SEMINAR NR. 3, REZOLVĂRI
Matematici Speciale, AIA

2.2. Câmp vectorial: flux, divergenţă, circulaţie, rotor

Comentariul 1.

-*fluxul* unui câmp vectorial $\overrightarrow{\mathbf{v}}$ într-un punct P_0 este numărul

$$\boxed{\Phi\left(\overrightarrow{\mathbf{v}}\right)(P_0) = \iint_S \overrightarrow{\mathbf{v}}(P_0)\cdot \overrightarrow{\mathbf{n}}(P_0)\, ds;}$$

-*divergenţa* unui câmp vectorial $\overrightarrow{\mathbf{v}}$ într-un punct P_0 este numărul

$$\boxed{\operatorname{div}\left(\overrightarrow{\mathbf{v}}\right)(P_0) = \lim_{\Delta V\to 0}\frac{\Phi\left(\overrightarrow{\mathbf{v}}\right)(P_0)}{\Delta V}.}$$

Teorema 1. (expresia divergenţei în coordonate carteziene). Fie $\overrightarrow{\mathbf{v}} : D\subseteq\mathbb{R}^3\to\mathbb{V}_3$,

$\overrightarrow{\mathbf{v}}(x,y,z) = \overrightarrow{\mathbf{v}}(P) = v_1(x,y,z)\,\overrightarrow{\mathbf{i}} + v_2(x,y,z)\,\overrightarrow{\mathbf{j}} + v_3(x,y,z)\,\overrightarrow{\mathbf{k}}$

un câmp vectorial de clasă $\mathcal{C}^1$ pe un domeniu D şi $P_0\in D$. Atunci

$$\boxed{\operatorname{div}\overrightarrow{\mathbf{v}}(P_0) = \frac{\partial v_1}{\partial x}(P_0) + \frac{\partial v_2}{\partial y}(P_0) + \frac{\partial v_3}{\partial z}(P_0).} \tag{1}$$

Observaţie. $\boxed{\operatorname{div}\overrightarrow{\mathbf{v}}(P) = \nabla\cdot\overrightarrow{\mathbf{v}}(P), \forall P\in D}$

Definiţia 1. Un câmp vectorial $\overrightarrow{\mathbf{v}} : D\subseteq\mathbb{R}^3\to\mathbb{V}_3$ de clasă $\mathcal{C}^1$ pe un domeniu D se numeşte *câmp solenoidal* dacă

$\operatorname{div}\overrightarrow{\mathbf{v}}(P) = 0, \forall P\in D.$

Exerciţiul 1. Să se calculeze, pe un domeniu D a.î. $O\notin D$,

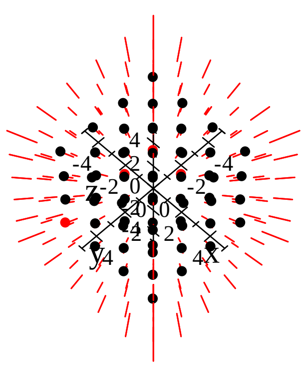
a) $\operatorname{div}\left(r^2\overrightarrow{\mathbf{r}}\right)$; **b)** $\operatorname{div}\left(\overrightarrow{\mathbf{c}}\times\overrightarrow{\mathbf{r}}\right)$,

unde $\overrightarrow{\mathbf{r}}(P) = x\,\overrightarrow{\mathbf{i}} + y\,\overrightarrow{\mathbf{j}} + z\,\overrightarrow{\mathbf{k}}$ este vectorul de poziţie a punctului $P(x,y,z)$ şi $r = \left\|\overrightarrow{\mathbf{r}}\right\|$,
iar $\overrightarrow{\mathbf{c}} = c_1\,\overrightarrow{\mathbf{i}} + c_2\,\overrightarrow{\mathbf{j}} + c_3\,\overrightarrow{\mathbf{k}}$ este un vector constant.

Rezolvare. a) $r(P) = \sqrt{x^2+y^2+z^2} \Rightarrow$

$\overrightarrow{\mathbf{v}}(P) = x\left(x^2+y^2+z^2\right)\overrightarrow{\mathbf{i}} + y\left(x^2+y^2+z^2\right)\overrightarrow{\mathbf{j}} + z\left(x^2+y^2+z^2\right)\overrightarrow{\mathbf{k}}.$

Are reprezentarea spaţială



Conform (1) $\Rightarrow$

$$\operatorname{div}\overrightarrow{\mathbf{v}}(P) = \frac{\partial}{\partial x}\left(x\left(x^2+y^2+z^2\right)\right) + \frac{\partial}{\partial y}\left(y\left(x^2+y^2+z^2\right)\right) + \frac{\partial}{\partial z}\left(z\left(x^2+y^2+z^2\right)\right) =$$

$$= \left(x^2+y^2+z^2\right) + x\cdot 2x + \left(x^2+y^2+z^2\right) + y\cdot 2y + \left(x^2+y^2+z^2\right) + z\cdot 2z =$$

$= 5 \cdot \left(x^2 + y^2 + z^2\right) > 0, \forall P \in D \Rightarrow$

Câmpul vectorial "emite" în orice punct $P \in D$.

Deci $\boxed{\operatorname{div}\left(r^2 \overrightarrow{\mathbf{r}}\right) = 5r^2.}$

b) $\overrightarrow{\mathbf{v}}(P) = \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ c_1 & c_2 & c_3 \\ x & y & z \end{vmatrix} = (c_2 z - c_3 y)\,\overrightarrow{\mathbf{i}} - (c_1 z - c_3 x)\,\overrightarrow{\mathbf{j}} + (c_1 y - c_2 x)\,\overrightarrow{\mathbf{k}}.$

Conform (1) $\Rightarrow$

$$\operatorname{div} \overrightarrow{\mathbf{v}}(P) = \frac{\partial}{\partial x}(c_2 z - c_3 y) + \frac{\partial}{\partial y}\left(-(c_1 z - c_3 x)\right) + \frac{\partial}{\partial z}(c_1 y - c_2 x) =$$
$$= 0 + 0 + 0 = 0, \forall P \in D \Rightarrow$$

Câmpul vectorial "nu emite" şi "nu absoarbe" în niciun punct $P \in D$, este solenoidal.

