CONVERGENCE THEOREMS FOR NONLINEAR SEMIGROUPS ASSOCIATED TO SEMILINEAR EVOLUTION EQUATIONS UNDER EQUICONTINUITY PROPERTIES

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Abstract. Given a semilinear problem of the form

$$\text{(SP)} \qquad u'(t) = (A+B)u(t), t > 0; \qquad u(0) = x \in D \subset X,$$

where X is a real Banach space and B is its nonlinear perturbation, whose continuity is localized using a lower semicontinuous functional φ, a sequence of approximating semilinear problems is formulated and the convergence of the associated approximating nonlinear semigroups is treated under appropriate consistency and stability conditions. Under these assumptions, it is shown that the convergence is achieved if and only if the family of approximating semigroups satisfies a certain equicontinuity property. A feature of our argument is that no convexity properties are assumed on the set D or on the functional φ.

1 Introduction and main result

Let X be a real Banach space with norm $|\cdot|$. We define the semi-inner products $[\cdot,\cdot]_-$ and $[\cdot,\cdot]_+$ on X by $[x,y]_- = \lim_{h\uparrow 0}\left(|x+hy| - |x|\right)/h$, respectively by $[x,y]_+ = \lim_{h\downarrow 0}\left(|x+hy| - |x|\right)/h$. Let D be a subset of X and let $\varphi : D(\varphi) \subset X \to [0,\infty)$ be a lower semicontinuous functional on X such that $D \subset D(\varphi) = \{x \in X; \varphi(x) < \infty\}$. We denote by $D_\alpha = \{x \in D; \varphi(x) \leq \alpha\}$ a generic level set of D with respect to φ.

[1]Research supported by a Postdoctoral Fellowship for Foreign Researchers from the Japan Society for the Promotion of Science.

2000 Mathematics Subject Classification: 37L05, 47H20.

Key words and phrases: Convergence theorems, semilinear evolution equations, nonlinear semigroups, discrete schemes.

A one-parameter family $S = \{S(t); t \geq 0\}$ of (possibly nonlinear) operators from D into itself is called a (nonlinear) semigroup on D if it satisfies the following conditions.

(S1) $S(t)S(s)x = S(t+s)x$, $S(0)x = x$ for all $t, s \geq 0$ and $x \in D$.

(S2) $S(\cdot)x \in C([0,\infty); X)$ for all $x \in D$.

If, in addition to (S1) and (S2), the semigroup S satisfies the following hypothesis

(S3) For all $\alpha \geq 0$ and $\tau > 0$ there is $\omega = \omega(\alpha, \tau) \in \mathbb{R}$ such that

$$|S(t)x - S(t)y| \leq e^{\omega t}|x - y|$$

for $t \in [0, \tau]$ and $x, y \in D_\alpha$,

then S is said to be locally Lipschitzian on D.

We consider the semilinear problem

$$\text{(SP)} \qquad u'(t) = (A+B)u(t), t > 0; \qquad u(0) = x \in D \subset X.$$

It is assumed that the operators A and B satisfy the following hypotheses.

(A) $A : D(A) \subset X \to X$ is linear, closed and its resolvent $(I - \lambda A)^{-1}$ exists for all $\lambda \geq 0$, satisfying $\|(I - \lambda A)^{-1}\| \leq 1$.

(B) $D \subset \overline{D(A)}$, D_α is closed in X, $B : D \to X$ is nonlinear and continuous from D_α into X for all $\alpha \geq 0$.

Let us denote $Y = \overline{D(A)}$. Condition (A) implies that the part A_Y of A in Y generates a contraction semigroup $T_Y = \{T_Y(t); t \geq 0\}$ on Y. Condition (B) requires the closedness of the level sets of D rather than the closedness of D and also localizes the continuity of the operator B. Note also that we do not impose any convexity assumptions on the set D or on the functional φ.

Since the semilinear problem (SP) may not necessarily admit strong solutions, we need to define a generalized notion of solution.

A continuous function $u : [0, \infty) \to D$ is then said to be a mild solution to (SP) if $Bu(\cdot) \in C([0,\infty); X)$ and $u(\cdot)$ satisfies the equation

$$u(t) = T_Y(t)x + \lim_{\lambda \downarrow 0} \int_0^t T_Y(t-s)(I - \lambda A)^{-1} Bu(s)ds$$

for all $t \geq 0$ and $x \in D$.

It is then assumed that the semilinear problem (SP) is well-posed in the sense of semigroups, that is, there exists a nonlinear semigroup $S = \{S(t); t \geq 0\}$ on D satisfying

$$\text{(MS)} \qquad S(t)x = T_Y(t)x + \lim_{\lambda \downarrow 0} \int_0^t T_Y(t-s)(I - \lambda A)^{-1} BS(s)x ds$$

and

$$\text{(GC)} \qquad \varphi(S(t)x) \leq e^{at}(\varphi(x) + bt)$$

for all $t \geq 0$ and $x \in D$, where $a, b \geq 0$. In other words, the nonlinear semigroup S provides mild solutions to (SP) in the sense mentioned above and satisfies an exponential growth condition.

Regarding the existence of the semigroup S which satisfies conditions (MS) and (GC), the following result has been established by Matsumoto and Shitaoka ([5, Theorem 3.1]) as an extension of a previous result obtained by Georgescu and Oharu ([2, Theorem 1]) to the case in which the linear and closed operator A is not necessarily densely defined.

