Facultatea de Hidrotehnică, Geodezie şi Ingineria Mediului
Analiza Matematică II, Semestrul II
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SEMINAR 4 – 5

Capitolul II. Extinderea noţiunii de integrală. Integrale improprii

1. Studiaţi, folosind definiţia, convergenţa următoarelor integrale improprii de specia I:

$$a) \int_e^\infty \frac{dx}{x(\ln x)^{3/2}}\ ,\quad b) \int_1^\infty \frac{e^x dx}{e^x-1}\ ,\quad c) \int_0^\infty e^{-2x}\sin 3x\ dx\ ,\quad d) \int_0^\infty e^{-3x}\cos 4x\ dx,$$

$$e) \int_1^\infty \frac{dx}{x^\alpha}\ ,\ \alpha>0\ ,\quad f) \int_{-\infty}^\infty \frac{dx}{1+x^2}\ ,\quad g) \int_0^\infty \frac{dx}{(1+x)^2}.$$

Rezolvare:

$a)$

$$\int_e^\infty \frac{1}{x(\ln x)^{3/2}}dx = (\text{subst. } \ln x = y) = \int_1^\infty y^{-3/2}dy = \frac{y^{-1/2}}{-1/2}\Big|_{y=1}^{y=\infty} = \frac{-1}{2\sqrt{y}}\Big|_{y=1}^{y=\infty} = \frac{1}{2}.$$

$b)$

$$\int_1^\infty \frac{e^x}{e^x-1}dx = (\text{subst. } e^x = y) = \int_e^\infty \frac{1}{y-1}dy = \ln|y-1|\ \Big|_{y=e}^{y=\infty} = \ln(+\infty) - \ln|e-1| = +\infty.$$

$c)$ aplicăm de două ori metoda de integrare prin părţi pentru a calcula primitiva

$$\begin{aligned} F(x) &= \int e^{-2x}\sin 3x\ dx = \frac{-1}{2}e^{-2x}\sin 3x + \frac{3}{-4}e^{-2x}\cos 3x - \frac{3}{2}\frac{3}{2}F(x) \\ &\Leftrightarrow F(x) = \frac{13}{4}\left(\frac{-1}{2}e^{-2x}\sin 3x + \frac{3}{-4}e^{-2x}\cos 3x\right) \\ &\Leftrightarrow F(x) = -\frac{2\sin 3x + 3\cos 3x}{2^2+3^2}\cdot e^{-2x}. \end{aligned}$$

Deci

$$\int_0^\infty e^{-2x}\sin 3x\ dx = -F(x)\,\Big|_{x=0}^{x=+\infty} = F(+\infty) - F(0).$$

Dar $F(0) = -\frac{3}{2^2+3^2}$ iar

$$F(+\infty) \overset{def}{=} \lim_{x\to+\infty}(-)\frac{2\sin 3x + 3\cos 3x}{2^2+3^2}\cdot e^{-2x} = 0$$

deoarece

$$\left|-\frac{2\sin 3x+3\cos 3x}{2^2+3^2}\right|=\left|\frac{2\sin 3x+3\cos 3x}{2^2+3^2}\right|\le\frac{|2\sin 3x|+|3\cos 3x|}{|2^2+3^2|}\le\frac{2+3}{2^2+3^2}$$

(adică este mărginit) iar

$$\lim_{x\to+\infty}e^{-2x}=e^{-\infty}=0$$

Am folosit rezultatul:

Lema 1 *Fie $f,g:I\to\mathbb{R}$ unde I este un interval. Presupunem că $\lim_{x\to a}f(x)=0$ şi $|g(x)|\le M$, $\forall x\in I$. Atunci*

$$\lim_{x\to a}[f(x)\cdot g(x)]=0$$

(adică produsul dintre o cantitate care tinde la zero şi o cantitate mărginită este o cantitate care tinde la zero).

Prin urmare

$$\int_0^\infty e^{-2x}\sin 3x\,dx=F(+\infty)-F(0)=\frac{3}{2^2+3^2}\ .$$

2. Studiaţi, folosind definiţia, convergenţa următoarelor integrale improprii de specia II:

$a)\ \int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}\ ,\quad b)\ \int_a^b\frac{dx}{(x-a)^\lambda}\ ,\lambda>0\ ,$

$c)\ \int_0^1\frac{e^x}{e^x-1}dx\ ,\quad d)\ \int_0^1\frac{dx}{x^\alpha}\ ,\alpha>0\ ,\quad e)\ \int_1^e\frac{1}{x(\ln x)^{3/2}}dx.$

3. Studiaţi, folosind criteriul de convergenţă în α, convergenţa următoarelor integrale improprii:

$a)\ \int_0^\infty\frac{x}{\sqrt{x^5+1}}dx\ ,\quad b)\ \int_0^\infty\frac{1}{2x+5+\sqrt[3]{x^2+1}}dx\ ,\quad c)\ \int_1^\infty\frac{\operatorname{arctg}x}{x}dx\ ,$

$d)\ \int_1^\infty\frac{1}{x\sqrt{1+x^4}}dx\ ,\quad e)\ \int_0^\infty\frac{dx}{\sqrt[3]{1+x^3+x^6}}.$