Deci $\boxed{\operatorname{div}\left(\overrightarrow{\mathbf{c}} \times \overrightarrow{\mathbf{r}}\right) = 0.}$

Exerciţiul 2. Să se calculeze $\operatorname{div}\left(f(r)\overrightarrow{\mathbf{v}}\right)(P)$, unde $\overrightarrow{\mathbf{v}}$ este un câmp vectorial de clasă $\mathcal{C}^1$ pe D a.î. $O \notin D$, f este o funcţie scalară de clasă $\mathcal{C}^1$, $\overrightarrow{\mathbf{r}}(P) = x\,\overrightarrow{\mathbf{i}} + y\,\overrightarrow{\mathbf{j}} + z\,\overrightarrow{\mathbf{k}}$ este vectorul de poziţie a punctului $P(x,y,z)$, cu $r = \left\|\overrightarrow{\mathbf{r}}\right\|$.

Cazuri particulare: $\operatorname{div}\left(r^2 \overrightarrow{\mathbf{c}}\right)$ şi $\operatorname{div}\left(r\overrightarrow{\mathbf{c}}\right)$, unde $\overrightarrow{\mathbf{c}} = c_1\overrightarrow{\mathbf{i}} + c_2\overrightarrow{\mathbf{j}} + c_3\overrightarrow{\mathbf{k}}$ este un vector constant.

Rezolvare. Fie $r(P) = \sqrt{x^2+y^2+z^2}$ şi $\overrightarrow{\mathbf{v}}(P) = v_1(P)\,\overrightarrow{\mathbf{i}} + v_2(P)\,\overrightarrow{\mathbf{j}} + v_3(P)\,\overrightarrow{\mathbf{k}} \Rightarrow$

$$f(r(x,y,z))\,\overrightarrow{\mathbf{v}}(x,y,z) \overset{f \text{ scalară}}{=}$$
$$= \underbrace{f(r(x,y,z))\,v_1(x,y,z)}_{\text{un } w_1}\overrightarrow{\mathbf{i}} + \underbrace{f(r(x,y,z))\,v_2(x,y,z)}_{\text{un } w_2}\overrightarrow{\mathbf{j}} + \underbrace{f(r(x,y,z))\,v_3(x,y,z)}_{\text{un } w_3}\overrightarrow{\mathbf{k}}.$$

Conform (1) $\overset{P(x,y,z)}{\Rightarrow}$

$$\operatorname{div}\left(f(r)\overrightarrow{\mathbf{v}}\right)(P) = \frac{\partial}{\partial x}\left(f(r)\cdot v_1\right)(P) + \frac{\partial}{\partial y}\left(f(r)\cdot v_2\right)(P) + \frac{\partial}{\partial z}\left(f(r)\cdot v_3\right)(P) =$$
$$= \frac{df}{dr}(r(P))\,\frac{\partial r}{\partial x}(P)\cdot v_1(P) + f(r(P))\cdot\frac{\partial v_1}{\partial x}(P) +$$
$$+ \frac{df}{dr}(r(P))\,\frac{\partial r}{\partial y}(P)\cdot v_2(P) + f(r(P))\cdot\frac{\partial v_2}{\partial z}(P) +$$
$$+ \frac{df}{dr}(r(P))\,\frac{\partial r}{\partial z}(P)\cdot v_3(P) + f(r(P))\cdot\frac{\partial v_3}{\partial z}(P) \overset{f \text{ scalară}}{=}$$
$$= f'(r(P))\,\frac{x}{\sqrt{x^2+y^2+z^2}}\cdot v_1(P) + f(r(P))\cdot\frac{\partial v_1}{\partial x}(P) +$$
$$+ f'(r(P))\,\frac{y}{\sqrt{x^2+y^2+z^2}}\cdot v_2(P) + f(r(P))\cdot\frac{\partial v_2}{\partial z}(P) +$$
$$+ f'(r(P))\,\frac{z}{\sqrt{x^2+y^2+z^2}}\cdot v_3(P) + f(r(P))\cdot\frac{\partial v_3}{\partial z}(P)$$
$$= f'(r(P))\,\frac{1}{\sqrt{x^2+y^2+z^2}}\left(xv_1(P) + yv_2(P) + zv_3(P)\right) +$$
$$+ \left(f(r(P))\right)\left(\frac{\partial v_1}{\partial x}(P) + \frac{\partial v_2}{\partial y}(P) + \frac{\partial v_3}{\partial z}(P)\right) =$$
$$= \frac{f'(r(P))}{r(P)}\left(\overrightarrow{\mathbf{r}}(P)\cdot\overrightarrow{\mathbf{v}}(P)\right) + f(r(P))\operatorname{div}\overrightarrow{\mathbf{v}}(P) \Rightarrow$$

$$\operatorname{div}\left(f(r)\overrightarrow{\mathbf{v}}\right)(P) = \frac{f'(r(P))}{r(P)}\left(\overrightarrow{\mathbf{r}}(P)\cdot\overrightarrow{\mathbf{v}}(P)\right) + f(r(P))\operatorname{div}\overrightarrow{\mathbf{v}}(P) \Rightarrow$$

$$\boxed{\operatorname{div}\left(f(r)\overrightarrow{\mathbf{v}}\right)=f'(r)\underbrace{\left(\frac{1}{r}\overrightarrow{\mathbf{r}}\right)}_{\operatorname{grad} r}\cdot\overrightarrow{\mathbf{v}}+f(r)\operatorname{div}\overrightarrow{\mathbf{v}}=\frac{f'(r)}{r}\left(\overrightarrow{\mathbf{r}}\cdot\overrightarrow{\mathbf{v}}\right)+f(r)\operatorname{div}\overrightarrow{\mathbf{v}}.}$$

În particular:

• $f(r)=r^2;\ \overrightarrow{\mathbf{v}}(P)=\overrightarrow{\mathbf{c}}\Rightarrow$

$$\operatorname{div}\left(r^2\overrightarrow{\mathbf{c}}\right)=\frac{2r}{r}\left(\overrightarrow{\mathbf{r}}\cdot\overrightarrow{\mathbf{c}}\right)+r^2\operatorname{div}\overrightarrow{\mathbf{c}}=2\left(\overrightarrow{\mathbf{r}}\cdot\overrightarrow{\mathbf{c}}\right)+r^2\left(\frac{\partial(c_1)}{\partial x}+\frac{\partial(c_2)}{\partial y}+\frac{\partial(c_3)}{\partial z}\right)=2\left(\overrightarrow{\mathbf{r}}\cdot\overrightarrow{\mathbf{c}}\right).$$