Theorem 1.1. *Let $a, b \geq 0$. Assume that A and B satisfy hypotheses* (A) *and* (B). *Then the following statements are equivalent.*

(I) *There exists a nonlinear semigroup $S = \{S(t); t \geq 0\}$ on D satisfying*

(I.a) *For $t \geq 0$ and $x \in D$,*

$$S(t)x = T_Y(t)x + \lim_{\lambda \downarrow 0} \int_0^t T_Y(t-s)(I - \lambda A)^{-1} BS(s)x ds.$$

(I.b) *For $t \geq 0$ and $x \in D$,*

$$\varphi(S(t)x) \leq e^{at}(\varphi(x) + bt).$$

(I.c) *For each $\alpha \geq 0$ and $\tau > 0$ there exists $w = w(\alpha, \tau) \in \mathbb{R}$ such that*

$$|S(t)x - S(t)y| \leq e^{wt} |x - y|$$

for all $x, y \in D_\alpha$ and $t \in [0, \tau]$.

(II) *For all $x \in D$ there are a null sequence of positive numbers $\{h_n\}_{n \geq 1}$ and a sequence $\{x_n\}_{n \geq 1}$ in D such that*

(II.a) $\lim_{n \to \infty} [(1/h_n) |T_Y(h_n)x + W(h_n)Bx - x_n|] = 0.$

(II.b) $\overline{\lim\limits_{n\to\infty}}\,[(1/h_n)\,|\varphi(x_n)-\varphi(x)|]\le a\varphi(x)+b.$

(II.c) *For all* $\alpha\ge 0$ *there is* $w_\alpha\in\mathbb{R}$ *such that*

$$\underline{\lim_{h\downarrow 0}}\,[(1/h)\,(|T_Y(h)(x-y)+W(h)(Bx-By)|-|x-y|)]\le w_\alpha\,|x-y|$$

for all $x,y\in D_\alpha$.

Hence there is a locally Lipschitzian semigroup S providing mild solutions to (SP) and satisfying an exponential growth condition if and only if a sequential version of Pavel's subtangential condition is satisfied, together with a semilinear stability condition of the type introduced by Iwamiya, Oharu and Takahashi in [6], but localized with respect to the lower semicontinuous functional φ. Here $W=\{W(t);t\ge 0\}$ is the integrated semigroup generated by A on X; see the next section for details.

We now consider a sequence of approximate problems (SP;n) defined by

$$\text{(SP;n)}\qquad u_n'(t)=(A_n+B_n)u_n(t),t>0;\qquad u_n(0)=x\in D_n\subset X,$$

where $\varphi_n:D(\varphi_n)\subset X\to[0,\infty)$ are proper lower semicontinuous functionals such that $D_n\subset D(\varphi_n)$. We also assume that the operators A_n and B_n, $n\ge 1$, satisfy the following conditions.

(A;n) $A_n:D(A_n)\subset X\to X$ is linear, closed and its resolvent $(I-\lambda A_n)^{-1}$ exists for all $\lambda\ge 0$, satisfying $\|(I-\lambda A_n)^{-1}\|\le 1$.

(B;n) $\overline{D(A_n)}=Y$, $D_n\subset Y$, $D_{n,\alpha}$ is closed in X and $B_n:D_n\to X$ is nonlinear and continuous from $D_{n,\alpha}$ into X for all $\alpha\ge 0$.

Conditions (A;n) imply that the parts $A_{n,Y}$ of A_n in Y generate C_0-semigroups on Y, denoted by $T_{n,Y}$. We also assume that the semilinear problems (SP;n) are well-posed in the sense of semigroups, that is, for all $n\ge 1$ there exists a nonlinear semigroup $S_n=\{S_n(t);n\ge 1\}$ on D_n satisfying

$$\text{(MS;n)}\qquad S_n(t)x=T_{n,Y}(t)x+\lim_{\lambda\downarrow 0}\int_0^t T_{n,Y}(t-s)(I-\lambda A_n)^{-1}B_nS_n(s)x\,ds$$

and

$$\text{(GC;n)}\qquad \varphi_n(S_n(t)x)\le e^{at}(\varphi_n(x)+bt),$$

for all $t\ge 0$ and $x\in D_n$.

In what follows, given a sequence $\{x_n\}_{n\ge 1}$ such that $x_n\in D_n$ for all $n\ge 1$, we say that $\{x_n\}_{n\ge 1}$ is $\{\varphi_n\}$-bounded if $\sup_{n\ge 1}\varphi_n(x_n)<\infty$. We also assume the following consistency conditions on the operators A_n, B_n and on the sets D_n, $n \geq 1$.

(C1) $(I - \lambda A_{n,Y})^{-1} y \to (I - \lambda A_Y)^{-1} y$ as $n \to \infty$ for $y \in Y$.

(C2) For all $\alpha \geq 0$ there is $\beta = \beta(\alpha) \geq 0$ such that for each $x \in D_\alpha$ there is $\{x_n\}_{n\geq 1}$ with $x_n \in D_{n,\beta}$ for all $n \geq 1$ and $x_n \to x$ as $n \to \infty$.

(C3) If $x \in D$, $x_n \in D_n$, $\overline{\lim}_{n\to\infty}\varphi_n(x_n) < \infty$, and $x_n \to x$ as $n \to \infty$, then $B_n x_n \to Bx$ in X as $n \to \infty$.

In the following we shall sometimes associate the subscript 0 to the semilinear problem (SP) and to the corresponding operators, functionals and sets, for the sake of clarity in the formulation of hypotheses. We introduce the following equicontinuity and stability conditions.