4. Studiaţi, folosind criteriul de convergenţă în λ, convergenţa următoarelor integrale improprii:

$a)\ \int_0^1\frac{1}{\sqrt[4]{1-x^4}}dx\ ,\quad b)\ \int_0^3\frac{1}{\sqrt[3]{x}+2\sqrt[4]{x}+x^3}dx\ ,$

$c)\ \int_0^1\frac{1}{x^3-5x^2}dx\ ,\quad d)\ \int_1^2\frac{1}{x\sqrt{x-1}}dx$

$e)\ \int_2^3\frac{x^2}{\sqrt[5]{(3-x)^2(x-2)}}dx\ ,\quad f)\ \int_2^5\frac{1}{\sqrt[3]{(2-x)^4(x^4+x^2+1)}}dx$

5. Studiaţi convergenţa următoarelor integrale improprii:

$a)\int_{-\infty}^{\infty}\frac{1}{x^2-2x+3}dx\ ,\quad b)\int_{-1}^{1}\frac{1}{\sqrt[5]{x^3}}dx\ ,$

$c)\int_{-\infty}^{\infty}x^2e^{-|x|}dx\ ,\quad d)\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx\ ,\quad e)\int_{-\infty}^{\infty}\frac{x}{1+x^2}dx$

Indicaţie: se scrie fiecare integrală ca sumă de alte două integrale. Eventual se poate folosi şi paritatea funcţiei de sub integrală şi faptul că $\int_{-a}^{a}f(x)\,dx=\begin{cases}2\int_0^a f(x)\,dx\ , & f \text{ este funcţie pară}\\ 0\ , & f \text{ este funcţie impară}\end{cases}$

6. Studiaţi convergenţa următoarelor integrale improprii:

$a)\int_{1}^{\infty}(\pi-2\mathrm{arctg}\,(x))\,dx\ ,\quad b)\int_{1}^{\infty}\frac{x+\cos x}{x^3+\sin x}dx\ ,\quad c)\int_{1}^{\infty}\frac{\ln x}{x}dx\ ,$

$d)\int_{2}^{\infty}\frac{1}{x\,(\ln x)^{\beta}}dx\ ,\ \beta\in\mathbb{R}\ ,\quad e)\int_{1}^{2}\frac{1}{x\,(\ln x)^{\beta}}dx\ ,\ \beta\in\mathbb{R}\ ,\quad f)\int_{1}^{2}\frac{\ln x}{x}dx\ .$

Rezolvare:

$a)$ Avem $f(x)=\pi-2\mathrm{arctg}\,(x)$. Trebuie să determinăm α astfel încât să existe limita $\lim\limits_{x\to\infty}x^{\alpha}f(x)$.

$$\begin{aligned}\lim_{x\to\infty}x^{\alpha}f(x) &= \lim_{x\to\infty}\left(x^{\alpha}\cdot(\pi-2\mathrm{arctg}\,(x))\right)=(\text{nedeterm. }\infty\cdot 0)=\lim_{x\to\infty}\frac{\pi-2\mathrm{arctg}\,(x)}{\frac{1}{x^{\alpha}}}\\ &= (\text{aleg }\alpha=1\text{ şi aplic L'Hospital})=\lim_{x\to\infty}\frac{0-2\frac{1}{1+x^2}}{\frac{-1}{x^2}}=\lim_{x\to\infty}\frac{2x^2}{1+x^2}=2.\end{aligned}$$

Deci pentru $\alpha=1\leq 1$ obţin $\lim\limits_{x\to\infty}x^{\alpha}f(x)=2\in(0,\infty)$ deci $\int_1^{\infty}(\pi-2\mathrm{arctg}\,(x))\,dx$ este (D).

$b)$

$$\begin{aligned}\lim_{x\to\infty}x^{\alpha}f(x) &= \lim_{x\to\infty}x^{\alpha}\cdot\frac{x+\cos x}{x^3+\sin x}=(\text{aleg }\alpha=2)=\lim_{x\to\infty}\frac{x^3+x^2\cos x}{x^3+\sin x}=\lim_{x\to\infty}\frac{x^3\left(1+\frac{\cos x}{x}\right)}{x^3\left(1+\frac{\sin x}{x^3}\right)}\\ &= \lim_{x\to\infty}\frac{1+\frac{\cos x}{x}}{1+\frac{\sin x}{x^3}}=\frac{1+0}{1+0}=1.\end{aligned}$$

Am folosit că

$$\lim_{x\to\infty}\frac{\cos x}{x}=0\quad\text{şi}\quad\lim_{x\to\infty}\frac{\sin x}{x^3}=0,$$

$c)$

$$\lim_{x\to\infty}x^{\alpha}f(x)=\lim_{x\to\infty}x^{\alpha}\cdot\frac{\ln x}{x}=(\text{aleg }\alpha=1)=\lim_{x\to\infty}\ln x=+\infty>0$$

deci limita este >0 iar $\alpha\leq 1$, prin urmare integrala este divergentă (Conform *Criteriului în* α).

