THE RANK OF A MATRIX

1 Find the rank of the following matrices

$$1.\ A = \begin{pmatrix} 2 & -1 & 3 & -2 & 4 \\ 4 & -2 & 5 & 1 & 7 \\ 2 & -1 & 1 & 8 & 2 \end{pmatrix}, \quad 2.\ A = \begin{pmatrix} 2 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 2 \\ 3 & 3 & -3 & -3 & 4 \end{pmatrix}.$$

2 Find $a \in \mathbb{R}$ so that the matrix $A = \begin{pmatrix} a+1 & 3 & 1 & 2 \\ -1 & 1 & -1 & 1 \\ a-2 & -2 & 2 & -2 \end{pmatrix}$ has rank 2.

3 Discuss the rank of the matrix $A = \begin{pmatrix} 1+a & a & a \\ a & 1+a & a \\ a & a & 1+a \end{pmatrix}$ with respect to the values of $a \in \mathbb{R}$.

4 Find $a \in \mathbb{R}$ so that the matrix $A = \begin{pmatrix} a-1 & 1 & a & a-2 \\ 1 & 2a & -1 & a+3 \\ -a+3 & 4a-1 & -a-2 & a+8 \end{pmatrix}$ has maximal rank.

Hint: Prove first that $\operatorname{rank}(A) \geq 2$.

5 Let $A \in M_2(\mathbb{C})$ be such that $\operatorname{rank}(A) = \operatorname{rank}(A^2)$. Prove that $\operatorname{rank}(A) = \operatorname{rank}(A^n)$ for all $n \in \mathbb{N}^*$.

Hint: Analyze the cases $\operatorname{rank} A = 2$, $\operatorname{rank} A = 1$ and $\operatorname{rank} A = 0$ separately.

6 Let $A, B \in M_n(\mathbb{C})$ such that $\operatorname{rank}(AB) = \operatorname{rank}((AB)^2)$, $\operatorname{rank}(BA) = \operatorname{rank}((BA)^2)$. Prove that $\operatorname{rank}(AB) = \operatorname{rank}(BA)$.

Hint: $\operatorname{rank}(AB) = \operatorname{rank}((AB)^2) = \operatorname{rank}(ABAB) = \operatorname{rank}(A \cdot BA \cdot B) \leq ...$

7 Let $A \in M_{3,2}(\mathbb{C})$ and $B \in M_{2,3}(\mathbb{C})$ such that $AB = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. Prove that $\operatorname{rank}(BA) = 2$.

Hint: $\operatorname{rank}((AB)^2) = 2$. See then above.

8 If $A \in M_{n+1,n}(\mathbb{C})$, then $\det(AA^t) = 0$.

Hint: $\operatorname{rank}(AA^t) \leq \min(\operatorname{rank} A, \operatorname{rank} A^t) \leq n$.

THE INVERSE OF A MATRIX

9 Prove that the matrix $A = \begin{pmatrix} 1 & -a & b \\ a & 1 & -c \\ -b & c & 1 \end{pmatrix}$ is invertible for any $a, b, c \in \mathbb{R}$ and find its inverse.

10 Find the inverse of the following matrices
1. $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix}$, 2. $A = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 1 & -3 \\ 1 & 5 & 5 \end{pmatrix}$, 3. $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 2 \end{pmatrix}$, 4. $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$.

11 Prove that $A = \begin{pmatrix} 2 & 2 & 3 & 1 \\ 1 & -2 & -3 & 2 \\ 4 & -2 & -3 & 5 \\ 1 & 3 & 1 & 1 \end{pmatrix}$ is not invertible.

Hint: Use Gauss-Jordan elimination.

12 Find the inverse for the matrices $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$. Use

1. Gauss-Jordan elimination
2. block writing

Hint: For $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, one sees that $C^2 = I_2$. Also, $A = \begin{pmatrix} C & O_2 \\ O_2 & C \end{pmatrix}$ and $B = \begin{pmatrix} 2I_2 & C \\ C & 2I_2 \end{pmatrix}$.

Hint: Let $D = \begin{pmatrix} 2I_2 & -C \\ -C & 2I_2 \end{pmatrix}$. Find A^2 and BD.

13 Let $A \in M_m(\mathbb{C})$, A invertible, $B \in M_{m,n}(\mathbb{C})$, $C \in M_{n,m}(\mathbb{C})$, $D \in M_{n,n}(\mathbb{C})$. Prove that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \cdot \det(D - CA^{-1}B)$$

Hint: Perform elementary row operations on the block form.

14 Let $A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{pmatrix}$.

1. Prove that $A^2 + 15I_3 = 16A$.
2. From the above equality, prove that A is invertible and find A^{-1}.

15 Let $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $B = A - I_4$.

1. Prove that $(I_4 - A)(I_4 + A + A^2 + A^3) = I_4$.
2. Prove that B is invertible and find B^{-1}.