• $f(r)=r;\ \overrightarrow{\mathbf{v}}(P)=\overrightarrow{\mathbf{c}}\Rightarrow$

$$\operatorname{div}\left(r\overrightarrow{\mathbf{c}}\right)=\frac{1}{r}\left(\overrightarrow{\mathbf{r}}\cdot\overrightarrow{\mathbf{c}}\right)+r\operatorname{div}\overrightarrow{\mathbf{c}}=\frac{1}{r}\left(\overrightarrow{\mathbf{r}}\cdot\overrightarrow{\mathbf{c}}\right)+r\left(\frac{\partial(c_1)}{\partial x}+\frac{\partial(c_2)}{\partial y}+\frac{\partial(c_3)}{\partial z}\right)=\frac{1}{r}\left(\overrightarrow{\mathbf{r}}\cdot\overrightarrow{\mathbf{c}}\right).$$

$$\boxed{\operatorname{div}\overrightarrow{\mathbf{c}}=0 \text{ şi } \operatorname{div}\left(r^2\overrightarrow{\mathbf{c}}\right)=2\left(\overrightarrow{\mathbf{r}}\cdot\overrightarrow{\mathbf{c}}\right) \text{ şi } \operatorname{div}\left(r\overrightarrow{\mathbf{c}}\right)=\frac{1}{r}\left(\overrightarrow{\mathbf{r}}\cdot\overrightarrow{\mathbf{c}}\right).}$$

Exerciţiul 3.Fie f o funcţie scalară de clasă $\mathcal{C}^1$, $\overrightarrow{\mathbf{r}}(P)=x\overrightarrow{\mathbf{i}}+y\overrightarrow{\mathbf{j}}+z\overrightarrow{\mathbf{k}}\neq\overrightarrow{\mathbf{0}}$ vectorul de poziţie a punctului $P(x,y,z)$ pe D a.î. $O\notin D$, cu $r=\left\|\overrightarrow{\mathbf{r}}\right\|$. Să se determine f astfel încât:

a) $\operatorname{div}(\operatorname{grad} f(r))=0$; **b)** $2r\operatorname{div}(\operatorname{grad} f(r))=\operatorname{div}\left(\frac{1}{r}\overrightarrow{\mathbf{r}}\right)$.

Rezolvare. Fie $r(x,y,z)=\sqrt{x^2+y^2+z^2}$ şi $\varphi(x,y,z)=f(r(x,y,z))$. Atunci

$$\operatorname{grad}\varphi(P)=\nabla\varphi(P)=\underbrace{\frac{\partial\varphi}{\partial x}(P)}_{v_1 \text{ pentru div}}\overrightarrow{\mathbf{i}}+\underbrace{\frac{\partial\varphi}{\partial y}(P)}_{v_2 \text{ pentru div}}\overrightarrow{\mathbf{j}}+\underbrace{\frac{\partial\varphi}{\partial z}(P)}_{v_3 \text{ pentru div}}\overrightarrow{\mathbf{k}}.$$

Conform (1) $\Rightarrow$

$$\boxed{\operatorname{div}(\operatorname{grad}\varphi(P))=\frac{\partial^2\varphi}{\partial x^2}(P)+\frac{\partial^2\varphi}{\partial y^2}(P)+\frac{\partial^2\varphi}{\partial z^2}(P)=\boldsymbol{\Delta}\varphi(P)}$$

Se observă că:

$$\frac{\partial\varphi}{\partial x}(P)=\frac{df}{dr}(r(P))\cdot\frac{\partial r}{\partial x}(P)=f'(r(P))\cdot\frac{x}{\sqrt{x^2+y^2+z^2}}.$$

$$\frac{\partial\varphi}{\partial y}(P)=\frac{df}{dr}(r(P))\cdot\frac{\partial r}{\partial y}(P)=f'(r(P))\cdot\frac{y}{\sqrt{x^2+y^2+z^2}}.$$

$$\frac{\partial\varphi}{\partial z}(P)=\frac{df}{dr}(r(P))\cdot\frac{\partial r}{\partial z}(P)=f'(r(P))\cdot\frac{z}{\sqrt{x^2+y^2+z^2}}.$$

Mai mult:

$$\frac{\partial^2\varphi}{\partial x^2}(P)=\frac{df'}{dr}(r(P))\cdot\frac{\partial r}{\partial x}(P)\cdot\frac{x}{\sqrt{x^2+y^2+z^2}}+$$
$$+f'(r(P))\cdot\frac{1\cdot\sqrt{x^2+y^2+z^2}-x\cdot\frac{x}{\sqrt{x^2+y^2+z^2}}}{\left(\sqrt{x^2+y^2+z^2}\right)^2}.$$

$$\frac{\partial^2\varphi}{\partial y^2}(P)=\frac{df'}{dr}(r(P))\cdot\frac{\partial r}{\partial y}(P)\cdot\frac{y}{\sqrt{x^2+y^2+z^2}}+$$
$$+f'(r(P))\cdot\frac{1\cdot\sqrt{x^2+y^2+z^2}-y\cdot\frac{y}{\sqrt{x^2+y^2+z^2}}}{\left(\sqrt{x^2+y^2+z^2}\right)^2}.$$

$$\frac{\partial^2\varphi}{\partial z^2}(P)=\frac{df'}{dr}(r(P))\cdot\frac{\partial r}{\partial z}(P)\cdot\frac{z}{\sqrt{x^2+y^2+z^2}}+$$

$$+f'(r(P))\cdot\frac{1\cdot\sqrt{x^2+y^2+z^2}-z\cdot\dfrac{z}{\sqrt{x^2+y^2+z^2}}}{\left(\sqrt{x^2+y^2+z^2}\right)^2}.$$