(EC) If $x \in D$ and $\{x_n\}_{n\geq 1}$ is a $\{\varphi_n\}$-bounded sequence such that $x_n \in D_n$ for $n \geq 1$ and $x_n \to x$ as $n \to \infty$, then

$$\sup_{n\geq 1} |S_n(t)x_n - x_n| \to 0 \quad \text{as } t \downarrow 0.$$

(S) There is a separately nondecreasing function $\omega : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that

$$|S_n(t)\, x_n - S_n(t)\, y_n| \leq e^{\omega(\alpha,t)t} \, |x_n - y_n|$$

for $t \geq 0$, $\alpha \geq 0$, x_n, $y_n \in D_\alpha$ and $n \geq 0$.

Under these conditions it is seen that the following result, which is our main theorem, holds.

Theorem 1.2. *Let* $\{S_n\}_{n\geq 1}$ *be a sequence of nonlinear semigroups satisfying* (MS;n) *and* (GC;n) *for* $n \geq 1$. *Suppose that the consistency conditions* (C1), (C2), (C3) *and the stability condition* (S) *are satisfied. Then the following statements are equivalent.*

(I) *Condition* (EC) *holds.*

(II) *If* $x \in D$ *and* $\{x_n\}_{n\geq 1}$ *is a* $\{\varphi_n\}$*-bounded sequence such that* $x_n \in D_n$ *for* $n \geq 1$ *and* $x_n \to x$ *as* $n \to \infty$, *then* $S_n(t)x_n \to S(t)x$ *as* $n \to \infty$, *uniformly on bounded subintervals of* $[0,\infty)$.

If A is densely defined, then $Y = X$ and we obtain [3, Theorem 1]. In this sense, our main theorem may be regarded as a generalization of that result. The argument employed for the proof of the main theorem is based on the approach devised in [3], to which this paper is related.

We now indicate a condition which insures that (EC) is satisfied.

Lemma 1.1. *Suppose that for every $\beta > 0$ there are δ, λ_0, $C_{1,\beta}$ and $C_{2,\beta} > 0$ such that*

$$[x - y, T_{n,Y}(s)(I - \lambda A)^{-1}(B_n z - B_n y)]_- \leq C_{1,\beta}|z - y| + C_{2,\beta}|x - y| \tag{1.1}$$

for all $x, y, z \in D_\beta$, $s \in [0, \delta]$, $\lambda \in [0, \lambda_0]$ and $n \geq 1$. Then (EC) *is satisfied.*

Proof. Let $\alpha > 0$, $x \in D$ and let $\{x_n\}_{n\geq 1}$ be a $\{\varphi_n\}$-bounded sequence such that $x_n \in D_{n,\alpha}$ for $n \geq 1$ and $x_n \to x$ as $n \to \infty$. Let $\beta > \alpha$. Then there are δ, λ_0, $C_{1,\beta}$ and $C_{2,\beta} > 0$ such that (1.1) holds, and we may suppose that δ is small enough, so that $e^{a\delta}(\alpha + b\delta) \leq \beta$. By (GC;n), $S_n(s)x_n \in D_{n,\beta}$ for all $s \in [0, \delta]$ and $n \geq 1$, which implies that

$$\begin{aligned}&[S_n(h)x_n - x_n, T_{n,Y}(h - s)(I - \lambda A)^{-1}(B_n S_n(s)x_n - B_n x_n)]_- \\ &\quad \leq C_{1,\beta}\,|S_n(s)x_n - x_n| + C_{2,\beta}\,|S_n(h)x_n - x_n|\end{aligned}$$

for $0 \leq s \leq h \leq \delta$, $\lambda \in [0, \lambda_0]$ and $n \geq 1$. One then sees that

$$\begin{aligned}&|S_n(h)x_n - x_n| \\ &\quad \leq [S_n(h)x_n - x_n, T_{n,Y}(h)x_n - x_n]_+ \\ &\quad + \left[S_n(h)x_n - x_n, \lim_{\lambda\downarrow 0}\int_0^h T_{n,Y}(h - s)(I - \lambda A)^{-1}(B_n S_n(s)x_n - B_n x_n)ds\right]_- \\ &\quad + \left[S_n(h)x_n - x_n, \lim_{\lambda\downarrow 0}\int_0^h T_{n,Y}(h - s)(I - \lambda A)^{-1}B_n x_n ds\right]_+,\end{aligned}$$

from which we deduce that

$$\begin{aligned}|S_n(h)x_n - x_n| \leq |T_{n,Y}(h)x_n - x_n| &+ C_{1,\beta}\int_0^h |S_n(s)x_n - x_n|\,ds \\ &+ hC_{2,\beta}\,|S_n(h)x_n - x_n| + h\sup_{n\geq 1}|B_n x_n|\,.\end{aligned}$$

By Gronwall's lemma, it is seen that

$$|S_n(h)x_n - x_n| \leq \left[\sup_{h\in[0,h_0]}\left(\sup_{n\geq 1}|T_{n,Y}(h)x_n - x_n|\right) + h_0\sup_{n\geq 1}|B_n x_n|\right]e^{2C_{1,\beta}h},$$

for $0 < h \leq h_0 < 1/2C_{2,\beta}$.