$d)$ Calculăm primitiva

$$\int \frac{1}{x(\ln x)^{\beta}}dx = \int (\ln x)^{-\beta}(\ln x)'\,dx = (\text{pp. } \beta \neq 1 \text{ şi fac subst. } \ln x = t)$$
$$\int t^{-\beta}dt = \frac{t^{-\beta+1}}{-\beta+1} + C = \frac{(\ln x)^{-\beta+1}}{-\beta+1} + C.$$

Dacă $\beta = 1$, atunci

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{\ln x}(\ln x)'\,dx = (\text{subst. } \ln x = t) = \int \frac{1}{t}dt = \ln t + C = \ln|\ln x| + C.$$

Prin urmare, pentru $\beta \neq 1$,

$$\begin{aligned}\int_2^{\infty} \frac{1}{x(\ln x)^{\beta}}dx &= \frac{(\ln x)^{-\beta+1}}{-\beta+1}\Big|_{x=2}^{x=\infty} = \frac{1}{1-\beta}\left(\lim_{x\to\infty}(\ln x)^{-\beta+1} - (\ln 2)^{-\beta+1}\right)\\ &= \frac{1}{1-\beta}\left((\ln(+\infty))^{-\beta+1} - (\ln 2)^{-\beta+1}\right)\\ &= \begin{cases} \frac{1}{1-\beta}\left(+\infty - (\ln 2)^{-\beta+1}\right) = +\infty, & \text{dacă } \beta < 1,\\ \frac{1}{1-\beta}\left(\frac{1}{(+\infty)^{\beta-1}} - \frac{1}{(\ln 2)^{\beta-1}}\right) = \frac{1}{\beta-1}\frac{1}{(\ln 2)^{\beta-1}}\ , & \text{dacă } \beta > 1.\end{cases}\end{aligned},$$

Pentru $\beta = 1$,

$$\int_1^{\infty} \frac{1}{x\ln x}dx = \ln|\ln x|\ \Big|_{x=1}^{x=\infty} = \lim_{x\to\infty}\ln|\ln x| - \ln|\ln 1| = \ln\ln(+\infty) - \ln 0_+ = \ln(+\infty) - (-\infty) = +\infty.$$

7. Studiaţi convergenţa următoarelor integrale improprii (calculându-le, eventual, în prealabil):

$a)\ \int_0^{\infty} \frac{dx}{1+x^3}\ ,\quad b)\ \int_1^{\infty} \frac{\ln x}{x^3}dx\ ,\quad c)\ \int_1^{\infty} \frac{\sqrt{x}}{(1+x)^2}dx\ ,$

$d)\ \int_0^{\infty} \frac{x^3dx}{(1+x^3)^2}\ ,\quad e)\ \int_1^{\infty} \frac{x\ln x}{(1+x^2)^2}dx\ .$

Rezolvare:

$a)$ Mai întâi,

$$\frac{1}{1+x^3} = \frac{a}{1+x} + \frac{bx+c}{1-x+x^2} \Leftrightarrow 1 = (a+b)x^2 + (-a+b+c)x + (a+c)$$
$$\Leftrightarrow a = 1/3,\ b = -1/3,\ c = 2/3$$

şi deci

$$\int \frac{dx}{1+x^3} = \frac{1}{3}\int \frac{1}{1+x}dx + \frac{1}{3}\int \frac{-x+2}{1-x+x^2}dx = \frac{1}{3}\ln|1+x| + \frac{1}{3}\int \frac{1}{-2}\frac{2x-4}{1-x+x^2}dx$$
$$= \frac{1}{3}\ln|1+x| + \frac{1}{3}\ \frac{1}{-2}\int \frac{2x-1-3}{1-x+x^2}dx$$
$$= \frac{1}{3}\ln|1+x| - \frac{1}{6}\int \frac{\left(1-x+x^2\right)'}{1-x+x^2}dx - \frac{1}{6}\int \frac{-3}{1-x+x^2}dx$$
$$= \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln\left|1-x+x^2\right| + \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dx =$$
$$= \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln\left|1-x+x^2\right| + \frac{1}{2}\frac{1}{\sqrt{3}/2}\operatorname{arctg}\frac{x-1/2}{\sqrt{3}/2} + C$$
$$= \frac{1}{6}\left(2\ln|1+x| - \ln\left|1-x+x^2\right|\right) + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x-1/2}{\sqrt{3}/2} + C$$
$$= \frac{1}{6}\ln\frac{|1+x|^2}{|1-x+x^2|} + \frac{1}{2}\operatorname{arctg}\frac{x-1/2}{\sqrt{3}/2} = \frac{1}{6}\ln\frac{x^2+x+1}{x^2-x+1} + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x-1/2}{\sqrt{3}/2} + C\ .$$

