Department of Computer Science, University of Verona
Degree in Applied Mathematics
Probability, Monday, September 26^{th}, 2016

Exercises

1. (a) Jim takes an oral exam in *Probability* by answering to the questions written on an examination card. There are 20 such examination cards and Jim will receive one of them drawn at random. Of the 20 there are 8 favorable cards. Jim will get an A if he answers to the questions on the card correctly. What is the probability that Jim gets an A if he draws first a card ? But if is the second who draw a card ? But if he is the third ?

   (b) Starting at a fixed time, we observe the gender of each newborn child at a certain hospital until a boy is born. We suppose that the probability to born a boy is p. Assume that the successive births are independent. Find the distribution of the random variable X whose values are the total number of the births observed.

   Also find the cumulative distribution function associated to X and compute the probability of having to examine at most five births to see the first boy.

   Compute the mean $\mathbb{E}\left(\frac{1}{1+X}\right)$.

   Solution:

   (a) We have

   $\mathbb{P}(A_1) = \frac{8}{20}$ **(0.5 points)**,

   $\mathbb{P}(A_2) = \frac{8}{20} \cdot \frac{7}{19} + \frac{12}{20} \cdot \frac{8}{19} = \frac{8}{20}$ **(1 points)**,

   $\mathbb{P}(A_3) = \frac{8}{20} \cdot \frac{7}{19} \cdot \frac{6}{18} + \frac{8}{20} \cdot \frac{12}{19} \cdot \frac{7}{18} + \frac{12}{20} \cdot \frac{11}{19} \cdot \frac{8}{18} + \frac{12}{20} \cdot \frac{8}{19} \cdot \frac{7}{18} = \frac{8}{20}$ **(1 points)**.

   Hence $\mathbb{P}(A_1) = \mathbb{P}(A_2) = \mathbb{P}(A_3) = \frac{8}{20}$.

   (b) We have

   $\mathbb{P}(X = 1) = \mathbb{P}(1st - B) = p$

   $\mathbb{P}(X = 2) = \mathbb{P}(\{1st - G, 2nd - B\}) = \mathbb{P}(\{1st - G\} \cap \{2nd - B\}) = \mathbb{P}(1st - G) \cdot \mathbb{P}(2nd - B) = q \cdot p$,

   $\mathbb{P}(X = 3) = \mathbb{P}(\{1st - G, 2nd - G, 3rd - B\}) = \ldots = q \cdot q \cdot p$

   an so on.

   We obtain $\mathbb{P}(X = k) = pq^{k-1}$**,** $k \in \mathbb{N}^*$ **(2 points)**, hence $X \sim \mathcal{G}(p)$.

   Then $F_X(x) = \mathbb{P}(X \leq x)$, with $x \in \mathbb{R}$.

   For $n \in \mathbb{N}^*$, $F_X(n) = \mathbb{P}(X \leq n) = \ldots = 1 - q^n$ **(1 points)**.

And $\mathbb{P}(X \le 5) = F_X(5) = p + pq + pq^2 + pq^3 + pq^4 = p\left(1 + q + q^2 + q^3 + q^4\right) = p\dfrac{1-q^5}{1-q} = 1 - q^5$ **(0.5 points)**.

We compute $\mathbb{E}\left(\dfrac{1}{1+X}\right) = \sum_{k=1}^{\infty} \dfrac{1}{1+k} pq^{k-1} = pq^2 \sum_{k=1}^{\infty} \dfrac{1}{1+k} q^{k+1}$.

But $\sum_{k=0}^{\infty} q^k = \dfrac{1}{1-q}$, hence $\sum_{k=0}^{\infty} \dfrac{q^{k+1}}{k+1} = -\ln|1-q|$.

Therefore $\mathbb{E}\left(\dfrac{1}{1+X}\right) = pq^2\left(-\ln|1-q| - \frac{q^2}{2}\right)$ **(1 points)**.

2. Let $p \in (0,1)$. Let the independent random variables X_i given by $\mathbb{P}(X_i = 0) = q$ and $\mathbb{P}(X_i = 2) = p$, with $i \in \mathbb{N}^*$.

   (a) Find the mean and the variance of X_i .

   (b) Find the type of the distribution of the r.v. $X_i/2$ and compute the variance of this r.v..

   (c) Find the type of the distribution of the r.v. $\bar{X}_k := \dfrac{\sum_{i=1}^{k} X_i}{2}$.