Since $\sup_{h\in[0,h_0]}\left[\left(\sup_{n\geq 1}|T_{n,Y}(h)x_n - x_n|\right)\right] \to 0$ as $h_0 \downarrow 0$ by Trotter-Kato theorem and $\sup_{n\geq 1}|B_n x_n| < \infty$ by (C3), this implies the required conclusion. □

Remark 1.1. *If the family of nonlinear operators* $\{B_n\}_{n\geq 1}$ *is uniformly Lipschitz on level sets, in the sense that for all* $\alpha > 0$ *there is* $\gamma_\alpha \in \mathbb{R}_+$ *such that*

$$|B_n x_n - B_n y_n| \leq \gamma_\alpha \, |x_n - y_n|$$

for all $x_n, y_n \in D_{n,\alpha}$ *and* $n \geq 1$*, then* (1.1) *holds, since*

$$\left| \left[x - y, T_{n,Y}(s)(I - \lambda A)^{-1}(B_n z - B_n y) \right]_- \right| \leq |B_n z - B_n y| \leq \gamma_\alpha \, |z - y|$$

for all $x, y \in D_\alpha$*,* $s \geq 0$*,* $\lambda \geq 0$ *and* $n \geq 1$*.*

We also note that if (1.1) *is satisfied, then* $\{B_n\}_{n\geq 1}$ *is uniformly quasidissipative on level sets, in the sense that for all* $\beta \geq 0$ *one has*

$$[x_n - y_n, B_n x_n - B_n y_n]_- \leq (C_{1,\beta} + C_{2,\beta}) \, |x_n - y_n|$$

for all $x_n, y_n \in D_{n,\beta}$ *and* $n \geq 1$*. Then there is no need to assume* (S) *a priori, since* (1.1) *implies*

$$|S_n(t)x_n - S_n(t)y_n| \leq e^{(C_{1,\gamma}+C_{2,\gamma})t} \, |x_n - y_n|$$

for all $\tau > 0$*,* $\beta \geq 0$*,* $x_n, y_n \in D_{n,\beta}$*,* $t \in [0, \tau]$*,* $\gamma \geq e^{a\tau}(\beta + b\tau)$ *and* $n \geq 1$*, and hence* (S) *is satisfied. Also, if suitable subtangential condtions of type* (II.a) *and* (II.b) *are assumed, then there is no need to assume the existence of* $\{S_n\}_{n\geq 1}$ *a priori; see Theorem 1.1.*

2 Integrated semigroups

In this section we state some basic facts about integrated semigroups and their generators.

A one-parameter family of bounded linear operators on X is said to be a once integrated semigroup on X, or, in short, an integrated semigroup on X, if it satisfies the following two properties.

(I1) $W(0) = O_X$ and $W(\cdot)x \in C([0, \infty); X)$ for $x \in X$.

(I2) $W(s)W(t)x = \displaystyle\int_0^s [W(r + t)x - W(r)x] \, dr$ for $s, t \geq 0$ and $x \in X$.

By similarity with the classical theory of C_0-semigroups, it can be shown that, given a once integrated semigroup $W = \{W(t); t \geq 0\}$, there exists a closed linear operator A such that for each $x \in D(A)$ the function $t \to W(t)x$ is continuously differentiable and

$$\frac{d}{dt} W(t)x = x + W(t)Ax \qquad \text{for } t > 0.$$

The above-defined operator A is called the generator of W and it can be shown that an integrated semigroup W is uniquely determined by its generator.

If W is an integrated semigroup and A is its generator, the following properties are satisfied.

(a) $W(t)x \in D(A)$ and $AW(t)x = W(t)Ax$ for $x \in D(A)$ and $t \geq 0$.

(b) $\int_0^t W(t)xdt \in D(A)$ and $W(t)x = A\int_0^t W(s)xds + tx$, for $x \in X$ and $t \geq 0$.

In the following we shall mainly be concerned with the particular class of semigroups W whose generators A are Hille-Yosida operators, that is, satisfy condition (A). With regard to this assumption, it is seen that the following characterization theorem, which emphasizes the importance of this particular class of integrated semigroups, holds.

Theorem 2.1. *A closed linear operator A in X is the generator of a once integrated semigroup W on X such that*

$$\|W(t+h) - W(t)\| \leq h \quad \text{for all } t, h \geq 0$$

if and only if satisfies (A).

It has already been seen that the part A_Y of A in the Banach space $Y = \overline{D(A)}$ generates a C_0-semigroup $T_Y = \{T_Y(t); t \geq 0\}$ on Y. With this notation, one may obtain the following structure theorem.

Theorem 2.2. *Let A be a closed linear operator in X satisfying* (A) *and let $T_Y = \{T_Y(t); t \geq 0\}$ be the C_0-semigroup on Y generated by the part A_Y of A in Y. Then the integrated semigroup generated by A on X can be represented by*

$$W(t)x = \lim_{\lambda \downarrow 0} \int_0^t T_Y(s)(I - \lambda A)^{-1}xds \quad \text{for } t \geq 0 \text{ and } x, y \in X.$$

Using the above theorem, it may be proved that

$$|W(t)x - W(t)y| \leq t\,|x - y| \ \text{ for } t \geq 0 \text{ and } x \in D. \tag{2.1}$$

For other results regarding integrated semigroups, see [1], [4] or [7].

3 The local uniformity of the subtangential condition

This section is devoted to establishing that a version of Pavel's subtangential condition holds uniformly in a certain sense.

First, let us denote by W_n the integrated semigroups generated by A_n on X, $n \geq 1$. By an elementary argument, one may establish the following result ([3, Lemma 4.2]), which yields that the family of nonlinear operators $\{B_n\}_{n\geq 1}$ is equicontinuous in a local sense.