Atunci $\operatorname{div}(\operatorname{grad} f(r))(P) =$

$$= f''(r(P))\cdot\left(\frac{x^2}{x^2+y^2+z^2}+\frac{y^2}{x^2+y^2+z^2}+\frac{z^2}{x^2+y^2+z^2}\right)+$$
$$+f'(r(P))\cdot\frac{3\left(x^2+y^2+z^2\right)-\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)\sqrt{x^2+y^2+z^2}}=$$
$$= f''(r(P))+f'(r(P))\cdot\frac{2}{r(P)}.$$

a) $\operatorname{div}(\operatorname{grad} f(r))=0 \Leftrightarrow f''(r)+f'(r)\cdot\frac{2}{r}=0, r\in D$ a.î. $r\neq 0\Leftrightarrow$

$r^2 f''(r)+2rf'(r)=0, r\in D, r\neq 0 \Leftrightarrow \left(r^2 f'(r)\right)'=0, r\in D, r\neq 0\Leftrightarrow$

$r^2 f'(r)=c_1, r\in D, r\neq 0, c_1\in\mathbb{R}\Leftrightarrow f'(r)=\frac{c_1}{r^2}, r\in D, r\neq 0, c_1\in\mathbb{R}\Leftrightarrow$

$f(r)=-\frac{c_1}{r}+c_2, r\in D, r\neq 0, c_1\in\mathbb{R}, c_2\in\mathbb{R}.$

b) $$\frac{1}{r(P)}\overrightarrow{\mathbf{r}}(P)=\underbrace{\frac{x}{\sqrt{x^2+y^2+z^2}}}_{v_1 \text{ pentru div}}\overrightarrow{\mathbf{i}}+\underbrace{\frac{y}{\sqrt{x^2+y^2+z^2}}}_{v_2 \text{ pentru div}}\overrightarrow{\mathbf{j}}+\underbrace{\frac{z}{\sqrt{x^2+y^2+z^2}}}_{v_3 \text{ pentru div}}\overrightarrow{\mathbf{k}}.$$

Conform (1) $\Rightarrow \operatorname{div}\left(\frac{1}{r}\overrightarrow{\mathbf{r}}\right)(P)=$

$$=\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)+\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)+\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)=$$
$$=\frac{1\cdot\sqrt{x^2+y^2+z^2}-x\cdot\dfrac{x}{\sqrt{x^2+y^2+z^2}}}{\left(\sqrt{x^2+y^2+z^2}\right)^2}+\frac{1\cdot\sqrt{x^2+y^2+z^2}-y\cdot\dfrac{y}{\sqrt{x^2+y^2+z^2}}}{\left(\sqrt{x^2+y^2+z^2}\right)^2}+$$
$$+\frac{1\cdot\sqrt{x^2+y^2+z^2}-z\cdot\dfrac{z}{\sqrt{x^2+y^2+z^2}}}{\left(\sqrt{x^2+y^2+z^2}\right)^2}=\frac{2}{r(P)}.$$

Atunci

$2r\operatorname{div}(\operatorname{grad} f(r))=\operatorname{div}\left(\frac{1}{r}\overrightarrow{\mathbf{r}}\right)\Leftrightarrow 2r\left(f''(r)+2f'(r)\cdot\frac{1}{r}\right)=\frac{2}{r}, r\in D$ a.î. $r\neq 0\Leftrightarrow$

$r^2 f''(r)+2rf'(r)=1, r\in D\Leftrightarrow \left(r^2 f'(r)\right)'=1, r\in D, r\neq 0\Leftrightarrow$

$r^2 f'(r)=r+c_1, r\in D, r\neq 0, c_1\in\mathbb{R}\Leftrightarrow f'(r)=\frac{1}{r}+\frac{c_1}{r^2}, r\in D, r\neq 0, c_1\in\mathbb{R}\Leftrightarrow$

$f(r)=\ln r-\frac{c_1}{r}+c_2, r\in D, r\neq 0, c_1\in\mathbb{R}, c_2\in\mathbb{R}.$

Comentariul 2.

-*circulaţia* unui câmp vectorial $\overrightarrow{\mathbf{v}}$ într-un punct P_0 este numărul

$$\boxed{\textit{circulaţia}\left(\overrightarrow{\mathbf{v}}\right)(P_0)=\int_\gamma \overrightarrow{\mathbf{v}}(P_0)\cdot d\overrightarrow{\mathbf{r}}(P_0);}$$

-*rotorul* unui câmp vectorial $\overrightarrow{\mathbf{v}}$ într-un punct P_0 este vectorul rot $\overrightarrow{\mathbf{v}} \overset{\text{not.}}{=}$ curl $\overrightarrow{\mathbf{v}}$ dat de formula

$$\boxed{\overrightarrow{\mathbf{n}}(P_0)\cdot\operatorname{rot}\left(\overrightarrow{\mathbf{v}}\right)(P_0)=\lim_{\Delta S\to 0}\frac{\text{circulaţia}\left(\overrightarrow{\mathbf{v}}\right)(P_0)}{\Delta S}.}$$

Teorema 3. (expresia rotorului în coordonate carteziene). Fie $\overrightarrow{\mathbf{v}} : D \subseteq \mathbb{R}^3 \to \mathbb{V}_3$,

$\overrightarrow{\mathbf{v}}(x,y,z) = \overrightarrow{\mathbf{v}}(P) = v_1(x,y,z)\,\overrightarrow{\mathbf{i}} + v_2(x,y,z)\,\overrightarrow{\mathbf{j}} + v_3(x,y,z)\,\overrightarrow{\mathbf{k}}$

un câmp vectorial de clasă $\mathcal{C}^1$ pe un domeniu D şi $P_0 \in D$. Atunci

$$\boxed{\operatorname{rot}\overrightarrow{\mathbf{v}}(P_0) \overset{\text{not.}}{=} \operatorname{curl}\overrightarrow{\mathbf{v}}(P_0) = \left|\begin{array}{ccc} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{array}\right|_{\text{formal}} (P_0) =}$$

$$\boxed{= \left(\frac{\partial v_3}{\partial y}(P_0) - \frac{\partial v_2}{\partial z}(P_0)\right)\overrightarrow{\mathbf{i}} - \left(\frac{\partial v_3}{\partial x}(P_0) - \frac{\partial v_1}{\partial z}(P_0)\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial v_2}{\partial x}(P_0) - \frac{\partial v_1}{\partial y}(P_0)\right)\overrightarrow{\mathbf{k}}} \quad (3)$$

