

THE DYNAMICS OF AN IMPULSIVELY CONTROLLED TWO-PREY ONE-PREDATOR FOOD WEB MODEL

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Abstract A two-prey one-predator food web model with an abstract functional response of the predator is considered as being subject to impulsive controls, in the form of periodic predator release and pesticide spraying. Sufficient conditions for the local and global stability of the prey-free periodic solution are established by using Floquet theory and comparison arguments in terms of integral estimates with biological significance, the permanence of the system being also discussed.

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1. INTRODUCTION

A general model of predator-prey interaction, constructed to address several shortcomings of the celebrated Lotka-Volterra model, can be expressed in the following canonical form

$$\begin{cases} N'(t) = N(t)f(N(t)) - P(t)F(N(t), P(t)) - d_N N(t), \\ P'(t) = P(t)G(N(t), P(t)) - d_P P(t), \end{cases} \quad (1)$$

where $N = N(t)$ is the density of the prey population and $P = P(t)$ is the density of the predator population. That is, in order to describe such a model, three functions have to be given, namely: $f(N)$ - the per capita growth rate of the prey population in the absence of the predation, $F(N, P)$ - the predator's functional (behavioral) response, which represents the number of prey individuals consumed per unit area and unit time per single predator, that is, the dependence of the rate of consumption upon the density of the prey, and $G(N, P)$ - the predator's numerical response, which represents the per capita growth rate of the predator population, that is, the dependence of the reproductive rate upon the density of the prey. The simplest form of a numerical response is $G(N, P) = eF(N, P)$, where e is a conversion coefficient, expressing the idea that the growth of the predator class depends on the surplus of energy caused by prey consumption. Also, the functional response can be expressed as $F(N, P) = NF_1(N, P)$, where $F_1(N, P)$ represents the per capita death rate of prey due to predation. In the

above model, d_N and d_P are the natural mortalities of the prey and predator, respectively. If predation is the most important cause of fatalities for prey, then d_N can be neglected. See also Yodzis [9] for biological considerations on the above concepts, with applications in the management of fisheries.

Integrated pest management (IPM) is an ecological approach which represents a synthesis of techniques of various natures (mechanical, chemical, biological and not only) to control pests, with an accent on those which are potentially less damaging to the environment, minimizing pesticide use. Here, by pests are meant organisms which are detrimental to human health or to crops, such as spiders, mites or rodents. One of the usual approaches to biological control is to release natural predators of the given pest, with the purpose to keep the size of the pest population under a certain level, called the economic injury level. See also Maredia, Dakouo and Mota-Sanchez [5] for an overview of various IPM concepts and techniques.

Our purpose is to construct a model of concurring predator-prey interactions, in the form of a food web model. It is assumed that the pest population consists in two distinct species which compete for the same resources and are regulated by the same predator, able to feed on both. The model intends to describe a pest control strategy which uses both biological controls (periodic release of natural predators) and chemical controls (periodic pesticide spraying). It is assumed that the pest population consists in two distinct species which compete for the same resources and are regulated by the same predator, able to feed on both.

Since for certain pesticides the effects are detectable shortly after application and the size of the predator population grows immediately after each release of predators, we describe this situation by using controls which are employed in an impulsive and periodic fashion, with the same periodicity but not simultaneously. It is assumed that a fixed amount of predators are released each time, while as a result of pesticide spraying fixed proportions of the pest and predator populations are removed from the environment.

2. THE MODEL

On the basis of the biological assumptions mentioned above, we may formulate the following impulsively perturbed model

$$\left\{ \begin{array}{ll}
 x'(t) = x(t)[r_1 - a_{11}x(t) - a_{12}y(t)] & t \neq (n+l-1)T, t \neq nT; \\
 \quad - a_1x(t)z(t)\varphi(x(t), z(t)), & \\
 y'(t) = y(t)[r_2 - a_{21}x(t) - a_{22}y(t)] & t \neq (n+l-1)T, t \neq nT; \\
 \quad - a_2y(t)z(t)\varphi(y(t), z(t)), & \\
 z'(t) = e_1a_1x(t)z(t)\varphi(x(t), z(t)) & t \neq (n+l-1)T, t \neq nT; \\
 \quad + e_2a_2y(t)z(t)\varphi(y(t), z(t)) - dz(t), & \\
 \Delta x(t) = -\delta_1x(t), & t = (n+l-1)T; \\
 \Delta y(t) = -\delta_2y(t), & t = (n+l-1)T; \\
 \Delta z(t) = -\delta_3x(t), & t = (n+l-1)T; \\
 \Delta x(t) = 0, & t = nT; \\
 \Delta y(t) = 0, & t = nT; \\
 \Delta z(t) = \mu, & t = nT.
 \end{array} \right. \quad (S)$$