Lemma 3.1. *Suppose that conditions* (C3) *and* (B;n)*, $n \geq 1$, are satisfied. Let $\varepsilon > 0$, $\alpha > 0$, $x \in D$ and let $\{x_n\}_{n\geq 1} \subset D_{n,\alpha}$ be such that $x_n \to x$ in X as $n \to \infty$. Then there is a number $r = r(\varepsilon, \alpha, \{x_n\}_{n\geq 1}, x) > 0$ such that*

$$\sup_{n\geq 1} |B_n x_n - B_n y_n| \leq \varepsilon \tag{3.1}$$

for any $\{\varphi_n\}$-bounded sequence $\{y_n\}_{n\geq 1}$ such that $y_n \in D_{n,\alpha}$ for $n \geq 1$ and $\sup_{n\geq 1} |y_n - x_n| \leq r$.

Using this result, one may establish the follwing theorem, which yields that a version of Pavel's subtangential condition holds uniformly with respect to $n \geq 1$.

Theorem 3.1. *Let $\{S_n\}_{n\geq 1}$ be a sequence of locally Lipschitzian semigroups satisfying* (MS;n) *and* (EC;n) *for $n \geq 1$. Suppose that conditions* (C1), (C3), (EC) *and* (A;n), (B;n)*, $n \geq 1$ are satisfied. Let $x \in D$ and let $\{x_n\}_{n\geq 1}$ be a $\{\varphi_n\}$-bounded sequence such that $x_n \in D_n$ for $n \geq 1$ and $x_n \to x$ as $n \to \infty$. Then*

$$\lim_{h\downarrow 0} \left[\sup_{n\geq 1} [(1/h) |S_n(h)x_n - T_{n,Y}(h)x_n - W_n(h)B_n x_n|] \right] = 0.$$

Proof. Let $\varepsilon > 0$, $\delta_0 > 0$ and $\beta > e^{a\delta_0}\big(\sup_{n\geq 1} \varphi_n(x_n) + b\delta_0\big)$. Then

$$\varphi_n(S_n(t)x_n) \leq e^{at}(\varphi_n(x) + bt) < \beta \quad \text{for } t \in [0, \delta_0].$$

Using Lemma 3.1, one may find $r = r(\varepsilon, \beta, \{x_n\}_{n\geq 1}, x) > 0$ such that $\sup_{n\geq 1} |B_n y_n - B_n x_n| < \varepsilon$ for all $\{y_n\}_{n\geq 1}$ such that $y_n \in D_{n,\beta}$ for $n \geq 1$ and $\sup_{n\geq 1} |y_n - x_n| \leq r$. Choose $h_0 \in (0, \delta_0)$ such that $\sup_{n\geq 1} |S_n(t)x_n - x_n| < r$ for $t \in [0, h_0]$ (this is possible since (EC) is satisfied). By Theorem 2.2, one has

$$\begin{aligned}
&(1/h) |S_n(h)x_n - T_{n,Y}(h)x_n - W_n(h)B_n x_n| \\
&\quad = (1/h) \lim_{\lambda\downarrow 0} \left| \int_0^h T_{n,Y}(h-s)(I - \lambda A)^{-1}(B_n S_n(s)x_n - B_n x_n) ds \right|
\end{aligned}$$

and so

$$\overline{\lim_{h\downarrow 0}}\left[\sup_{n\geq 1}\left[(1/h)\left|S_n(h)x_n - T_{n,Y}(h)x_n - W_n(h)B_nx_n\right|\right]\right]$$
$$\leq \overline{\lim_{h\downarrow 0}}\left[(1/h)\int_0^h \left|B_nS_n(s)x_n - B_nx_n\right| ds\right] \leq \varepsilon.$$

Since ε is arbitrary, one then obtains the required conclusion. □

Remark 3.1. *Under the above assumptions, the following conditions are equivalent to* (EC).

(LU) *If $x \in D$ and if $\{x_n\}_{n\geq 1}$ is a $\{\varphi_n\}$-bounded sequence such that $x_n \in D_n$ for $n \geq 1$ and $x_n \to x$ as $n \to \infty$, then*

$$\lim_{h\downarrow 0}\left[\sup_{n\geq 1}\left[(1/h)\left|S_n(h)x_n - T_{n,Y}(h)x_n - W_n(h)B_nx_n\right|\right]\right] = 0.$$

(LUB) *If $x \in D$ and if $\{x_n\}_{n\geq 1}$ is a $\{\varphi_n\}$-bounded sequence such that $x_n \in D_n$ for $n \geq 1$ and $x_n \to x$ as $n \to \infty$, then there are $\delta > 0$ and $M_\delta > 0$ such that*

$$\sup_{n\geq 1}\left[(1/h)\left|S_n(h)x_n - T_{n,Y}(h)x_n - W_n(h)B_nx_n\right|\right] \leq M_\delta \quad \textit{for } h \in [0,\delta].$$

Proof. The implication from (EC) to (LU) has already been proved. It is obvious that (LU) implies (LUB). For the implication from (LUB) to (EC), let $x \in D$ and let $\{x_n\}_{n\geq 1}$ be a $\{\varphi_n\}$-bounded sequence such that $x_n \in D_n$ for $n \geq 1$ and $x_n \to x$ as $n \to \infty$. Then

$$|S_n(h)x_n - x_n| \leq |T_{n,Y}(h)x_n - x_n| + hM_\delta + |W_n(h)B_nx_n|$$

and the conclusion follows from Trotter-Kato theorem and Theorem 2.2. □

4 The convergence argument

We devote this section to the proof of our main theorem.