Integrala improprie este atunci

$$\int_0^\infty \frac{dx}{1+x^3} = \frac{1}{6}\ln\frac{x^2+x+1}{x^2-x+1}\Big|_{x=0}^{x=\infty} + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x-1/2}{\sqrt{3}/2}\Big|_{x=0}^{x=\infty}$$
$$= \frac{1}{6}\lim_{x\to\infty}\ln\frac{x^2+x+1}{x^2-x+1} - \frac{1}{3}\ln\frac{0^2+0+1}{0^2-0x+1} + \frac{1}{\sqrt{3}}\lim_{x\to\infty}\operatorname{arctg}\frac{x-1/2}{\sqrt{3}/2} - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{0-1/2}{\sqrt{3}/2}$$
$$= \frac{1}{3}\ln 1 - \frac{1}{3}\ln 1 + \frac{1}{\sqrt{3}}\operatorname{arctg}(\infty) - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{-1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}\frac{\pi}{2} + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{1}{\sqrt{3}}$$
$$= \frac{\pi}{2\sqrt{3}} + \frac{\pi}{6\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}\ .$$

b) Calculăm mai întâi primitiva

$$\int \frac{\ln x}{x^3}dx = \int x^{-3}\ln x dx = \int \left(\frac{x^{-2}}{-2}\right)'\ln x dx = \frac{x^{-2}}{-2}\ln x - \int \frac{x^{-2}}{-2}\frac{1}{x}dx = \frac{x^{-2}}{-2}\ln x + \frac{1}{2}\int x^{-3}dx$$
$$= \frac{1}{-2}\frac{\ln x}{x^2} + \frac{1}{2}\frac{x^{-2}}{-2} + C.$$

Integrala improprie este atunci

$$\int_1^\infty \frac{\ln x}{x^3}dx = \frac{-1}{2}\frac{\ln x}{x^2}\Big|_{x=1}^{x=\infty} - \frac{1}{4}\frac{1}{x^2}\Big|_{x=1}^{x=\infty}$$
$$= \frac{-1}{2}\lim_{x\to\infty}\frac{\ln x}{x^2} - \frac{-1}{2}\frac{\ln 1}{1^2} - \frac{1}{4}\lim_{x\to\infty}\frac{1}{x^2} + \frac{1}{4}\frac{1}{1^2} = \frac{1}{4}\ .$$

deoarece

$$\lim_{x\to\infty}\frac{\ln x}{x^2} = (\text{L'Hospital}) = \lim_{x\to\infty}\frac{1/x}{2x} = 0.$$

$c)$ Calculăm mai întâi primitiva făcând substituţia

$$x = t^2 \Leftrightarrow t = \sqrt{x} \Rightarrow dx = 2tdt$$

(vezi Integrale din funcţii iraţionale):

$$\int \frac{\sqrt{x}}{(1+x)^2}dx = \int \frac{t}{(1+t^2)^2}2tdt.$$

Dar

$$\frac{t}{(1+t^2)^2} = \frac{1}{2}\frac{(1+t^2)'}{(1+t^2)^2} = \frac{1}{2}\left(1+t^2\right)^{-2}\left(1+t^2\right)' = \frac{1}{2}\left(\frac{\left(1+t^2\right)^{-1}}{-1}\right)' = -\frac{1}{2}\left(\frac{1}{1+t^2}\right)'$$

deci

$$\int \frac{\sqrt{x}}{(1+x)^2}dx = 2\int t\left(\frac{-1}{2}\frac{1}{1+t^2}\right)'dt = 2t\left(\frac{-1}{2}\frac{1}{1+t^2}\right) - 2\int 1\left(\frac{-1}{2}\frac{1}{1+t^2}\right)dt = -\frac{t}{1+t^2} + \text{arctg}t + C$$

$$= -\frac{\sqrt{x}}{1+x} + \text{arctg}\sqrt{x} + C.$$

Integrala improprie este atunci

$$\int_1^\infty \frac{\sqrt{x}}{(1+x)^2}dx = -\frac{\sqrt{x}}{1+x}\Big|_{x=1}^{x=\infty} + \text{arctg}\sqrt{x}\Big|_{x=1}^{x=\infty}$$

$$= -\lim_{x\to\infty}\frac{\sqrt{x}}{1+x} + \frac{\sqrt{1}}{1+1} + \lim_{x\to\infty}\text{arctg}\sqrt{x} - \text{arctg}\sqrt{1} = \frac{1}{2} + \frac{\pi}{2} - \frac{\pi}{4} = \frac{1}{2} + \frac{\pi}{4}.$$

$d)$ Observăm mai întâi că

$$\frac{x^2}{(1+x^3)^2} = \frac{1}{3}\frac{(1+x^3)'}{(1+x^3)^2} = \frac{1}{3}\left(1+x^3\right)^{-2}\left(1+x^3\right)' = \frac{1}{3}\left(\frac{\left(1+x^3\right)^{-1}}{-1}\right)' = -\frac{1}{3}\left(\frac{1}{1+x^3}\right)'$$

deci

$$\int \frac{x^3}{(1+x^3)^2}dx = \int x\frac{x^2}{(1+x^3)^2}dx = \int x\left(-\frac{1}{3}\frac{1}{1+x^3}\right)'dx = x\left(-\frac{1}{3}\frac{1}{1+x^3}\right) - \int 1\left(-\frac{1}{3}\frac{1}{1+x^3}\right)dx$$

$$= -\frac{1}{3}\frac{x}{1+x^3} + \frac{1}{3}\int\frac{1}{1+x^3}dx$$

Integrala improprie este atunci (vezi şi punctul $a)$)

$$\int_0^\infty \frac{x^3dx}{(1+x^3)^2} = -\frac{1}{3}\frac{x}{1+x^3}\Big|_{x=0}^{x=\infty} + \frac{1}{3}\int_0^\infty\frac{1}{1+x^3}dx$$

$$= -\frac{1}{3}\lim_{x\to\infty}\frac{x}{1+x^3} + \frac{1}{3}\frac{0}{1+0^3} + \frac{1}{3}\frac{2\pi}{3\sqrt{3}} = \frac{2\pi}{9\sqrt{3}}.$$