   Solution:

   (a) We have $\mathbb{E}(X_i) = 2p, \quad \mathbb{E}\left(X_i^2\right) = 4p, \quad \mathrm{Var}(X_i) = 4pq$ **(2 points)**.

   (b) We have $X_i/2 : \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}$, hence $X_i/2 \sim Bernoulli(p) = Binomial(1,p)$ and $\mathrm{Var}(X_i/2) = \mathrm{Var}(X_i)/4 = pq$ **(2 points)**.

   (c) It should be proved (see the course) that

   $$\text{“If } X, Y \sim Binomial(1,p), \text{ then } X + Y \sim Binomial(2,p)\text{”},$$

   or, more generally,

   $$\text{“If } X \sim Binomial(n,p) \text{ and } Y \sim Binomial(m,p), \text{ then } X + Y \sim Binomial(n+m,p)\text{”}.$$

   Therefore we obtain that $\bar{X}_k := \dfrac{\sum_{i=1}^{k} X_i}{2} = \sum_{i=1}^{k} \dfrac{X_i}{2} \sim Binomial(k,p)$ **(3 points)**.

3. Let X and Y be two random variables with the distributions

   $$\mathbb{P}(X = -3) = a, \quad \mathbb{P}(X = -2) = 5/32, \quad \mathbb{P}(X = -1) = b, \quad \mathbb{P}(X = 0) = 5/16$$

   and

   $$\mathbb{P}(X = 1) = c, \quad \mathbb{P}(X = 2) = 1/32.$$

   We know that

   $$\mathbb{E}(X) = -1/2, \quad \mathrm{Var}(X) = 5/4.$$

   (a) Find a, b, c.

   (b) Compute $\mathbb{E}(3 + 2X)$, $\mathbb{E}\left[(3 + 2X)^2\right]$, $\mathrm{Var}(3 + 2X)$.

   (c) Find the law (the table) of the r.v. $Y := X^2$ and $Z := |X|$.

Solution:

(a) We have $\begin{cases} a+b+c=\dfrac{1}{2} \\ -3a-b+c=\dfrac{-1}{4} \\ 9a+b+c=\dfrac{3}{4} \end{cases}$ and therefore $a=1/32, b=10/32, c=5/32$ **(1 points)**.

(b) We can find the table of $2X+3$: $\begin{pmatrix} -3 & -1 & 1 & 3 & 5 & 7 \\ 1/32 & 5/32 & 10/32 & 10/32 & 5/32 & 1/32 \end{pmatrix}$ and $(2X+3)^2$: $\begin{pmatrix} 1 & 9 & 25 & 49 \\ 15/32 & 11/32 & 5/32 & 1/32 \end{pmatrix}$ Therefore $\mathbb{E}(2X+3)=2$, $\mathbb{E}\left[(2X+3)^2\right]=9$ and $\mathrm{Var}(2X+3)=5$ **(3 points)**.

Or
$\mathbb{E}(2X+3)=2\mathbb{E}(X)+3=2\frac{-1}{2}+3=2,\quad \mathbb{E}\left[(2X+3)^2\right]=\mathbb{E}\left[4X^2+12X+9\right]=4\mathbb{E}\left[X^2\right]+12\mathbb{E}[X]+9=\ldots=9$ and $\mathrm{Var}(2X+3)=5$.

(c) We have X^2 : $\begin{pmatrix} 9 & 4 & 1 & 0 & 1 & 4 \\ 1/32 & 5/32 & 10/32 & 10/32 & 5/32 & 1/32 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 4 & 9 \\ 10/32 & 15/32 & 6/32 & 1/32 \end{pmatrix}$

and $|X|$: $\begin{pmatrix} 3 & 2 & 1 & 0 & 1 & 2 \\ 1/32 & 5/32 & 10/32 & 10/32 & 5/32 & 1/32 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 10/32 & 15/32 & 6/32 & 1/32 \end{pmatrix}$,

since, for instance,

$\mathbb{P}\left(X^2=9\right)=\mathbb{P}(X=-3)=1/32$ and

$\mathbb{P}\left(X^2=4\right)=\mathbb{P}(\{X=-2\}\cup\{X=2\})=\mathbb{P}(X=-2)+\mathbb{P}(X=2)=6/32$ and so on **(3 points)**.