(I)→(II) Assume that condition (EC) holds. Let $x \in D$ and let $x_n \in D_n$ be a $\{\varphi_n\}$-bounded sequence such that $x_n \in D_n$ for $n \geq 1$ and $x_n \to x$ as $n \to \infty$. Let $\varepsilon > 0$, $\tau > 0$ and $\alpha > e^{a\tau}(\varphi(x)+b\tau)$. Since the growth condition (GC) is satisfied, one sees that $S(t)x \in D_\alpha$ for $t \in [0,\tau]$.

Let $\beta = \beta(\alpha)$ a number given by (C2). We may suppose that $\beta > e^{a\tau}(\sup_{n\geq 1}\varphi_n(x_n) + b\tau)$, so that $S_n(t)x_n \in D_{n,\beta}$ for $n \geq 1$ and $t \in [0,\tau]$.

We aim to establish our convergence result using discrete schemes. To this purpose, we construct a time-discretizing sequence $\{t_k\}_{k=0}^N$ and an associate sequence of approximating sequences $\left\{\left\{y_n^{(k)}\right\}_{n\geq 1}\right\}_{k=0}^N$ satisfying

(1) $t_0 = 0$, $y_n^{(0)} = x_n$ for $n \geq 1$ and $t_N = \tau$;

(2) $0 < t_{k+1} - t_k < \varepsilon$ for $0 \leq k \leq N-1$;

(3) $y_n^{(k)} \in D_{n,\beta}$ for $n \geq 1$ and $y_n^{(k)} \to S(t_k)x$ as $n \to \infty$;

(4) $\overline{\lim\limits_{n\to\infty}} \left|T_{n,Y}(t_{k+1}-t_k)y_n^{(k)} + W_n(t_{k+1}-t_k)B_n y_n^{(k)} - S_n(t_{k+1}-t_k)y_n^{(k)}\right|$
$\leq (t_{k+1}-t_k)\varepsilon$

and

$|T_Y(t_{k+1}-t_k)S(t_k)x + W(t_{k+1}-t_k)BS(t_k)x - S(t_{k+1})x|$
$\leq (t_{k+1}-t_k)\varepsilon \quad$ for $0 \leq k \leq N-1$.

To initialize the discrete scheme, we first set $t_0 = 0$ and $\left\{y_n^{(0)}\right\}_{n\geq 1} = \{x_n\}_{n\geq 1}$. Let us now suppose that $\{t_j\}_{j=0}^k$ and $\left\{\left\{y_n^{(j)}\right\}_{n\geq 1}\right\}_{j=0}^k$ have been constructed in such a way that (2), (3), (4) and the first half of (1) are satisfied for $0 \leq j \leq k-1$.

If $t_k < \tau$, we define

$$\hat{h}_k = \sup\{h \in (0,\varepsilon] \cap (0, \tau - t_k];\ (4.2), (4.3) \text{ hold}\}, \tag{4.1}$$

where

$$\overline{\lim_{n\to\infty}} \left|T_{n,Y}(h)\, y_n^{(k)} + W_n(h)B_n y_n^{(k)} - S_n(h)\, y_n^{(k)}\right| \leq h\varepsilon \tag{4.2}$$

and

$$|T_Y(h)\, S(t_k)\, x + W(h)BS(t_k)\, x - S(t_k + h)\, x| \leq h\varepsilon. \tag{4.3}$$

As noticed in Theorem 3.1, one has

$$\lim_{h\downarrow 0}\left[\sup_{n\geq 1}\left[(1/h)\left|S_n(h)y_n^{(k)} - T_{n,Y}(h)y_n^{(k)} - W_n(h)B_n y_n^{(k)}\right|\right]\right] = 0,$$

since $\left\{y_n^{(k)}\right\}_{n\geq 1}$ is $\{\varphi_n\}$-bounded and $y_n^{(k)} \to S(t_k)x$ as $n \to \infty$. Also,

$$\lim_{h\downarrow 0} |T_Y(h)S(t_k)x + W(h)BS(t_k)x - S(h+t_k)x| = 0$$

and hence $\hat{h}_k > 0$. We then choose $h_k \in (\hat{h}_k/2, \hat{h}_k)$, put $t_{k+1} = t_k + h_k$ and apply (C2) to find $\left\{y_n^{(k+1)}\right\}_{n\geq 1}$ such that $\left\{y_n^{(k+1)}\right\}_{n\geq 1} \to S(t_{k+1})x$ as $n \to \infty$.

It is easily seen that (2), (3) and (4) are satisfied for this choice of h_k. We now need to show that the time-discretizing sequence $\{t_k\}_{k\geq 1}$ reaches τ in a finite number of steps.

We shall argue by contradiction. Suppose that $t_k < \tau$ for all $1 \leq k < \infty$. Then there is $s \leq \tau$ such that $t_k \to s$ as $k \to \infty$. By (C2), there is a sequence $\{z_n\}_{n\geq 1}$ such that $z_n \to S(s)x$ as $n \to \infty$.

Using Theorem 3.1, one may deduce that there is $h \in (0, \varepsilon]$ such that

$$\sup_{n\geq 1} \left[(1/h) \left|T_{n,Y}(h)\, z_n + W_n(h)B_n z_n - S_n(h)\, z_n\right|\right] < \varepsilon/3 \tag{4.4}$$

and

$$(1/h) \left|S(h+s)\, x - W(h)BS(s)\, x - T_Y(h)\, S(s)\, x\right| < \varepsilon/3. \tag{4.5}$$

Since $\sum_{k=0}^{\infty} h_k = s$, there is $N \geq 1$ such that $\hat{h}_k < h$ for $k \geq N$. From the definition of $\hat{h}_k$, it is seen that either

$$\overline{\lim_{n\to\infty}} \left|T_{n,Y}(h)\, y_n^{(k)} + W_n(h)B_n y_n^{(k)} - S_n(h)\, y_n^{(k)}\right| > h\varepsilon$$

for infinitely many $k \geq N$, or

$$\left|T_Y(h)\, S(t_k)\, x + W(h)BS(t_k)\, x - S(t_k + h)\, x\right| > h\varepsilon$$

for infinitely many $k \geq N$.