$e)$ Observăm mai întâi că

$$\frac{x}{(1+x^2)^2} = \frac{1}{2}\frac{(1+x^2)'}{(1+x^2)^2} = \frac{1}{2}\left(1+x^2\right)^{-2}\left(1+x^2\right)' = \frac{1}{2}\left(\frac{\left(1+x^2\right)^{-1}}{-1}\right)' = -\frac{1}{2}\left(\frac{1}{1+x^2}\right)'$$

deci

$$\int \frac{x\ln x}{\left(1+x^2\right)^2}dx = \int \ln x\left(-\frac{1}{2}\frac{1}{1+x^2}\right)' dx = \ln x\left(-\frac{1}{2}\frac{1}{1+x^2}\right) - \int \frac{1}{x}\left(-\frac{1}{2}\frac{1}{1+x^2}\right)dx$$
$$= -\frac{1}{2}\frac{\ln x}{1+x^2} + \frac{1}{2}\int \frac{1}{x\left(1+x^2\right)}dx.$$

Pentru a calcula integrala $\int \frac{1}{x(1+x^2)}dx$, trebuie sa descompunem fracţia în fracţii simple:

$$\frac{1}{x\left(1+x^2\right)} = \frac{a}{x} + \frac{bx+c}{1+x^2} \Leftrightarrow a=1,\ b=-1,\ c=0.$$

Deci

$$\int \frac{x\ln x}{\left(1+x^2\right)^2}dx = -\frac{1}{2}\frac{\ln x}{1+x^2} + \frac{1}{2}\int \frac{1}{x}dx + \frac{1}{2}\int \frac{-x}{1+x^2}dx$$
$$= -\frac{1}{2}\frac{\ln x}{1+x^2} + \frac{1}{2}\ln x - \frac{1}{4}\int \frac{2x}{1+x^2}dx = -\frac{1}{2}\frac{\ln x}{1+x^2} + \frac{1}{2}\ln x - \frac{1}{4}\ln\left(1+x^2\right) + C$$
$$= -\frac{1}{2}\frac{\ln x}{1+x^2} + \frac{1}{4}\ln x^2 - \frac{1}{4}\ln\left(1+x^2\right) + C = -\frac{1}{2}\frac{\ln x}{1+x^2} + \frac{1}{4}\ln\frac{x^2}{1+x^2} + C$$

Integrala improprie este atunci

$$\int_1^\infty \frac{x\ln x}{\left(1+x^2\right)^2}dx = -\frac{1}{2}\frac{\ln x}{1+x^2}\Big|_{x=1}^{x=\infty} + \frac{1}{4}\ln\frac{x^2}{1+x^2}\Big|_{x=1}^{x=\infty}$$
$$= -\frac{1}{2}\lim_{x\to\infty}\frac{\ln x}{1+x^2} + \frac{1}{2}\frac{\ln 1}{1+1^2} + \frac{1}{4}\lim_{x\to\infty}\ln\frac{x^2}{1+x^2} - \frac{1}{4}\ln\frac{1^2}{1+1^2} = -\frac{1}{4}\ln\frac{1}{2} = \frac{1}{4}\ln 2,$$

deoarece

$$\lim_{x\to\infty}\frac{\ln x}{1+x^2} = (\text{L'Hospital}) = \lim_{x\to\infty}\frac{1/x}{2x} = 0$$
$$\lim_{x\to\infty}\ln\frac{x^2}{1+x^2} = \ln\lim_{x\to\infty}\frac{x^2}{1+x^2} = \ln 1 = 0.$$

8. Arătaţi că următoarele integrale improprii sunt divergente:

$a)\ \int_0^{+\infty}\cos x\,dx\ ,\quad b)\ \int_0^{+\infty}\sin x\,dx\ ,\quad c)\ \int_0^{+\infty}x\sin x\,dx\ .$

9. Studiaţi convergenţa următoarelor integrale improprii:

$a)\ \int_0^1 \ln(1-x)\,dx\ ,\quad b)\ \int_1^3 \frac{dx}{4-x^2}\ ,\quad c)\ \int_2^3 \frac{x^2dx}{\sqrt[5]{(3-x)^2(x-2)}}\ ,$

$d)\ \int_a^b \frac{xdx}{\sqrt{(x-a)(b-x)}}\ ,\quad e)\ \int_2^\infty \frac{dx}{\sqrt[3]{(2-x)^4(x^4+x^2+1)}}\ ,\quad f)\ \int_1^2 \frac{dx}{x\sqrt{x-1}}\ ,$

$g)\ \int_1^\infty \frac{dx}{1-x^2}\ ,\quad h)\ \int_0^1 \frac{dx}{1-x^2}\ ,\quad i)\ \int_1^\infty \frac{5x+1}{x^3-x-6}dx.$

Rezolvare:

