MATRICES

1 If $A = \begin{pmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, find out which of the products AB and BA are well-defined and determine that product explicitly.

2
1. Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. Find AB and discuss the result.
2. Is it necessarily true that if $A, B \in M_n(\mathbb{C})$, $AB = O_n$, then necessarily $A = O_n$ or $B = O_n$?

3 Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Find A^n, $n \in \mathbb{N}$. Can you state a general rule about the powers of a diagonal matrix?

4 Explain why the equalities

$$(A+B)^2 = A^2 + 2AB + B^2, \quad (A+B)(A-B) = A^2 - B^2$$

(which are true when A, B are complex numbers) do not necessarily remain valid for $A, B \in M_n(\mathbb{C})$. Do they remain valid when $AB = BA$?

5 If $A = \begin{pmatrix} 4 & 9 \\ -1 & -2 \end{pmatrix}$, find A^n, $n \geq 2$.

Hint: Write $A = I_2 + B$, with $B^2 = O_2$. Use the binomial formula.

6 If $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, find A^n, $n \geq 3$.

Hint: Write $A = I_3 + B$, with $B^3 = O_3$. Use the binomial formula.

7 If $A = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$ and $P \in \mathbb{R}[X]$, prove that

$$P(A) = \begin{pmatrix} P(a) & P'(a) & \frac{1}{2}P''(a) \\ 0 & P(a) & P'(a) \\ 0 & 0 & P(a) \end{pmatrix}.$$

Hint: Write $A = aI_3 + B$, with $B^3 = O_3$. Use the binomial formula to compute the powers of A.

8 Let $R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, $\alpha \in \mathbb{R}$.

1. Prove that $R(\alpha)R(\beta) = R(\alpha+\beta)$, $\forall \alpha, \beta \in \mathbb{R}$.
2. Find $R(\alpha)^n$, $n \geq 2$.

9 For $A \in M_n(\mathbb{R})$, let us denote by $\operatorname{Tr} A$ (trace of A) the sum of the elements on the main diagonal of A.

1. Prove that $\operatorname{Tr}(A+B) = \operatorname{Tr} A + \operatorname{Tr} B$ and $\operatorname{Tr}(aA) = a \operatorname{Tr} A$, for any $a \in \mathbb{R}$ and any $A, B \in M_n(\mathbb{R})$.
2. Let $M = \{X \in M_n(\mathbb{R}); \operatorname{Tr} A = 0\}$. Prove that the identity matrix I_n cannot be written as a sum of matrices belonging to M.

Hint: How much is the trace of the sum?

3. If $\operatorname{Tr}(AA^t) = 0$, prove that $A = O_n$.

DETERMINANTS

10 Find $D = \begin{vmatrix} 1 & \varepsilon & \varepsilon^2 \\ \varepsilon & \varepsilon^2 & 1 \\ a & b & c \end{vmatrix}$, where $a, b, c, \varepsilon \in \mathbb{C}$, $\varepsilon^3 = 1$, $\varepsilon \neq 1$.

11 Find $D = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$, provided that

1. $a_1, a_2, \ldots, a_9$ is an arithmetic progression;
2. $a_1, a_2, \ldots, a_9$ is a geometric progression.

12 Find $D = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \\ x_3 & x_1 & x_2 \end{vmatrix}$, where x_1, x_2, x_3 are the roots of the equation $x^3 + px + q = 0$.

13 Find $D = \begin{vmatrix} a+b & b+c & c+a \\ a^2+b^2 & b^2+c^2 & c^2+a^2 \\ a^3+b^3 & b^3+c^3 & c^3+a^3 \end{vmatrix}$, where $a, b, c \in \mathbb{C}$.

Hint: Use elementary column operations.

14 Let $A = \begin{pmatrix} x & y & z \\ y & z & x \\ z & x & y \end{pmatrix} \in M_3(\mathbb{C})$. By computing $\det A$ in two distinct ways, prove that

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz).$$

15 Find $D = \begin{vmatrix} x & a & b & c \\ a & x & b & c \\ a & b & x & c \\ a & b & c & x \end{vmatrix}$, where $a, b, c, x \in \mathbb{C}$..

Hint: Use elementary row operations first.

16 Find $D = \begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix}$, where $a, b, c \in \mathbb{C}$.

Hint: Expand the squares and use elementary column operations.

17 Find $D = \begin{vmatrix} a & b & -a & -b \\ -b & a & b & -a \\ c & d & c & d \\ -d & c & -d & -c \end{vmatrix}$, where $a, b, c, d \in \mathbb{C}$.

Hint: Use Laplace rule (expand over the first two rows).

18 Find $V(a, b, c) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$, where $a, b, c \in \mathbb{C}$..

19 Find $V(a_1, a_2, \ldots, a_n) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ \ldots & \ldots & \ldots & \ldots \\ a_1^{n-1} & a_2^{n-1} & \ldots & a_n^{n-1} \end{vmatrix}$.

20 If $A, B \in M_2(\mathbb{C})$, prove that

1. $\det(A + B) + \det(A - B) = 2(\det(A) + \det(B))$;
2. $\det(AB) = \det(A)\det(B)$.

Hint: Usually, $\det(A + B) \neq \det A + \det B$!

21 Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$. Prove that

1. The following equality holds

$$A^2 - (a+d)A + (ad-bc)I_2 = O_2.$$

2. If $\det A = 0$, then $A^n = (a+d)^{n-1}A$.
3. If there is $n \in \mathbb{N}^*$, $n \geq 2$, such that $A^n = O_2$, then $A^2 = O_2$.

22 Let $A \in M_2(\mathbb{R})$ such that $A^2 = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$.

1. Prove that $\det A \in \{-1, 1\}$.
2. Find all matrices A that satisfy the above equality.

23 Let $A \in M_m(\mathbb{C})$, $B \in M_{m,n}(\mathbb{C})$, $C \in M_{n,m}(\mathbb{C})$, $D \in M_n(\mathbb{C})$. Prove that

$$\det \begin{pmatrix} A & B \\ O & D \end{pmatrix} = \det \begin{pmatrix} A & O \\ C & D \end{pmatrix} = \det A \cdot \det D.$$

Hint: Use Laplace Rule (expand over the first m rows).

24 Let $A = \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}$, $a \in \mathbb{R}$. Prove that $A^n \neq O_3$, for any $n \in \mathbb{N}^*$.

Hint: If $A^n = O_3$, how much $\det A$ should be? Find then the corresponding values of a. Discuss each case.