Observaţie. $\boxed{\operatorname{rot}\overrightarrow{\mathbf{v}}(P) = \nabla \times \overrightarrow{\mathbf{v}}(P), \forall P \in D.}$

Definiţia 2. Un câmp vectorial $\overrightarrow{\mathbf{v}} : D \subseteq \mathbb{R}^3 \to \mathbb{V}_3$ de clasă $\mathcal{C}^1$ pe un domeniu D se numeşte *câmp irotaţional* dacă

$\operatorname{rot}\overrightarrow{\mathbf{v}}(P) = \overrightarrow{\mathbf{0}}, \forall P \in D.$

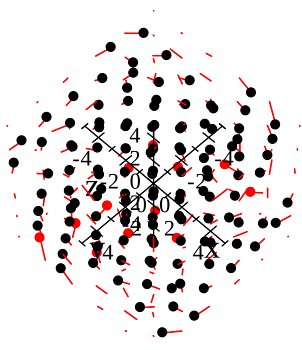
Definiţia 3. Un câmp vectorial $\overrightarrow{\mathbf{v}} : D \subseteq \mathbb{R}^3 \to \mathbb{V}_3$ de clasă $\mathcal{C}^1$ pe un domeniu D se numeşte *câmp armonic* dacă este simultan solenoidal şi irotaţional, adică:

$\operatorname{div}\overrightarrow{\mathbf{v}}(P) = 0, \forall P \in D$ şi $\operatorname{rot}\overrightarrow{\mathbf{v}}(P) = \overrightarrow{\mathbf{0}}, \forall P \in D.$

Exerciţiul 4. Să se determine rotorul următoarelor câmpuri vectoriale:

a) $\overrightarrow{\mathbf{v}}(P) = (z^2y)\,\overrightarrow{\mathbf{i}} + (x^2z)\,\overrightarrow{\mathbf{j}} + (y^2x)\,\overrightarrow{\mathbf{k}}.$

Rezolvare. Câmpul vectorial are reprezentarea



Conform (3) $\Rightarrow$

$$\operatorname{rot}\overrightarrow{\mathbf{v}}(P) = \left|\begin{array}{ccc} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2y & x^2z & y^2x \end{array}\right| =$$

$$= \left(\frac{\partial}{\partial y}(y^2x) - \frac{\partial}{\partial z}(x^2z)\right)\overrightarrow{\mathbf{i}} - \left(\frac{\partial}{\partial x}(y^2x) - \frac{\partial}{\partial z}(z^2y)\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial}{\partial x}(x^2z) - \frac{\partial}{\partial y}(z^2y)\right)\overrightarrow{\mathbf{k}} =$$

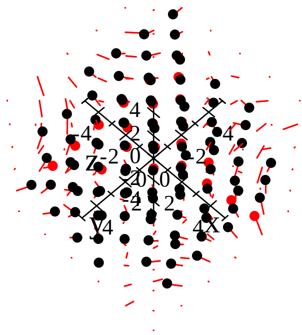
$$= \left(x\cdot 2y - x^2\right)\overrightarrow{\mathbf{i}} - \left(y^2 - y\cdot 2z\right)\overrightarrow{\mathbf{j}} + \left(z\cdot 2x - z^2\right)\overrightarrow{\mathbf{k}} = (x(2y-x))\,\overrightarrow{\mathbf{i}} + (y(2z-y))\,\overrightarrow{\mathbf{j}} + (z(2x-z))\,\overrightarrow{\mathbf{k}},$$

$\forall P \in D.$

Câmpul vectorial

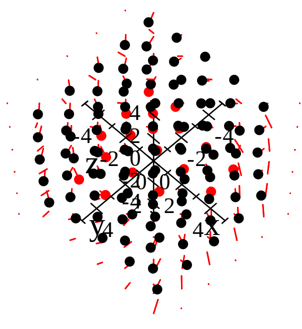
$\overline{\mathbf{w}}(P) = (x(2y-x))\,\overrightarrow{\mathbf{i}} + (y(2z-y))\,\overrightarrow{\mathbf{j}} + (z(2x-z))\,\overrightarrow{\mathbf{k}}$

este un câmp de "vârtejuri", de rotori, deoarece provine din $\operatorname{rot}\overrightarrow{\mathbf{v}} = \overline{\mathbf{w}}$. Deoarece $\operatorname{rot}\overrightarrow{\mathbf{v}}(P) \neq \overrightarrow{\mathbf{0}}$, în $P \neq O$, $\overrightarrow{\mathbf{v}}$ este câmp cu rotaţie în $P \neq O$.



b) $\overrightarrow{\mathbf{v}}(P) = (x^2+y^2)\,\overrightarrow{\mathbf{i}} + (y^2+z^2)\,\overrightarrow{\mathbf{j}} + (z^2+x^2)\,\overrightarrow{\mathbf{k}}$.

Rezolvare. Câmpul vectorial are reprezentarea



Conform (3) $\Rightarrow$

$$\operatorname{rot}\overrightarrow{\mathbf{v}}(P) = \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+y^2 & y^2+z^2 & z^2+x^2 \end{vmatrix} =$$

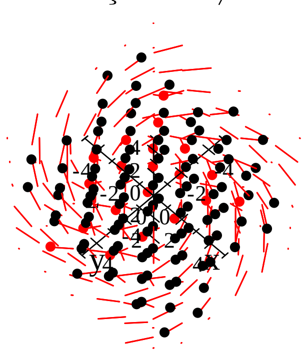
$$= \left(\frac{\partial}{\partial y}(z^2+x^2) - \frac{\partial}{\partial z}(y^2+z^2)\right)\overrightarrow{\mathbf{i}} - \left(\frac{\partial}{\partial x}(z^2+x^2) - \frac{\partial}{\partial z}(x^2+y^2)\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial}{\partial x}(y^2+z^2) - \frac{\partial}{\partial y}(x^2+y^2)\right)\overrightarrow{\mathbf{k}}$$

$$= (0-2z)\,\overrightarrow{\mathbf{i}} - (2x-0)\,\overrightarrow{\mathbf{j}} + (0-2y)\,\overrightarrow{\mathbf{k}} = -2z\,\overrightarrow{\mathbf{i}} - 2x\,\overrightarrow{\mathbf{j}} - 2y\,\overrightarrow{\mathbf{k}}, \forall P \in D.$$