$a)$ Calculăm mai întâi primitiva

$$\begin{aligned}\int \ln(1-x)\,dx &= x\ln(1-x) - x - \ln(1-x) + C\\ &= -(1-x)\ln(1-x) - x + C,\ C\in\mathbb{R}.\end{aligned}$$

Deci

$$\begin{aligned}\int_0^1 \ln(1-x)\,dx &= \left[-(1-x)\ln(1-x) - x\right]\Big|_{x=0}^{x=1}\\ &= -\lim_{\substack{x\to 1\\ x<1}}(1-x)\ln(1-x) + 1\ln 1 - 1 + 0 = (\text{notând } 1-x=y)\\ &= -\lim_{y\to 0_+} y\ln y - 1 = \left(\text{notând } \frac{1}{y}=z\right) = \lim_{z\to\infty}\frac{\ln z}{z} - 1 = 0-1.\end{aligned}$$

$b)$

$$\int_1^3 \frac{dx}{4-x^2} = \int_1^3 \frac{dx}{(2-x)(2+x)} = \int_1^2 \frac{dx}{4-x^2} + \int_2^3 \frac{dx}{4-x^2} = \cdots$$

$c)$

$$\int_2^3 \frac{x^2dx}{\sqrt[5]{(3-x)^2(x-2)}} = \int_2^{5/2} \frac{x^2dx}{\sqrt[5]{(3-x)^2(x-2)}} + \int_{5/2}^3 \frac{x^2dx}{\sqrt[5]{(3-x)^2(x-2)}} = \cdots$$

$d)$ Facem substituţia

$$x - \frac{a+b}{2} = y \Leftrightarrow x = y + \frac{a+b}{2} \Rightarrow dx = dy$$

şi obţinem

$$\begin{aligned} I &= \int_{\frac{a-b}{2}}^{\frac{b-a}{2}} \frac{y+\frac{a+b}{2}}{\sqrt{\left(\frac{b-a}{2}\right)^2 - y^2}}dy = \int_{\frac{a-b}{2}}^{\frac{b-a}{2}} \frac{y}{\sqrt{\left(\frac{b-a}{2}\right)^2 - y^2}}dy + \int_{\frac{a-b}{2}}^{\frac{b-a}{2}} \frac{\frac{a+b}{2}}{\sqrt{\left(\frac{b-a}{2}\right)^2 - y^2}}dy\\ &= \int_{\frac{a-b}{2}}^{\frac{b-a}{2}} (-)\frac{\left(\left(\frac{b-a}{2}\right)^2 - y^2\right)'}{2\sqrt{\left(\frac{b-a}{2}\right)^2 - y^2}}dy + \frac{a+b}{2}\int_{\frac{a-b}{2}}^{\frac{b-a}{2}} \frac{1}{\sqrt{\left(\frac{b-a}{2}\right)^2 - y^2}}dy\\ &= -\sqrt{\left(\frac{b-a}{2}\right)^2 - y^2}\ \Bigg|_{x=\frac{a-b}{2}}^{x=\frac{b-a}{2}} + \frac{a+b}{2}\arcsin\frac{y}{\frac{b-a}{2}}\ \Bigg|_{x=\frac{a-b}{2}}^{x=\frac{b-a}{2}} = \cdots\end{aligned}$$

$e)$ vom lua $\lambda = 4/3$.
$f)$ vom lua $\lambda = 1/2$.
$g)$

$$\int \frac{dx}{1-x^2} = -\int \frac{dx}{x^2-1} = -\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.$$

Acum

$$\begin{aligned}
\int_1^\infty \frac{dx}{1-x^2} &= \int_1^2 \frac{dx}{1-x^2} + \int_2^\infty \frac{dx}{1-x^2} \\
&= (-)\frac{1}{2}\ln\left|\frac{2-1}{2+1}\right| - \lim_{\substack{a\to 1\\ a>1}} (-)\frac{1}{2}\ln\left|\frac{a-1}{a+1}\right| + \lim_{c\to\infty} (-)\frac{1}{2}\ln\left|\frac{c-1}{c+1}\right| - (-)\frac{1}{2}\ln\left|\frac{2-1}{2+1}\right| \\
&= \frac{1}{2}\lim_{\substack{a\to 1\\ a>1}} \ln\left|\frac{a-1}{a+1}\right| + \lim_{c\to\infty} (-)\frac{1}{2}\ln 1 = \cdots
\end{aligned}$$

$i)$ Calculăm mai întâi primitiva

$$\int \frac{5x+1}{x^3-x-6}dx = \int \frac{5x+1}{(x-2)(x^2+2x+3)}dx = \text{desc. în fracţii simple} = \cdots$$

Deci

$$\int_1^\infty \frac{5x+1}{x^3-x-6}dx = \cdots$$

10. Studiaţi convergenţa următoarelor integrale improprii:

$a)\ \int_0^\pi \frac{\sin x}{7+6\cos x-2\sin x}dx$, $\quad b)\ \int_0^{\pi/2} \frac{dx}{a^2\sin^2 x+b^2\cos x}$, $\quad c)\ \int_{-1}^1 \frac{dx}{(3-x)\sqrt{1-x^2}}$,

