracţii din tabel (chiar simple) $\Rightarrow$

$$F(s) = -\frac{1}{s-1} + \frac{1}{s-2} \overset{\text{tabel}}{\Rightarrow}$$
$$F(s) = -\mathcal{L}\left\{e^t\cdot\eta(t)\right\}(s) + \mathcal{L}\left\{e^{2t}\cdot\eta(t)\right\}(s) \overset{T1,\text{lin}}{=} \mathcal{L}\left\{-e^t\cdot\eta(t) + e^{2t}\cdot\eta(t)\right\}(s)$$
$$\Rightarrow f(t) = \left(-e^t + e^{2t}\right)\cdot\eta(t) \text{ sau } f(t) = -e^t + e^{2t}, t \geq 0$$

<u>modul 2.</u> $F(s) = 2\dfrac{\frac{1}{2}}{\left(s-\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \underset{\text{sau } T3, \text{ a deplasării}}{\overset{\text{tabel}}{=}}$

$$= 2\mathcal{L}\left\{e^{\frac{3}{2}t}\,\text{sh}\left(\tfrac{1}{2}t\right)\cdot\eta(t)\right\}(s) \overset{T1,\text{lin}}{=} \mathcal{L}\left\{2e^{\frac{3}{2}t}\,\text{sh}\left(\tfrac{1}{2}t\right)\cdot\eta(t)\right\}(s)$$
$$\Rightarrow f(t) = 2e^{\frac{3}{2}t}\frac{e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}}{2}\cdot\eta(t) = \left(-e^t + e^{2t}\right)\cdot\eta(t) \text{ sau } f(t) = -e^t + e^{2t}, t \geq 0$$

<u>modul 3.</u> Se foloseşte $T10$, a lui Borel:

$$F(s) = \underbrace{\frac{1}{s-1}}_{\widetilde{F}(s)}\cdot\underbrace{\frac{1}{s-2}}_{\widetilde{G}(s)}.$$
$$\widetilde{F}(s) = \frac{1}{s-1} \overset{\text{tabel}}{\Rightarrow} \widetilde{f}(t) = e^t\cdot\eta(t).$$
$$\widetilde{G}(s) = \frac{1}{s-2} \overset{\text{tabel}}{\Rightarrow} \widetilde{g}(t) = e^{2t}\cdot\eta(t).$$

Atunci $F(s) = \widetilde{F}(s)\cdot\widetilde{G}(s) \underset{\text{convenţie}}{\overset{T10,\text{a lui Borel}}{=}} \mathcal{L}\left\{\int_0^t \widetilde{f}(\tau)\,\widetilde{g}(t-\tau)\,d\tau\right\}(s) = \mathcal{L}\left\{\int_0^t e^\tau\cdot e^{2(t-\tau)}d\tau\right\}(s).$

$$\Rightarrow f(t) = \int_0^t e^{2t-\tau}d\tau = \left.\frac{e^{2t-\tau}}{-1}\right|_{\tau=0}^{\tau=t} = \frac{e^{2t-t}}{-1} - \frac{e^{2t-0}}{-1} = -e^t + e^{2t}, t \geq 0$$

sau $f(t) = \left(-e^t + e^{2t}\right)\cdot\eta(t)$.

i) $F(s) = \dfrac{s}{s^4+5s^2+4}$;

<u>modul 1.</u> Se descompune F în fracţii din tabel (chiar simple) $\Rightarrow$

$$F(s) = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}.$$
$$\overset{\text{tabel}}{\Rightarrow} F(s) = \frac{1}{3}\mathcal{L}\left\{(\cos t)\cdot\eta(t)\right\}(s) - \frac{1}{3}\mathcal{L}\left\{(\cos 2t)\cdot\eta(t)\right\}(s)$$
$$\overset{T1,\text{lin}}{=} \mathcal{L}\left\{\frac{1}{3}(\cos t)\cdot\eta(t) - \frac{1}{3}(\cos 2t)\cdot\eta(t)\right\}(s)$$
$$\Rightarrow f(t) = \left(\frac{1}{3}\cos t - \frac{1}{3}\cos 2t\right)\cdot\eta(t) \text{ sau } f(t) = \frac{1}{3}\cos t - \frac{1}{3}\cos 2t, t \geq 0.$$

<u>modul 2.</u> Se foloseşte $T10$, a lui Borel:

$$F(s) = \underbrace{\frac{1}{s^2+1}}_{\widetilde{F}(s)}\cdot\underbrace{\frac{s}{s^2+4}}_{\widetilde{G}(s)}.$$
$$\widetilde{F}(s) = \frac{1}{s^2+1} \overset{\text{tabel}}{\Rightarrow} \widetilde{f}(t) = (\sin t)\cdot\eta(t).$$
$$\widetilde{G}(s) = \frac{s}{s^2+4} \overset{\text{tabel}}{\Rightarrow} \widetilde{g}(t) = (\cos 2t)\cdot\eta(t).$$