Câmpul vectorial

$\overline{\mathbf{w}}(P) = -2z\overrightarrow{\mathbf{i}} - 2x\overrightarrow{\mathbf{j}} - 2y\overrightarrow{\mathbf{k}}$

este un câmp de "vârtejuri", deoarece provine din rot $\overrightarrow{\mathbf{v}} = \overline{\mathbf{w}}$. Deoarece rot $\overrightarrow{\mathbf{v}}(P) \neq \overrightarrow{\mathbf{0}}$, în $P \neq O$, $\overrightarrow{\mathbf{v}}$ este câmp cu rotaţie în $P \neq O$.



c) $\overrightarrow{\mathbf{v}}(P) = \left(r^n \overrightarrow{\mathbf{r}}\right)(P)$, $n \in \mathbb{N}^*$, unde $\overrightarrow{\mathbf{r}}(P) = x\overrightarrow{\mathbf{i}} + y\overrightarrow{\mathbf{j}} + z\overrightarrow{\mathbf{k}}$ este vectorul de poziţie a punctului $P(x, y, z)$ pe D a.î. $O \notin D$ şi $r = \left\|\overrightarrow{\mathbf{r}}\right\|$.

Rezolvare. Fie $r(P) = \sqrt{x^2 + y^2 + z^2} \Rightarrow r^n(P) = \left(x^2 + y^2 + z^2\right)^{\frac{n}{2}}$

$$\overrightarrow{\mathbf{v}}(P) = \left(x^2+y^2+z^2\right)^{\frac{n}{2}} x\overrightarrow{\mathbf{i}} + \left(x^2+y^2+z^2\right)^{\frac{n}{2}} y\overrightarrow{\mathbf{j}} + \left(x^2+y^2+z^2\right)^{\frac{n}{2}} z\overrightarrow{\mathbf{k}}.$$

Conform (3) $\Rightarrow$

$$\operatorname{rot}\overrightarrow{\mathbf{v}}(P) = \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x\left(x^2+y^2+z^2\right)^{\frac{n}{2}} & y\left(x^2+y^2+z^2\right)^{\frac{n}{2}} & z\left(x^2+y^2+z^2\right)^{\frac{n}{2}} \end{vmatrix} =$$

$$= \left(\frac{\partial}{\partial y}\left(z\cdot\left(x^2+y^2+z^2\right)^{\frac{n}{2}}\right) - \frac{\partial}{\partial z}\left(y\cdot\left(x^2+y^2+z^2\right)^{\frac{n}{2}}\right)\right)\overrightarrow{\mathbf{i}} -$$

$$-\left(\frac{\partial}{\partial x}\left(z\cdot\left(x^2+y^2+z^2\right)^{\frac{n}{2}}\right) - \frac{\partial}{\partial z}\left(x\cdot\left(x^2+y^2+z^2\right)^{\frac{n}{2}}\right)\right)\overrightarrow{\mathbf{j}} +$$

$$+\left(\frac{\partial}{\partial x}\left(y\cdot\left(x^2+y^2+z^2\right)^{\frac{n}{2}}\right) - \frac{\partial}{\partial y}\left(x\cdot\left(x^2+y^2+z^2\right)^{\frac{n}{2}}\right)\right)\overrightarrow{\mathbf{k}} =$$

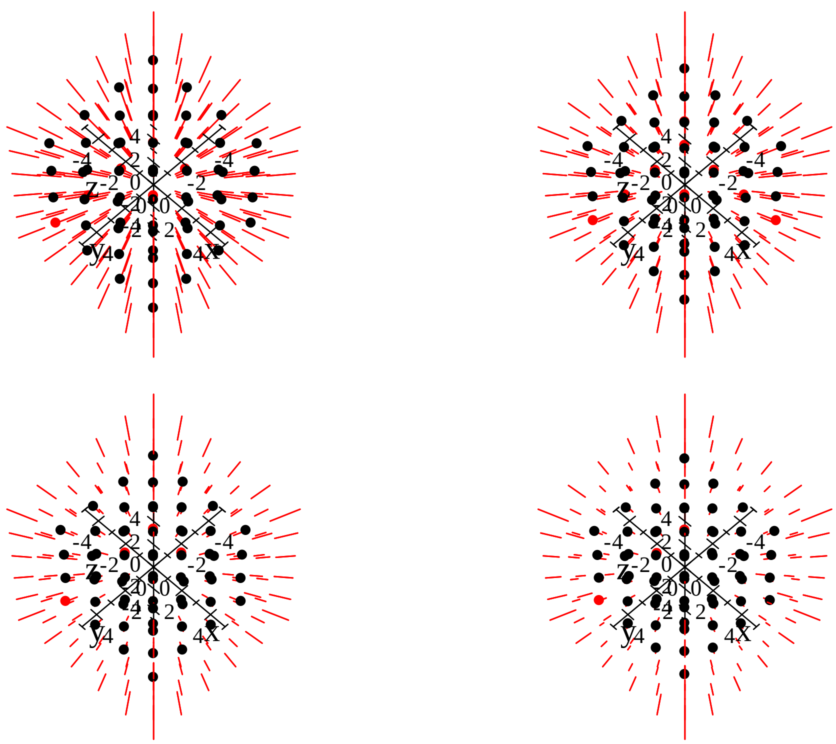
$$= \left(z\cdot\tfrac{n}{2}\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}(2y) - y\cdot\tfrac{n}{2}\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}(2z)\right)\overrightarrow{\mathbf{i}} -$$

$$-\left(z\cdot\tfrac{n}{2}\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}(2x) - x\cdot\tfrac{n}{2}\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}(2z)\right)\overrightarrow{\mathbf{j}} +$$

$$+\left(y\cdot\tfrac{n}{2}\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}(2x) - x\cdot\tfrac{n}{2}\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}(2y)\right)\overrightarrow{\mathbf{k}} =$$

$$= 0\overrightarrow{\mathbf{i}} + 0\overrightarrow{\mathbf{j}} + 0\overrightarrow{\mathbf{k}} = \overrightarrow{\mathbf{0}}, \forall P \in D \Rightarrow$$

Câmpul vectorial $\overrightarrow{\mathbf{v}}$ este unul irotaţional. Şi din reprezentările spaţiale ale $\overrightarrow{\mathbf{v}}$ pentru $n = 0, 1, 2, 3$ se observă lipsa rotaţiei.