$d)\ \int_a^b \frac{x^2dx}{\sqrt{(x-a)(b-x)}}$, $\quad e)\ \int_a^b \sqrt{\frac{b-x}{x-a}}dx$, $\quad f)\ \int_0^{2\pi} \frac{dx}{2\sin x+3\cos x+4}$

Rezolvare:

$a)$ Subst. $\mathrm{tg}\frac{x}{2}=t$

$$I = \int_0^\infty \frac{4tdt}{(t^2+1)(t^2-4t+13)} = \cdots$$

$b)$ Subst. $\mathrm{tg}x=t$ şi formulele $\sin x = \frac{t}{\sqrt{1+t^2}}$ şi $\cos x = \frac{1}{\sqrt{1+t^2}}$. Deci

$$I = \int_0^\infty \frac{dt}{a^2t^2+b^2} = \cdots$$

$c)$ $x=\sin t$

$d)$ $\lambda = 1/2$ şi $\lambda = 1/2$

$e)$ Subst. $\sqrt{\frac{b-x}{x-a}} = t$

$$I = \int_0^\infty \frac{2(b-a)t^2}{(t^2+1)^2}dt = \cdots$$

$f)$ Subst. $\mathrm{tg}\frac{x}{2}=t$ dar dacă $x\in[0,2\pi] \Leftrightarrow \frac{x}{2}\in[0,\pi]$ iar tangenta nu este definită în $\pi/2$. Deci

$$\begin{aligned}
\int_0^{2\pi} \frac{dx}{2\sin x+3\cos x+4} &= \int_0^\pi \frac{dx}{2\sin x+3\cos x+4} + \int_\pi^{2\pi} \frac{dx}{2\sin x+3\cos x+4} \\
&= \int_0^\infty \frac{2dt}{t^2+4t+7} + \int_{-\infty}^0 \frac{2dt}{t^2+4t+7} = \cdots
\end{aligned}$$

Capitolul II. Extinderea noţiunii de integrală. Integrale cu parametru
(calcul de integrale folosind derivarea integralelor cu parametru)

Teoremă. Fie funcţia $f : [a,b] \times [c,d] \to \mathbb{R}$ continuă. Să notăm cu $I(y) \stackrel{def}{=} \int_a^b f(x,y)\,dx$ integrala cu parametru. Să mai presupunem că există derivata parţială $f'_y(x,y)$ şi că ea este continuă ca funcţie de x şi y. Atunci pentru orice $y \in [c,d]$ integrala cu parametru $I(y)$ este derivabilă şi are loc

$$I'(y) = \int_a^b f'_y(x,y)\,dx.$$

11. Calculaţi următoarea integrală cu parametru (derivând-o în prealabil)

$$I(y) = \int_0^1 \frac{\operatorname{arctg}(xy)}{x\sqrt{1-x^2}}dx$$

Rezolvare:

Derivăm

$$\begin{aligned} I'(y) &= \int_0^1 \left(\frac{\operatorname{arctg}(xy)}{x\sqrt{1-x^2}}\right)'_y dx = \int_0^1 \frac{1}{x\sqrt{1-x^2}}\left(\operatorname{arctg}(xy)\right)'_y dx = \int_0^1 \frac{1}{x\sqrt{1-x^2}}\frac{1}{1+x^2y^2}xdx \\ &= \int_0^1 \frac{1}{\sqrt{1-x^2}}\frac{1}{1+x^2y^2}dx \end{aligned}$$

Deoarece apare cantitatea $\sqrt{1-x^2}$ este utilă subsituţia $x = \sin t \Leftrightarrow t = \arcsin x$ şi $dx = \cos t dt$. Deci

$$I'(y) = \int_0^{\pi/2} \frac{1}{\sqrt{1-\sin^2 t}}\frac{1}{1+y^2\sin^2 t}\cos t dt = \int_0^{\pi/2} \frac{1}{1+y^2\sin^2 t}dt.$$

Acum se face subst. $\operatorname{tg}(t) = r \Leftrightarrow t = \operatorname{arctg}(r) \Rightarrow dt = \frac{1}{1+r^2}dr$. Deci, folosind formula $\sin t = \frac{r}{\sqrt{1+r^2}}$,

$$\begin{aligned} I'(y) &= \int_0^{+\infty} \frac{1}{1+y^2\frac{r^2}{1+r^2}}\frac{1}{1+r^2}dr = \int_0^{+\infty} \frac{1}{1+(y^2+1)r^2}dr = \frac{1}{y^2+1}\int_0^{+\infty} \frac{1}{\left(\frac{1}{\sqrt{y^2+1}}\right)^2 + r^2}dr \\ &= \frac{1}{y^2+1}\frac{1}{\frac{1}{\sqrt{y^2+1}}}\operatorname{arctg}\frac{r}{\frac{1}{\sqrt{y^2+1}}}\Bigg|_{r=0}^{r=\infty} = \frac{1}{\sqrt{y^2+1}}(\pi/2-0) = \frac{\pi}{2}\frac{1}{\sqrt{y^2+1}}. \end{aligned}$$

Deci

$$I'(y) = \frac{\pi}{2}\frac{1}{\sqrt{y^2+1}} \Rightarrow I(y) = \frac{\pi}{2}\ln\left(y+\sqrt{y^2+1}\right) + C$$