Here, $x = x(t)$ and $y = y(t)$ represent the densities of the prey populations, while $z = z(t)$ represents the density of the predator population. The constants r_1 and r_2 denote the intrinsic birth rates of the prey populations, a_{11} , a_{22} are used to describe the effects of intraspecific competition, while a_{12} , a_{21} represent the effects of interspecific competition between the prey populations. Also, $n \in \mathbb{N}^*$, e_1 and e_2 represent the efficiencies of prey conversion into newborn predators, $T > 0$ is the periodicity of the impulsive perturbations, $0 < l < 1$ is a parameter used to describe the time lag between pesticide spraying and predator release, $\Delta\psi(t) = \psi(t+) - \psi(t)$ for $\psi \in \{x, y, z\}$ and $t > 0$ represent the instantaneous jumps in the population sizes after the use of impulsive controls, $0 \leq \delta_1, \delta_2, \delta_3 < 1$ are the fixed proportions of the prey and predator populations which are removed from the environment due to pesticide spraying, and μ is the constant amount of predators which are released each time.

The function $\varphi : [0, \infty)^2 \rightarrow [0, \infty)$, $\varphi \in C([0, \infty)^2)$ is assumed to satisfy the following assumptions.

(H1) $x \rightarrow \varphi(x, z)$ is decreasing for $z \in [0, \infty)$, $z \rightarrow \varphi(x, z)$ is decreasing for $x \in [0, \infty)$.

(H2) $x \rightarrow x\varphi(x, z)$ is increasing for $z \in [0, \infty)$, $z \rightarrow z\varphi(x, z)$ is increasing for $x \in [0, \infty)$.

Note that the above monotonicity hypotheses are satisfied for several functional responses in common use, namely Holling type II (for which $\varphi(x, z) = \frac{a}{1+bx}$), Ivlev (for which $\varphi(x, z) = \frac{a(1-e^{-kx})}{x}$, $x \neq 0$, $\varphi(0) = ak$) and Beddington-deAngelis (for

which $\varphi(x, z) = \frac{b}{a+k_1x+k_2z}$). Investigations regarding the dynamics of impulsively perturbed two-prey one-predator systems which are related to ours have been performed in Wang, Wang and Lin [8], Baek [1], Yu et. al. [10]. While [1] and [10] use a Beddington-deAngelis functional response which depends on both prey populations, our proofs can be adapted to cover their corresponding results. Also, [8] uses a Watt functional response ($xz\varphi(x, z) = z(1 - e^{-\frac{cx}{z^m}})$) which fits our framework provided that $0 < m \leq 1$. For further results on the impulsive control of predator-prey interaction and on integrated pest management strategies, see Georgescu and Zhang [4], Nundloll, Mailleret and Grogard [6], Tang, Tang and Cheke [7].

3. PRELIMINARIES

In this section we shall introduce a few definitions, notations and auxiliary results relating to Floquet theory for impulsively perturbed systems of ordinary differential equations, while also establishing the biological well-posedness of the Cauchy problem associated to our system (S) for strictly positive initial data. We first indicate a result which provides estimations for the solution of a system of differential inequalities.

Lemma 3.1. ([2]) *Let the function $u \in PC^1(\mathbb{R}_+, \mathbb{R})$ satisfy the inequalities*

$$\begin{cases} \frac{du}{dt} \leq (\geq) p(t)u(t) + f(t), & t \neq \tau_k, t > 0; \\ u(\tau_k+) \leq (\geq) d_k u(\tau_k) + h_k, & k \geq 0; \\ u(0+) \leq (\geq) u_0, \end{cases} \quad (2)$$

where $p, f \in PC(\mathbb{R}_+, \mathbb{R})$ and $d_k \geq 0, h_k$ and u_0 are constants and $(\tau_k)_{k \geq 0}$ is a strictly increasing sequence of positive real numbers. Then, for $t > 0$,

$$\begin{aligned} u(t) \leq (\geq) & u_0 \left(\prod_{0 < \tau_k < t} d_k \right) e^{\int_0^t p(s) ds} + \int_0^t \left(\prod_{0 \leq \tau_k < s} d_k \right) e^{\int_s^t p(\tau) d\tau} f(s) ds \\ & + \sum_{0 < \tau_k < t} \left(\prod_{\tau_k < \tau_j < t} d_j \right) e^{\int_{\tau_k}^t p(\tau) d\tau} h_k. \end{aligned}$$

In the above, by $PC(\mathbb{R}_+, \mathbb{R})$ ($PC^1(\mathbb{R}_+, \mathbb{R})$) is meant the class of real piecewise continuous (real piecewise continuously differentiable) functions defined on \mathbb{R}_+ . For other results on impulsive differential equations, see Bainov and Simeonov [2].