d) $\overrightarrow{\mathbf{v}}(P) = \overrightarrow{\mathbf{c}} \times \overrightarrow{\mathbf{r}}(P)$, unde $\overrightarrow{\mathbf{r}}(P) = x\overrightarrow{\mathbf{i}} + y\overrightarrow{\mathbf{j}} + z\overrightarrow{\mathbf{k}}$ este vectorul de poziţie a punctului $P(x, y, z)$ pe D a.î. $O \notin D$ şi $r = \left\|\overrightarrow{\mathbf{r}}\right\|$, iar $\overrightarrow{\mathbf{c}} = c_1\overrightarrow{\mathbf{i}} + c_2\overrightarrow{\mathbf{j}} + c_3\overrightarrow{\mathbf{k}}$ este un vector constant.

$$\overrightarrow{\mathbf{v}}(x,y,z)=\overrightarrow{\mathbf{v}}(P)=\begin{vmatrix}\overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}}\\ c_1 & c_2 & c_3\\ x & y & z\end{vmatrix}=(c_2z-c_3y)\overrightarrow{\mathbf{i}}-(c_1z-c_3x)\overrightarrow{\mathbf{j}}+(c_1y-c_2x)\overrightarrow{\mathbf{k}}.$$

Conform (3) $\Rightarrow$

$$\operatorname{rot}\overrightarrow{\mathbf{v}}(P)=\begin{vmatrix}\overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ c_2z-c_3y & -(c_1z-c_3x) & c_1y-c_2x\end{vmatrix}=$$

$$=\left(\frac{\partial}{\partial y}(c_1y-c_2x)-\frac{\partial}{\partial z}(-(c_1z-c_3x))\right)\overrightarrow{\mathbf{i}}-\left(\frac{\partial}{\partial x}(c_1y-c_2x)-\frac{\partial}{\partial z}(c_2z-c_3y)\right)\overrightarrow{\mathbf{j}}+$$

$$+\left(\frac{\partial}{\partial x}(-(c_1z-c_3x))-\frac{\partial}{\partial y}(c_2z-c_3y)\right)\overrightarrow{\mathbf{k}}=$$

$$=2c_1\overrightarrow{\mathbf{i}}+2c_2\overrightarrow{\mathbf{j}}+2c_3\overrightarrow{\mathbf{k}}=2\overrightarrow{\mathbf{c}},\forall P\in D\Rightarrow$$

Câmpul vectorial este unul rotaţional.

Se poate scrie: $\boxed{\operatorname{rot}(\overrightarrow{\mathbf{c}}\times\overrightarrow{\mathbf{r}})=2\overrightarrow{\mathbf{c}}.}$

Exerciţiul 5. Fie câmpul vectorial

$$\overrightarrow{\mathbf{v}}(P)=f(r)\overrightarrow{\mathbf{r}}(P)+\overrightarrow{\mathbf{a}}\times\overrightarrow{\mathbf{r}}(P),$$

unde $\overrightarrow{\mathbf{r}}(P)=x\overrightarrow{\mathbf{i}}+y\overrightarrow{\mathbf{j}}+z\overrightarrow{\mathbf{k}}$ este vectorul de poziţie a punctului $P(x,y,z)$ pe D a.î. $O\notin D$, $r=\left\|\overrightarrow{\mathbf{r}}\right\|$, $\overrightarrow{\mathbf{a}}=a_1\overrightarrow{\mathbf{i}}+a_2\overrightarrow{\mathbf{j}}+a_3\overrightarrow{\mathbf{k}}$ este un vector constant, iar f este o funcţie scalară de clasă $\mathcal{C}^1$. Să se determine legea de asociere pentru f astfel încât:

$$\boxed{\operatorname{div}\left(f(r)\overrightarrow{\mathbf{a}}\right)=\overrightarrow{\mathbf{v}}\cdot\operatorname{rot}\overrightarrow{\mathbf{v}}.}$$

Rezolvare. $\bullet\bullet\, r(P)=\sqrt{x^2+y^2+z^2}$

$$f(r(P))\overrightarrow{\mathbf{a}}=f(r(P))\left(a_1\overrightarrow{\mathbf{i}}+a_2\overrightarrow{\mathbf{j}}+a_3\overrightarrow{\mathbf{k}}\right)=\underbrace{f(r(P))a_1}_{v_1\text{ pentru div}}\overrightarrow{\mathbf{i}}+\underbrace{f(r(P))a_2}_{v_2\text{ pentru div}}\overrightarrow{\mathbf{j}}+\underbrace{f(r(P))a_3}_{v_3\text{ pentru div}}\overrightarrow{\mathbf{k}}.$$

Conform (1) $\Rightarrow$

$$\operatorname{div}\left(f(r)\overrightarrow{\mathbf{a}}\right)(P)=$$

$$=\frac{\partial}{\partial x}(a_1f(r(P)))+\frac{\partial}{\partial y}(a_2f(r(P)))+\frac{\partial}{\partial z}(a_3f(r(P)))=$$

$$=a_1\frac{df}{dr}(r(P))\cdot\frac{\partial r}{\partial x}(P)+a_2\frac{df}{dr}(r(P))\cdot\frac{\partial r}{\partial y}(P)+a_3\frac{df}{dr}(r(P))\cdot\frac{\partial r}{\partial z}(P)=$$

$$=a_1f'(r(P))\cdot\frac{x}{\sqrt{x^2+y^2+z^2}}+a_2f'(r(P))\cdot\frac{y}{\sqrt{x^2+y^2+z^2}}+a_3f'(r(P))\cdot\frac{z}{\sqrt{x^2+y^2+z^2}}=$$

$$=\frac{f'(r(P))}{r(P)}\cdot(a_1x+a_2y+a_3z)=\frac{f'(r(P))}{r(P)}\left(\overrightarrow{\mathbf{a}}\cdot\overrightarrow{\mathbf{r}}(P)\right).$$