Dar

$$I(0) = 0 = C.$$

12. Calculaţi următoarea integrală cu parametru (derivând-o în prealabil)

$$I(a) = \int_0^{\pi/2} \ln\left(a^2 - \sin^2 x\right) dx \ , \ a > 1$$

Rezolvare:

Derivăm

$$I'(a) = \int_0^{\pi/2} \left(\ln\left(a^2 - \sin^2 x\right)\right)_a' dx = \int_0^{\pi/2} \frac{1}{a^2 - \sin^2 x} 2a dx = 2a \int_0^{\pi/2} \frac{dx}{a^2 - \sin^2 x}.$$

Se face subst. $\operatorname{tg}(x) = t \Leftrightarrow x = \operatorname{arctg}(t) \Rightarrow dx = \frac{1}{1+t^2} dt$. Deci, folosind formula $\sin x = \frac{t}{\sqrt{1+t^2}}$

$$\begin{aligned} I'(a) &= 2a \int_0^{\pi/2} \frac{dx}{a^2 - \sin^2 x} = 2a \int_0^{+\infty} \frac{1}{a^2 - \frac{t^2}{1+t^2}} \frac{1}{1+t^2} dt \\ &= 2a \int_0^{+\infty} \frac{1}{(a^2-1)t^2 + a^2} dt = 2a \frac{1}{a^2-1} \int_0^{+\infty} \frac{1}{t^2 + \left(\frac{a}{\sqrt{a^2-1}}\right)^2} dt \\ &= \frac{2a}{a^2-1} \frac{1}{\frac{a}{\sqrt{a^2-1}}} \operatorname{arctg} \frac{t}{\frac{a}{\sqrt{a^2-1}}} \Bigg|_{r=0}^{r=\infty} = \frac{2}{\sqrt{a^2-1}} (\pi/2 - 0) = \frac{\pi}{\sqrt{a^2-1}}. \end{aligned}$$

Deci

$$I'(a) = \frac{\pi}{\sqrt{a^2-1}} \Rightarrow I(a) = \pi \ln\left(a + \sqrt{a^2+1}\right) + C.$$

13. Să se calculeze integrala cu parametru folosind *substituţia universală*

$$I(a) = \int_0^{\pi} \frac{1}{a + b\cos x} dx \ , \ a, b > 0$$

Apoi derivând în raport cu a integrala cu parametru $I(a)$ să se calculeze

$$\int_0^{\pi} \frac{1}{(a + b\cos x)^2} dx \ , \ \int_0^{\pi} \frac{1}{(a + b\cos x)^3} dx$$

Rezolvare:

$$\begin{aligned} I(a) &= \int_0^{\pi} \frac{1}{a + b\cos x} dx = \int_0^{\infty} \frac{1}{a + b\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt = \frac{2}{a-b} \int_0^{\infty} \frac{1}{\left(\sqrt{\frac{a+b}{a-b}}\right)^2 + t^2} dt \\ &= \frac{2}{a-b} \frac{1}{\sqrt{\frac{a+b}{a-b}}} \operatorname{arctg} \frac{t}{\sqrt{\frac{a+b}{a-b}}} \Bigg|_{x=0}^{x=\infty} = \frac{2}{\sqrt{a^2-b^2}} \frac{\pi}{2}. \end{aligned}$$

Temă: calculaţi şi $\int_0^{2\pi} \frac{1}{a+b\cos x}dx = \int_0^{\pi} \frac{1}{a+b\cos x}dx + \int_{\pi}^{2\pi} \frac{1}{a+b\cos x}dx = \cdots = \frac{2\pi}{\sqrt{a^2-b^2}}$

Derivăm şi obţinem

$$I'(a) = \int_0^{\pi} \left(\frac{1}{a+b\cos x}\right)'_a dx = \int_0^{\pi} \frac{-1}{(a+b\cos x)^2}(1+0)\,dx = \int_0^{\pi} \frac{-1}{(a+b\cos x)^2}dx.$$

Pe de altă parte

$$I'(a) = \left(\frac{\pi}{\sqrt{a^2-b^2}}\right)'_a = \pi\left[\left(a^2-b^2\right)^{-1/2}\right]'_a = \pi\frac{-1}{2}\left(a^2-b^2\right)^{-3/2}2a = \cdots$$

14. Să se calculeze integrala cu parametru

$$I(a) = \int_0^{\infty} \frac{1}{a+x^2}dx$$

Apoi derivând în raport cu a integrala cu parametru $I(a)$ să se calculeze

$$\int_0^{\infty} \frac{1}{(a+x^2)^2}dx\ ,\ \int_0^{\infty} \frac{1}{(a+x^2)^3}dx$$

15. Să se calculeze integrala cu parametru

$$I(a) = \int_0^{c} \frac{1}{a+x^2}dx$$

Apoi derivând în raport cu a integrala cu parametru $I(a)$ să se calculeze

$$\int_0^{c} \frac{1}{(a+x^2)^2}dx\ ,\ \int_0^{c} \frac{1}{(a+x^2)^3}dx$$

16. Să se calculeze integrala

$$\int_0^{\infty} \frac{\sin x}{x}dx$$