Atunci

$$F(s) = \widetilde{F}(s)\cdot\widetilde{G}(s) \underset{\text{convenţie}}{\overset{T10,\text{a lui Borel}}{=}} \mathcal{L}\left\{\int_0^t \widetilde{f}(\tau)\,\widetilde{g}(t-\tau)\,d\tau\right\}(s) = \mathcal{L}\left\{\int_0^t \sin\tau\cdot\cos 2(t-\tau)\,d\tau\right\}(s).$$

$$\Rightarrow f(t) = \int_0^t \frac{\sin(\tau+2t-2\tau)+\sin(\tau-2t+2\tau)}{2}d\tau = \int_0^t\left(\frac{1}{2}\sin(-\tau+2t)+\frac{1}{2}\sin(3\tau-2t)\right)d\tau =$$

$$= \left(\frac{1}{2}\frac{1-\cos(-\tau+2t)}{-1}+\frac{1}{2}\frac{1-\cos(3\tau-2t)}{3}\right)\Bigg|_{\tau=0}^{\tau=t}$$

$$= \left(\frac{1}{2}\frac{1-\cos t}{-1}+\frac{1}{2}\frac{1-\cos t}{3}\right)-\left(\frac{1}{2}\frac{1-\cos 2t}{-1}+\frac{1}{2}\frac{1-\cos(-2t)}{3}\right) = \frac{1}{3}\cos t-\frac{1}{3}\cos 2t,\ t\geq 0$$

sau $f(t) = \left(\frac{1}{3}\cos t - \frac{1}{3}\cos 2t\right)\cdot\eta(t)$.

j) $F(s) = \dfrac{1}{s^4-s^2}$.

<u>modul 1.</u> Se descompune F în fracţii din tabel (chiar simple) $\Rightarrow$

$$F(s) = -\frac{1}{s^2}-\frac{1}{2}\frac{1}{s+1}+\frac{1}{2}\frac{1}{s-1}$$

$$\overset{\text{tabel}}{\Rightarrow} F(s) = -\mathcal{L}\{(-t)\cdot\eta(t)\}(s)-\frac{1}{2}\mathcal{L}\left\{e^{-t}\cdot\eta(t)\right\}(s)+\frac{1}{2}\mathcal{L}\left\{e^{t}\cdot\eta(t)\right\}(s)$$

$$\overset{T1,\text{lin}}{=} \mathcal{L}\left\{t\cdot\eta(t)-\frac{1}{2}e^{-t}\cdot\eta(t)+\frac{1}{2}e^{t}\cdot\eta(t)\right\}(s)$$

$$\Rightarrow f(t) = \left(t-\frac{1}{2}e^{-t}+\frac{1}{2}e^{t}\right)\cdot\eta(t) \text{ sau } f(t) = t-\frac{1}{2}e^{-t}+\frac{1}{2}e^{t},\ t\geq 0.$$

<u>modul 2.</u> Se foloseşte $T10$, a lui Borel:

$$F(s) = \underbrace{\frac{1}{s^2}}_{\widetilde{F}(s)}\cdot\underbrace{\frac{1}{s^2-1}}_{\widetilde{G}(s)}.$$

$$\widetilde{F}(s) = \frac{1}{s^2} \overset{\text{tabel}}{\Rightarrow} \widetilde{f}(t) = (-t)\cdot\eta(t).$$

$$\widetilde{G}(s) = \frac{1}{s^2-1} \overset{\text{tabel}}{\Rightarrow} \widetilde{g}(t) = \operatorname{sh} t\cdot\eta(t).$$

Atunci $F(s) = \widetilde{F}(s)\cdot\widetilde{G}(s) \underset{\text{convenţie}}{\overset{T10,\text{a lui Borel}}{=}} \mathcal{L}\left\{\int_0^t \widetilde{f}(\tau)\,\widetilde{g}(t-\tau)\,d\tau\right\}(s) = \mathcal{L}\left\{\int_0^t (-\tau)\cdot\operatorname{sh}(t-\tau)\,d\tau\right\}(s).$

$$\Rightarrow f(t) = -\int_0^t \tau\cdot\operatorname{sh}(t-\tau)\,d\tau = ..$$

<u>modul 2.1.</u> $f(t) = -\displaystyle\int_0^t \tau\cdot\frac{e^{(t-\tau)}-e^{-(t-\tau)}}{2}d\tau = \frac{1}{2}\int_0^t \tau\cdot\frac{d}{d\tau}\left(e^{(t-\tau)}+e^{-(t-\tau)}\right)d\tau =$

$$= \frac{1}{2}\left(\left(\tau\cdot\left(e^{(t-\tau)}+e^{-(t-\tau)}\right)\right)\Big|_{\tau=0}^{\tau=t}-\int_0^t\left(e^{(t-\tau)}+e^{-(t-\tau)}\right)d\tau\right) =$$

$$= \frac{1}{2}\left(t\cdot\left(e^0+e^{-0}\right)-0-\frac{e^{(t-\tau)}}{-1}+\frac{e^{-(t-\tau)}}{1}\Bigg|_{\tau=0}^{\tau=t}\right)$$

$$= \frac{1}{2}\left(t\cdot\left(e^0+e^{-0}\right)-0-\left(\frac{e^0}{-1}+\frac{e^{-0}}{1}\right)+\left(\frac{e^t}{-1}+\frac{e^{-t}}{1}\right)\right)$$

$$= t\cdot\frac{e^0+e^{-0}}{2}+\frac{e^0-e^{-0}}{2}-\frac{e^t-e^{-t}}{2} = t-\operatorname{sh} t,\ t\geq 0$$

sau $f(t) = (t-\operatorname{sh} t)\cdot\eta(t)$.