First, it is possible to prove that the Cauchy problem with strictly positive initial data is well-posed for our system (S), that is, solutions (x, y, z) starting with strictly positive initial data remain strictly positive and bounded on their whole domains.

Lemma 3.2. *The positive orthant $(\mathbb{R}_+^*)^3$ is an invariant region for the system (S).*

Proof. Let us consider $(x(\cdot), y(\cdot), z(\cdot)) : [0, T_0] \rightarrow \mathbb{R}^3$ a saturated solution for (S) which starts with strictly positive $x(0), y(0), z(0)$. It is seen that

$$\begin{cases} x'(t) = x(t)(r_1 - a_{11}x(t) - a_{12}y(t) - a_1z(t)\varphi(x(t), z(t))), \\ y'(t) = y(t)(r_2 - a_{21}x(t) - a_{22}y(t) - a_2z(t)\varphi(y(t), z(t))), \\ z'(t) = z(t)(e_1a_1x(t)\varphi(x(t), z(t)) + e_2a_2y(t)\varphi(y(t), z(t)) - d), \end{cases}$$

for $0 \leq t < T_0, t \neq (n + l - 1)T, t \neq nT$, as long as the solution remains positive component-wise. It then follows from Lemma 3.1 that

$$\begin{cases} x(t) \geq x(0) (1 - \delta_1) \left[\frac{t+(1-l)T}{T} \right] e^{\int_0^t p_1(s)ds}, & 0 \leq t < T_0; \\ y(t) \geq y(0) (1 - \delta_2) \left[\frac{t+(1-l)T}{T} \right] e^{\int_0^t p_2(s)ds}, \\ z(t) \geq z(0) (1 - \delta_3) \left[\frac{t+(1-l)T}{T} \right] e^{\int_0^t p_3(s)ds}, \end{cases}$$

where

$$\begin{cases} p_1(t) = r_1 - a_{11}x(t) - a_{12}y(t) - a_1z(t)\varphi(x(t), z(t)); \\ p_2(t) = r_2 - a_{21}x(t) - a_{22}y(t) - a_2z(t)\varphi(y(t), z(t)); \\ p_3(t) = e_1a_1x(t)\varphi(x(t), z(t)) + e_2a_2y(t)\varphi(y(t), z(t)) - d, \end{cases}$$

that is, x, y, z remain strictly positive on $[0, T_0)$. \square

Also, using Lemma 3.1, it is possible to show that all solutions of (S) starting in $(\mathbb{R}_+^*)^3$ remain bounded and are actually defined on the whole \mathbb{R}_+ .

Lemma 3.3. All solutions $(x(\cdot), y(\cdot), z(\cdot))$ of (S) with initial data $(x(0), y(0), z(0)) \in (\mathbb{R}_+^*)^3$ are bounded and defined on \mathbb{R}_+ .

Proof. Let us consider a solution $(x(\cdot), y(\cdot), z(\cdot))$ of (S) starting with strictly positive $x(0), y(0), z(0)$ and define $u_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$u_1(t) = e_1x(t) + e_2y(t) + z(t), \quad t \geq 0.$$

One then has

$$u_1'(t) + du_1(t) \leq e_1x(t)[(r_1 + d) - a_{11}x(t)] + e_2y(t)[(r_2 + d) - a_{22}y(t)],$$

for $t > 0, t \neq (n + l - 1)T, t \neq nT$, and consequently

$$u_1'(t) + du_1(t) \leq C = e_1(r_1 + d)^2/(4a_{11}) + e_2(r_2 + d)^2/(4a_{22}),$$

for $t > 0, t \neq (n + l - 1)T, t \neq nT$. Also

$$u_1((n + l - 1)T+) \leq (1 - \delta)u_1((n + l - 1)T),$$

where $\delta = \min(\delta_1, \delta_2, \delta_3)$, and

$$u_1(nT+) = u_1(nT) + \mu.$$

By Lemma 3.1, it follows that

$$u_1(t) \leq u_1(0+) + C \int_0^t e^{-d(t-s)} ds + \sum_{0 < nT < t} \mu e^{-d(t-nT)},$$

which yields

$$u_1(t) \leq u_1(0+)e^{-dt} + \frac{C(1 - e^{-dt})}{d} + \mu \frac{e^{dT}}{e^{dT} - 1}, \quad t > 0, \quad (3)$$

from which the boundedness of u_1 follows. Consequently, x, y, z are bounded and, by an easy continuability argument, defined on the whole \mathbb{R}_+ . \square

