Department of Computer Science, University of Verona
Degree in Applied Mathematics
Probability, Tuesday, July 19$^{\text{th}}$, 2016

Exercises

1. (a) A medical laboratory test ensures with a probability 0.95 the detection of a certain disease D, when the disease effectively exists. Meanwhile, the test also indicates a positive result for one percent of the people without the disease that have tried the test. We assume that 0.5 percent of the population has the disease D. We choose at random a person for which the test was positive. What is the probability that the person is ill ?

   (b) An aircraft engine fails with probability $1-p$ where $p \in (0,1)$, independently of the other engines. To complete its flight, the aircraft needs that the majority of its engines works. Calculate the probability to successfully complete the flight for an aircraft with 3 engines (i.e. at least 2 engines work).

2. Let $n, m \in \mathbb{N}^*$ and $p \in (0,1)$. Let the independent random variables $X \sim \textit{Binomial}\,(n,p)$, $Y \sim \textit{Binomial}\,(m,p)$, and the independent random variables X_i given by $\mathbb{P}\,(X_i = -1) = \mathbb{P}\,(X_i = 1) = 1/2$, with $i \in \mathbb{N}^*$.

   (a) Find the mean and the variance of X.

   (b) Find the type of the distribution of the random variable $X+Y$. Deduce the variance of $X+Y$.

   (c) Find the type of the distribution of the random variable $\bar{X}_k := \dfrac{k + \sum_{i=1}^{k} X_i}{2}$.

   Hint: the notation $X \sim \textit{Binomial}(n,p)$ means that X is Binomial distributed with parameters $n \in \mathbb{N}^*$ and $p \in (0,1)$, i.e. $\mathbb{P}(X=k) = C_n^k p^k q^{n-k}$, with $k = \overline{0,n}$, where $C_n^k = \frac{n!}{k!(n-k)!}$. It is useful also the expansion $(a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}$ and the relation $\sum_{i=0}^{k} C_n^i C_m^{k-i} = C_{n+m}^k$.

3. Let the function $F : \mathbb{R} \to \mathbb{R}$ be given by

$$F(x) = \begin{cases} 1+a, & \text{if } x < 0, \\ b, & \text{if } x \in [0,2), \\ c+2b & \text{if } x \in [2,3), \\ d & \text{if } x \geq 3. \end{cases}$$

   (a) Under which conditions on a, b, c, d function F is a cumulative distribution function?

   (b) If F is a cumulative distribution function associated to a random variable X, then compute $\mathbb{P}\,(X > 2.5)$ and $\mathbb{P}\,(-1 < X \leq 1.5)$, using F.

   (c) Find a, b, c, d if we know that $\mathbb{P}(X > 2.5) = 0.25$ and $\mathbb{P}(-1 < X \leq 1.5) = 0.25$.

   (d) Find the law (the table) of X.