Se putea obţine şi din Exerciţiul 2,

$$\boxed{\operatorname{div}\left(f(r)\overrightarrow{\mathbf{v}}\right)=\frac{f'(r)}{r}\left(\overrightarrow{\mathbf{r}}\cdot\overrightarrow{\mathbf{v}}\right)+f(r)\operatorname{div}\overrightarrow{\mathbf{v}}}\Rightarrow\boxed{\operatorname{div}\left(f(r)\overrightarrow{\mathbf{a}}\right)=\frac{f'(r)}{r}\left(\overrightarrow{\mathbf{r}}\cdot\overrightarrow{\mathbf{a}}\right)+f(r)\underbrace{\operatorname{div}\overrightarrow{\mathbf{a}}}_{0}.}$$

$$\bullet\bullet\,\overrightarrow{\mathbf{a}}\times\overrightarrow{\mathbf{r}}=\begin{vmatrix}\overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}}\\ a_1 & a_2 & a_3\\ x & y & z\end{vmatrix}=(a_2z-a_3y)\overrightarrow{\mathbf{i}}-(a_1z-a_3x)\overrightarrow{\mathbf{j}}+(a_1y-a_2x)\overrightarrow{\mathbf{k}}.$$

$$\overrightarrow{\mathbf{v}}(P)=f(r)\overrightarrow{\mathbf{r}}+\overrightarrow{\mathbf{a}}\times\overrightarrow{\mathbf{r}}=$$

$$=f(r)x\overrightarrow{\mathbf{i}}+f(r)y\overrightarrow{\mathbf{j}}+f(r)z\overrightarrow{\mathbf{k}}+(a_2z-a_3y)\overrightarrow{\mathbf{i}}-(a_1z-a_3x)\overrightarrow{\mathbf{j}}+(a_1y-a_2x)\overrightarrow{\mathbf{k}}=$$

$$=\underbrace{(f(r)x+(a_2z-a_3y))}_{v_1\text{ pentru rot}}\overrightarrow{\mathbf{i}}+\underbrace{(f(r)y-(a_1z-a_3x))}_{v_2\text{ pentru rot}}\overrightarrow{\mathbf{i}}+\underbrace{(f(r)z+(a_1y-a_2x))}_{v_3\text{ pentru rot}}\overrightarrow{\mathbf{i}}.$$

Conform (3) $\Rightarrow$

$$\operatorname{rot}\overrightarrow{\mathbf{v}}(P)=\begin{vmatrix}\overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ f(r)\,x+(a_2z-a_3y) & f(r)\,y-(a_1z-a_3x) & f(r)\,z+(a_1y-a_2x)\end{vmatrix}=$$

$$=\left(\frac{\partial}{\partial y}\left(f(r)\,z+(a_1y-a_2x)\right)-\frac{\partial}{\partial z}\left(f(r)\,y-(a_1z-a_3x)\right)\right)\overrightarrow{\mathbf{i}}-$$
$$-\left(\frac{\partial}{\partial x}\left(f(r)\,z+(a_1y-a_2x)\right)-\frac{\partial}{\partial z}\left(f(r)\,x+(a_2z-a_3y)\right)\right)\overrightarrow{\mathbf{j}}+$$
$$+\left(\frac{\partial}{\partial x}\left(f(r)\,y-(a_1z-a_3x)\right)-\frac{\partial}{\partial y}\left(f(r)\,x+(a_2z-a_3y)\right)\right)\overrightarrow{\mathbf{k}}=$$
$$=\left(a_1+z\frac{df}{dr}(r(P))\cdot\frac{\partial r}{\partial y}(P)+a_1-y\frac{df}{dr}(r(P))\cdot\frac{\partial r}{\partial z}(P)\right)\overrightarrow{\mathbf{i}}-$$
$$-\left(-a_2+z\frac{df}{dr}(r(P))\cdot\frac{\partial r}{\partial x}(P)-a_2-x\frac{df}{dr}(r(P))\cdot\frac{\partial r}{\partial z}(P)\right)\overrightarrow{\mathbf{j}}+$$
$$+\left(a_3+y\frac{df}{dr}(r(P))\cdot\frac{\partial r}{\partial x}(P)+a_3-x\frac{df}{dr}(r(P))\cdot\frac{\partial r}{\partial y}(P)\right)\overrightarrow{\mathbf{k}}=$$
$$=2a_1\overrightarrow{\mathbf{i}}+2a_2\overrightarrow{\mathbf{j}}+2a_3\overrightarrow{\mathbf{k}}=2\overrightarrow{\mathbf{a}}.$$

$\overrightarrow{\mathbf{v}}\cdot\operatorname{rot}\overrightarrow{\mathbf{v}}=2\overrightarrow{\mathbf{v}}\cdot\overrightarrow{\mathbf{a}}=2\left(f(r)\,\overrightarrow{\mathbf{r}}+\overrightarrow{\mathbf{a}}\times\overrightarrow{\mathbf{r}}\right)\cdot\overrightarrow{\mathbf{a}}=2f(r)\,\overrightarrow{\mathbf{r}}\cdot\overrightarrow{\mathbf{a}}+\left(\overrightarrow{\mathbf{a}}\times\overrightarrow{\mathbf{r}}\right)\cdot\overrightarrow{\mathbf{a}}=2f(r)\,\overrightarrow{\mathbf{r}}\cdot\overrightarrow{\mathbf{a}}+\overrightarrow{\mathbf{0}}=2f(r)\,\overrightarrow{\mathbf{r}}\cdot\overrightarrow{\mathbf{a}}$

•• Atunci $\operatorname{div}\left(f(r)\,\overrightarrow{\mathbf{a}}\right)=\overrightarrow{\mathbf{v}}\cdot\operatorname{rot}\overrightarrow{\mathbf{v}}\Leftrightarrow\dfrac{f'(r)}{r}\left(\overrightarrow{\mathbf{a}}\cdot\overrightarrow{\mathbf{r}}\right)=2f(r)\,\overrightarrow{\mathbf{r}}\cdot\overrightarrow{\mathbf{a}}\Leftrightarrow$

$$\Leftrightarrow\frac{f'(r)}{r}=2f(r)\,,r\in D,r\neq 0\Rightarrow\frac{f'(r)}{f(r)}=2r,r\in D,r\neq 0\Rightarrow$$
$$\Rightarrow\ln|f(r)|=r^2+c,r\in D,r\neq 0,c\in\mathbb{R}\Rightarrow f(r)=e^{r^2+c},r\in D,r\neq 0,c\in\mathbb{R}.$$