We now introduce a few basic results regarding the Floquet theory for impulsive systems of ordinary differential equations which will be used in the next section to discuss the local stability of the prey-free periodic solution. Let us consider the impulsive linear system

$$\begin{cases} X'(t) = A(t)X(t), & t \neq \tau_k, t \in \mathbb{R}; \\ \Delta X = B_k X, & t = \tau_k, \tau_k < \tau_{k+1}, k \in \mathbb{Z}. \end{cases} \quad (4)$$

under the following hypotheses.

(A1) $A(\cdot) \in PC(\mathbb{R}, M_n(\mathbb{R}))$ and there is $T > 0$ such that $A(t + T) = A(t)$ for all $t \geq 0$.

(A2) $B_k \in M_n(\mathbb{R})$, $\det(I_n + B_k) \neq 0$ for $k \in \mathbb{Z}$.

(A3) There is $q \in \mathbb{N}^*$ such that $B_{k+q} = B_k$, $\tau_{k+q} = \tau_k + T$ for $k \in \mathbb{Z}$.

Let $\Phi(t)$ be a fundamental matrix of (4). Then there is a unique nonsingular matrix $\mathcal{M} \in M_n(\mathbb{R})$ such that $\Phi(t + T) = \Phi(t)\mathcal{M}$ for all $t \in \mathbb{R}$, which is called the monodromy matrix of (4) corresponding to Φ . Actually, all monodromy matrices of (4) corresponding to different Φ 's are similar and consequently they have the same eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, which are called the Floquet multipliers of (4). Under these hypotheses, the following stability result holds, where by elementary divisors of a square matrix we understand the characteristic polynomials of its Jordan blocks.

Lemma 3.4. ([2]) *Suppose that conditions (A1)-(A3) hold. Then*

- (1) *The system (4) is stable if and only if all Floquet multipliers λ_k , $1 \leq k \leq n$ satisfy $|\lambda_k| \leq 1$ and if $|\lambda_k| = 1$, then to λ_k there corresponds a simple elementary divisor.*
- (2) *The system (4) is asymptotically stable if and only if all Floquet multipliers λ_k , $1 \leq k \leq n$ satisfy $|\lambda_k| < 1$.*
- (3) *The system (4) is unstable if there is a Floquet multiplier λ_k such that $|\lambda_k| > 1$.*

When the prey populations x and y are eradicated, we are led to investigate the properties of the subsystem

$$\begin{cases} z'(t) = -dz(t), & t \neq (n+l-1)T, t \neq nT \\ \Delta z(t) = -\delta_3 z(t), & t = (n+l-1)T; \\ \Delta z(t) = \mu, & t = nT; \\ z(0+) = z_0, \end{cases} \quad (5)$$

which describes the dynamics of the predator in the absence of prey. It is seen that the system constructed with the first three equations in (5) has a periodic solution to which all solutions of (5) starting with strictly positive z_0 tend as $t \rightarrow \infty$. It is seen that the following Lemma (Lemma 4.2 of Georgescu and Moroşanu [3]) holds.

Lemma 3.5. ([3]) *The system constructed with the first three equations in (5) has a T -periodic solution z_d^* . With this notation, the following properties are satisfied.*

- 1 $\int_0^T z_d^*(t)dt = \frac{\mu}{1-e^{-dT}(1-\delta_3)} [(1 - e^{-dT}) + (1 - \delta_3)(e^{-dT} - e^{-dT})]$.
- 2 $\lim_{t \rightarrow \infty} |z(t) - z_d^*(t)| = 0$ for all solutions $z(t)$ of (5) starting with strictly positive z_0 .
- 3 $\sup_{t \geq 0} |z_d^*(t) - z_d^*(t)| \leq f_2(d, \tilde{d}; T, a, \delta_3)$, with $\lim_{\tilde{d} \rightarrow d} f_2(\tilde{d}, d; T, a, \delta_3) = 0$.