<u>modul 2.2.</u> Folosim

$$\boxed{\begin{cases} \dfrac{d}{du}(\operatorname{ch} u) = \operatorname{sh} u, \forall u \in \mathbb{R} \\ \dfrac{d}{du}(\operatorname{sh} u) = \operatorname{ch} u, \forall u \in \mathbb{R} \end{cases}} \boxed{\begin{cases} \int (\operatorname{ch} u)\, du = \operatorname{sh} u + c, \forall u \in \mathbb{R}, \forall c \in \mathbb{R} \\ \int (\operatorname{sh} u)\, du = \operatorname{ch} u + c, \forall u \in \mathbb{R}, \forall c \in \mathbb{R} \end{cases}} \boxed{\begin{cases} \operatorname{ch} 0 = 1 \\ \operatorname{sh} 0 = 0 \end{cases}}$$

$$f(t) = -\int_0^t \tau \cdot \operatorname{sh}(t-\tau)\, d\tau = \int_0^t \tau \cdot \frac{d}{d\tau}(\operatorname{ch}(t-\tau))\, d\tau$$

$$= \left(\tau \cdot \operatorname{ch}(t-\tau)\big|_{\tau=0}^{\tau=t} - \int_0^t 1 \cdot \operatorname{ch}(t-\tau)\, d\tau \right) = \left(t \cdot \operatorname{ch} 0 - 0 - \frac{\operatorname{sh}(t-\tau)}{-1}\bigg|_{\tau=0}^{\tau=t} \right)$$

$$= \left(t \cdot \operatorname{ch} 0 - 0 - \frac{\operatorname{sh} 0}{-1} + \frac{\operatorname{sh} t}{-1} \right) = t - \operatorname{sh} t, t \geq 0 \text{ sau } f(t) = (t - \operatorname{sh} t) \cdot \eta(t).$$

Observaţie. Există o metodă de determinare a funcţiei original f atunci când se cunoaşte funcţia imagine F, bazată pe analiza matematică a funcţiilor complexe (se va studia la Matematici Speciale), şi anume:

Definiţie. Fie $G : D \subset \mathbb{C} \to \mathbb{C}$.

a) $s_0 \in \mathbb{C}$ este *pol simplu* (sau pol de ordin 1) pentru G dacă

$\lim\limits_{s\to s_0} G(s) = \infty$ şi $\lim\limits_{s\to s_0} (s - s_0)G(s) \in \mathbb{C}$.

În acest caz *reziduul lui G în s_0* este numărul complex notat $\operatorname{rez}(G; s_0)$ dat de

$$\boxed{\operatorname{rez}(G; s_0) = \lim_{s\to s_0} ((s - s_0)G(s)).}$$

b) $s_0 \in \mathbb{C}$ este *pol de ordin $k \in \mathbb{N}^*$* pentru G dacă

$\lim\limits_{s\to s_0} (s - s_0)^{k-1}G(s) = \infty$ şi $\lim\limits_{s\to s_0} (s - s_0)^k G(s) \in \mathbb{C}$.

În acest caz se reziduul lui G în s_0 este numărul complex notat $Rez(G; s_0)$ dat de

$$\boxed{\operatorname{rez}(G; s_0) = \frac{1}{(k-1)!} \lim_{s\to s_0} \left(\frac{d^{k-1}}{ds^{k-1}} \left((s - s_0)^k G(s) \right) \right).}$$

Teoremă. Fie o funcţie raţională

$F : D \subset \mathbb{C} \to \mathbb{C}$, $F(s) = \dfrac{P(s)}{Q(s)}$,

unde P şi Q sunt polinoame cu coeficienţi din $\mathbb{C}$, $\operatorname{grad} P < \operatorname{grad} Q$. Atunci F este imaginea Laplace a originalului Laplace

$f : \mathbb{R} \to \mathbb{C}$, $f(t) = \sum\limits_j \operatorname{rez}\left(e^{st} \cdot F(s); s_j\right)$,

unde suma este calculată după toţi polii s_j ai funcţiei F.

În particular, dacă F admite doar poli simpli $s_1, ..., s_m$ (sau, echivalent, Q are rădăcini distincte), atunci

$$\boxed{f(t) = \sum_{j=1}^{m} \frac{P(s_j)}{Q'(s_j)} e^{s_j t}, t \geq 0.}$$

Exerciţiul 13. Să se determine funcţiile original pentru :

a) $F(s) = \dfrac{s^2 + 1}{s^2 (s^2 + 4)}$.

Rezolvare. A se vedea Curs.