Department of Computer Science, University of Verona
Degree in Applied Mathematics
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Exercises

1. (a) If n students are in a classroom, what is the probability that at least two of them celebrate their birthday on the same day of the year? (we suppose that the year has 365 days)
   Write the associated probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
   (b) Suppose that we have an unbalanced coin which comes up heads with probability $p \in (0,1)$. Find the distribution of the random variable X whose values are the total number of the failures needed to obtain the first head. Compute the mean and the variance of X.

   Solution:

   (a) Let A be the following event

   $$A = \{\text{at least two students celebrate their birthday on the same day of the year}\}$$

   and therefore

   $$\bar{A} = \{\text{all the } n \text{ students celebrate their birthday in different days of the year}\}.$$

   If $n > 365$, then $\mathbb{P}(A) = 1$.
   Let now $n \leq 365$. We compute $\mathbb{P}\left(\bar{A}\right) = 1 - \mathbb{P}(A)$.
   Solution 1. We can define

   $$\Omega = \left\{(P1, P2, \dots, Pn) : Pi \in \{1, 2, \dots, 365\},\ i = \overline{1, n}\right\}$$

   and therefore $\operatorname{card}(\Omega) = 365^n$ and $\operatorname{card}\left(\bar{A}\right) = \dfrac{365!}{(365-n)!} = (365-n+1) \cdot (365-n+2) \cdot \ldots \cdot 365 \quad (= A_{365}^n)$.
   Hence $\mathbb{P}\left(\bar{A}\right) = \dfrac{\operatorname{card}\left(\bar{A}\right)}{\operatorname{card}(\Omega)} = \dfrac{A_{365}^n}{365^n}$.

   Solution 2. We can define

   $$\Omega = \{f : \{P1, P2, \dots, Pn\} \to \{1, 2, \dots, 365\} : f \text{ is a function}\}$$

   and therefore

   $$\bar{A} = \{f : \{P1, P2, \dots, Pn\} \to \{1, 2, \dots, 365\} : f \text{ is an injective function}\}$$

   and $\operatorname{card}\left(\bar{A}\right) = A_{365}^n = \frac{365!}{(365-n)!}$ (the total number of injective functions between two sets) and $\mathbb{P}\left(\bar{A}\right) = \dfrac{\operatorname{card}\left(\bar{A}\right)}{\operatorname{card}(\Omega)} = \dfrac{A_{365}^n}{365^n}$.

   (b) **It should be proved (see the proof made during my course)** that the r.v. X has the distribution $\mathbb{P}(X = k) = p \cdot q^k$, with $k \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}$ (hence X follows a geometric distribution type).

   Then, using that $\sum_{k=0}^{\infty} q^k = \dfrac{1}{1-q}$, we can deduce $\mathbb{E}(X) = \dfrac{1}{p}$ and $\operatorname{Var}(X) = \dfrac{q}{p^2}$.

2. Let $X \sim \mathcal{P}(\lambda)$, where $\lambda > 0$, and a sequence of independent random variables $(X_k)_{k\in\mathbb{N}^*}$ such that $X_k \sim \mathcal{P}(1)$.

(a) Find the mean and the variance of X.

(b) Find the type of the distribution of the random variable $\bar{X}_n := \frac{1}{n}\sum_{k=1}^{n} X_k$.

Hint: the notation $X \sim \mathcal{P}(\lambda)$ means that X is Poisson distributed with parameter λ, i.e. $\mathbb{P}(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}$.

Solution:

(a) After some computations we obtain, using that $\sum_{k=0}^{\infty} \dfrac{x^k}{k!} = e^x$, the results $\mathbb{E}(X) = \text{Var}(X) = \lambda$.

Also these results can be obtained more easily computinf first the *Characteristic Function* φ_X .

(b) First it should be proved that if $X_1 \sim \mathcal{P}(\lambda_1)$ and $X_2 \sim \mathcal{P}(\lambda_2)$ and X_1, X_2 are independent, then $X_1 + X_2 \sim \mathcal{P}(\lambda_1 + \lambda_2)$ (to see this we can compute directly $\mathbb{P}(X_1 + X_2 = k)$ or we can use the *Characteristic Function*).

Hence $\sum_{k=1}^{n} X_k \sim \mathcal{P}(n)$.

Afterthat, it should be proved that

$$\bar{X}_n : \begin{pmatrix} \dfrac{k}{n} \\ \dfrac{\lambda^k}{k!}e^{-\lambda} \end{pmatrix}_{k\in\mathbb{N}}$$

3. Let us consider a random vector (X, Y) whose distribution is described by the next table:

$X \backslash Y$	2	4
1	a	0.1
2	0.1	0.3
3	a	$3a$

(a) Find $a \in \mathbb{R}$ such that the above table is associated to a bidimensional random vector.

(b) Write the marginal distributions.

(c) Compute the probabilities $\mathbb{P}(X \geq 2,\, Y \geq 3)$, $\mathbb{P}(Y \leq 3 |\, X \geq 2)$ and $F_{(X,Y)}(2, 3)$ (where $F_{(X,Y)}$ is the cumulative distribution function associated to (X, Y)).

(d) Obtain (the table of) the random variable $W = 2X - 0.5Y + 1$.

(e) Check if the random variables X and Y are independente.

Hint: the notation $\mathbb{P}(A|\, B)$ stands for conditional probability.

Solution:

(a) $a = 0.1$.

(b) $X : \begin{pmatrix} 1 & 2 & 3 \\ 0.2 & 0.4 & 0.4 \end{pmatrix}$ and $Y : \begin{pmatrix} 2 & 4 \\ 0.3 & 0.7 \end{pmatrix}$.

(c) $F_{(X,Y)}(2,\,3) := \mathbb{P}\left(X \leq 2, Y \leq 3\right).$

(d) Since

$$2X = \begin{pmatrix} 2 & 4 & 6 \\ 0.2 & 0.4 & 0.4 \end{pmatrix} \quad \text{and} \quad \frac{1}{2}Y : \begin{pmatrix} 1 & 2 \\ 0.3 & 0.7 \end{pmatrix},$$

we obtain, using the operations with discrete r.v., $W = 2X - \dfrac{1}{2}Y + 1 : \begin{pmatrix} 2 & 4 & 6 \\ 0.2 & 0.4 & 0.4 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0.3 & 0.7 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 4 & 3 & 6 & 5 \\ p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{pmatrix}.$

After that, we compute, for instance

$$\begin{aligned} p_1 &= \mathbb{P}\left(W = 2\right) = \mathbb{P}\left(X = 1, Y = 2\right) = 0.1, \\ p_2 &= \mathbb{P}\left(W = 1\right) = \mathbb{P}\left(X = 1, Y = 4\right) = 0.1 \quad \text{and so on.} \end{aligned}$$

(e) $\mathbb{P}\left(X = a, Y = b\right) \neq \mathbb{P}\left(X = a\right)\mathbb{P}\left(Y = b\right)$ for some $a \in X\left(\Omega\right)$ and $b \in Y\left(\Omega\right).$