In the first case, there is a subsequence $k_l \to \infty$ such that

$$\overline{\lim_{n\to\infty}} \left|T_{n,Y}(h)\, y_n^{(k_l)} + W_n(h)B_n y_n^{(k_l)} - S_n(h)\, y_n^{(k_l)}\right| > h\varepsilon.$$

Consequently, there is a sequence $n_l \to \infty$ such that

$$\left|T_{n_l,Y}(h)\, y_{n_l}^{(k_l)} + W_{n_l}(h)B_{n_l} y_{n_l}^{(k_l)} - S_{n_l}(h)\, y_{n_l}^{(k_l)}\right| > h\varepsilon/2 \tag{4.6}$$

and

$$\left|y_{n_l}^{(k_l)} - S(t_{k_l})\, x\right| \leq 1/k_l \quad \text{for } l \geq 1. \tag{4.7}$$

Then $y_{n_l}^{(k_l)} \to S(s)x$ as $l \to \infty$ and hence $\left|y_{n_l}^{(k_l)} - z_{n_l}\right| \to 0$ as $l \to \infty$.

One sees that

$$\begin{aligned}
&\left|T_{n_l,Y}(h)z_{n_l} + W_{n_l}(h)B_{n_l}z_{n_l} - S_{n_l}(h)z_{n_l}\right| \\
&\quad \geq h\varepsilon/2 - \left|W_{n_l}(h)(B_{n_l}y_{n_l}^{(k_l)} - B_{n_l}z_{n_l})\right| - \left|T_{n_l,Y}(h)y_{n_l}^{(k_l)} - T_{n_l,Y}(h)z_{n_l}\right| \\
&\qquad - \left|S_{n_l}(h)y_{n_l}^{(k_l)} - S_{n_l}(h)z_{n_l}\right|.
\end{aligned}$$

Moreover,

$$\left|T_{n_l,Y}(h)y_{n_l}^{(k_l)} - T_{n_l,Y}(h)z_{n_l}\right| \leq \left|y_{n_l}^{(k_l)} - z_{n_l}\right|$$

since all the semigroups $T_{n,Y}$, $n \geq 1$, are contraction semigroups on Y. Also, from the stability condition (S),

$$\left|S_{n_l}(h)y_{n_l}^{(k_l)} - S_{n_l}(h)z_{n_l}\right| \leq e^{w(\beta,h)h}\left|y_{n_l}^{(k_l)} - z_{n_l}\right|.$$

Using Theorem 2.2, one may obtain

$$\left|W_{n_l}(h)(B_{n_l}y_{n_l}^{(k_l)} - B_{n_l}z_{n_l})\right| \leq h\left|B_{n_l}y_{n_l}^{(k_l)} - B_{n_l}z_{n_l}\right|$$

and it is seen using (C3) that $\left|B_{n_l}y_{n_l}^{(k_l)} - B_{n_l}z_{n_l}\right| \to 0$ as $l \to \infty$, since both $y_{n_l}^{(k_l)}$ and z_{n_l} tend to $S(s)x$ as $l \to \infty$. Hence, for l large enough, one has

$$(1/h)\left|T_{n_l,Y}(h)\,z_{n_l} + W_{n_l}(h)B_{n_l}z_{n_l} - S_{n_l}(h)\,z_{n_l}\right| \geq \varepsilon/3,$$

which contradicts (4.4).

In the second case, passing to limit as $l \to \infty$ in (4.6) one sees that

$$(1/h)\left|S(h+s)\,x - W(h)BS(s)\,x - T(h)\,S(s)\,x\right| \leq \varepsilon,$$

which contradicts (4.5). In conclusion, τ can be reached in a finite number of steps and the time-discretizing sequence $\{t_k\}_{k=0}^{N}$, respectively the associate sequence of approximating sequences $\left\{\left\{y_n^{(k)}\right\}_{n\geq 1}\right\}_{k=0}^{N}$, having properties (1) through (4) can be constructed as requested.

We now estimate $|S_n(t)x_n - S(t)x|$ for arbitrary $t \in (0,\tau]$. Let $0 \leq k \leq N-1$ such that $t \in (t_k, t_{k+1}]$. Using the stability condition (S), one sees that

$$\begin{aligned}
(4.8)\qquad & |S_n(t)x_n - S(t)x| \\
&\leq e^{w(\beta,\tau)(t_{k+1}-t_k)}(|S_n(t_{k+1}-t)x_n - x_n| \\
&\quad + |S(t_{k+1}-t)x - x|) + |S_n(t_{k+1})x_n - S(t_{k+1})x|,
\end{aligned}$$

and similarly

$$\begin{aligned}
(4.9)\qquad & |S_n(t_{k+1})x_n - S(t_{k+1})x| \\
&\leq e^{w(\beta,\tau)(t_{k+1}-t_k)}\left(|S_n(t_k)x_n - S(t_k)x| + \left|S(t_k)x - y_n^{(k)}\right|\right) \\
&\quad + \left|S_n(t_{k+1}-t_k)y_n^{(k)} - S(t_{k+1})x\right|.
\end{aligned}$$

Also,

$$
\begin{aligned}
&\left|S_n\left(t_{k+1}-t_k\right) y_n^{(k)}-S\left(t_{k+1}\right) x\right| \\
&\quad \leq\left|S_n\left(t_{k+1}-t_k\right) y_n^{(k)}-T_{n, Y}\left(t_{k+1}-t_k\right) y_n^{(k)}-W_n\left(t_{k+1}-t_k\right) B_n y_n^{(k)}\right| \\
&\quad\quad+\left|T_{n, Y}\left(t_{k+1}-t_k\right) y_n^{(k)}-T_Y\left(t_{k+1}-t_k\right) S\left(t_k\right) x\right| \\
&\quad\quad+\left|W_n\left(t_{k+1}-t_k\right)\left(B_n y_n^{(k)}-B S\left(t_k\right) x\right)\right| \\
&\quad\quad+\left|T_Y\left(t_{k+1}-t_k\right) S\left(t_k\right) x+W\left(t_{k+1}-t_k\right) B S\left(t_k\right) x-S\left(t_{k+1}\right) x\right| .
\end{aligned}
\tag{4.10}
$$

Passing to superior limit as $n \to \infty$ in (4.9), one deduces using (4.10) and Trotter-Kato theorem that

$$
\begin{aligned}
\varlimsup_{n\to\infty} |S_n(t_{k+1})x_n - S(t_{k+1})x| \leq e^{w(\beta,\tau)(t_{k+1}-t_k)} &\varlimsup_{n\to\infty} |S_n(t_k)x_n - S(t_k)x| \\
&+ 2\varepsilon(t_{k+1}-t_k).
\end{aligned}
$$

By an inductive argument, one obtains that

$$
\varlimsup_{n\to\infty} |S_n(t_{k+1})x_n - S(t_{k+1})x| \leq 2\varepsilon\tau e^{w(\beta,\tau)\tau}.
$$

Passing now to superior limit in (4.8) as $n \to \infty$, one sees that

$$
\begin{aligned}
&\varlimsup_{n\to\infty} |S_n(t)x_n - S(t)x| \\
&\quad \leq e^{w(\beta,\tau)\tau}\left[\varlimsup_{n\to\infty} |S_n(t_{k+1}-t)x_n - x_n| + |S(t_{k+1}-t)x - x| + 2\varepsilon\tau\right] \\
&\quad \leq e^{w(\beta,\tau)\tau}\left[\sup_{h\in[0,\varepsilon]}\left(\sup_{n\geq 1} |S_n(h)x_n - x_n|\right) + \sup_{h\in[0,\varepsilon]} |S(h)x - x| + 2\varepsilon\tau\right].
\end{aligned}
$$

Now, since

$$
\sup_{h\in[0,\varepsilon]}\left[\sup_{n\geq 1} |S_n(h)x_n - x_n|\right] \to 0 \quad \text{as } \varepsilon \downarrow 0
$$

by (EC) and

$$
\sup_{h\in[0,\varepsilon]} |S(h)x - x| \to 0 \quad \text{as } \varepsilon \downarrow 0,
$$

we conclude that

$$
\varlimsup_{n\to\infty} |S_n(t)x_n - S(t)x| = 0 \text{ uniformly on } [0,\tau].
$$

This finishes the proof of the first implication.

(II)→(I) Let $\mathcal{X} = \left\{\{x_n\}_{n\geq 0}\,;\{x_n\}_{n\geq 0} \subset X, x_n \to x_0 \text{ as } n \to \infty\right\}$ be the space of the sequences in X which converge to their first component, endowed with the norm $\|\cdot\|$ defined by $\left\|\{x_n\}_{n\geq 0}\right\| = \sup_{n\geq 0} |x_n|$. It is easy to see that $(\mathcal{X}, \|\cdot\|)$ is a Banach space.

Let $x \in D$ and let $\{x_n\}_{n\geq 1}$ be a $\{\varphi_n\}$-bounded sequence such that $x_n \in D_n$ for $n \geq 1$ and $x_n \to x$ as $n \to \infty$. For $t \geq 0$, define $\mathcal{V}(t) : \mathcal{X} \to \mathcal{X}$ by $\mathcal{V}(t) = \{S_n(t)x_n\}_{n\geq 0}$, using again the convention $S \equiv S_0$ and $x \equiv x_0$. From our hypothesis, it is seen that $\mathcal{V}(\cdot)$ is well-defined on $[0, \infty)$. For $N \geq 1$ and $t \geq 0$, define also $\mathcal{V}_N(t) : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{V}_N(t) = \{V_N^n(t)x_n\}_{n\geq 0}\,, \quad V_N^n(t)x_n = \begin{cases} S_n(t)x_n; & \text{for } 0 \leq n \leq N-1; \\ S(t)x; & \text{for } n \geq N. \end{cases}$$

Then $\mathcal{V}_N(\cdot)$ is well defined and continuous on $[0, \infty)$ for all $N \geq 1$. Since

$$\|\mathcal{V}_N(t) - \mathcal{V}(t)\| = \sup_{n\geq N} |S_n(t)x_n - S(t)x|\,,$$

it is seen that $\|\mathcal{V}_N(t) - \mathcal{V}(t)\| \to 0$ as $N \to \infty$, uniformly on compact subsets of $[0, \infty)$. Hence $\mathcal{V}(\cdot)$ is continuous on $[0, \infty)$, which implies that (EC) is satisfied. This ends the proof of our main result.

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