The notation z_d^* , emphasizing the d -dependence of the solutions of (5) has been chosen since systems of type (5) occur throughout this paper with different d 's but with the same δ_3 and μ . Note that

$$z_d^* = \begin{cases} \frac{\mu}{1-e^{-dT}(1-\delta_3)} e^{-d(t-nT)}, & t \in (nT, (n+l)T] \\ \frac{\mu}{1-e^{-dT}(1-\delta_3)} e^{-d(t-nT)}(1 - \delta_3), & t \in ((n+l)T, (n+1)T]. \end{cases} \quad (6)$$

Consequently, in the absence of prey, the system consisting of the first three equations of (S) has a periodic solution $E^* = (0, 0, z_d^*)$, which will be called in what follows the prey-free periodic solution. Note that the success of the IPM strategy can be expressed in terms of stability properties of E^* . In this sense, the global stability of E^* can be interpreted as an absolute success of the IPM strategy, as the pests (prey) will be ultimately eradicated irrespective of their initial population sizes.

4. THE STABILITY RESULTS

We shall now give sufficient conditions for the local and global stability of E^*

Theorem 4.1. *The prey-free periodic solution E^* is locally asymptotically stable if*

$$\int_0^T z_d^*(t) \varphi(0, z_d^*(t)) dt > \max \left(\frac{\ln(1 - \delta_1) + r_1 T}{a_1}, \frac{\ln(1 - \delta_2) + r_2 T}{a_2} \right). \quad (7)$$

Further, E^* is globally asymptotically stable if

$$\int_0^T z_d^*(t) \varphi\left(\frac{r_1}{a_{11}}, z_d^*(t)\right) dt > \frac{\ln(1 - \delta_1) + r_1 T}{a_1}, \quad (8)$$

$$\int_0^T z_d^*(t) \varphi\left(\frac{r_2}{a_{22}}, z_d^*(t)\right) dt > \frac{\ln(1 - \delta_2) + r_2 T}{a_2}. \quad (9)$$

Proof. First, we discuss the local stability of E^* by using the method of small amplitude perturbations. To this purpose, let us denote

$$x(t) = u(t), \quad y(t) = v(t), \quad z(t) = w(t) + z_d^*(t),$$

where u, v, w are understood to be small amplitude perturbations. The linearization of (S) reads as

$$\left\{ \begin{array}{ll} u'(t) = u(t) \left(r_1 - a_1 z_d^*(t) \varphi(0, z_d^*(t)) \right), & t \neq (n+l-1)T, t \neq nT; \\ v'(t) = v(t) \left(r_2 - a_2 z_d^*(t) \varphi(0, z_d^*(t)) \right), & t \neq (n+l-1)T, t \neq nT; \\ w'(t) = e_1 a_1 z_d^*(t) \varphi(0, z_d^*(t)) u(t), \\ \quad + e_2 a_2 z_d^*(t) \varphi(0, z_d^*(t)) v(t) - dw(t), & t \neq (n+l-1)T, t \neq nT; \\ \Delta u(t) = -\delta_1 x(t), & t = (n+l-1)T; \\ \Delta v(t) = -\delta_2 y(t), & t = (n+l-1)T; \\ \Delta w(t) = -\delta_3 x(t), & t = (n+l-1)T; \\ \Delta u(t) = \Delta v(t) = \Delta w(t) = 0, & t = nT. \end{array} \right. \quad (10)$$

Let Φ_L be a fundamental matrix of the differential system constructed with the first three equations of (10). Then Φ_L satisfies

$$\frac{d\Phi_L}{dt}(t) = \begin{pmatrix} r_1 - a_1 z_d^*(t) \varphi(0, z_d^*(t)) & 0 & 0 \\ 0 & r_2 - a_2 z_d^*(t) \varphi(0, z_d^*(t)) & 0 \\ e_1 a_1 z_d^*(t) \varphi(0, z_d^*(t)) & e_2 a_2 z_d^*(t) \varphi(0, z_d^*(t)) & -d \end{pmatrix} \Phi_L(t).$$

Consequently, a fundamental matrix Φ_L of (10) which satisfies $\Phi_L(0) = I_3$ is

$$\Phi_L(t) = \begin{pmatrix} e^{\int_0^t r_1 - a_1 z_d^*(s) \varphi(0, z_d^*(s)) ds} & 0 & 0 \\ 0 & e^{\int_0^t r_2 - a_2 z_d^*(s) \varphi(0, z_d^*(s)) ds} & 0 \\ p_{31}(t) & p_{32}(t) & e^{-dt} \end{pmatrix}, \quad (11)$$

where

$$p_{31}(t) = \int_0^t e_1 a_1 z_d^*(s) \varphi(0, z_d^*(s)) e^{\int_0^s r_1 - a_1 z_d^*(\tau) d\tau} e^{d(t-s)} ds$$

$$p_{32}(t) = \int_0^t e_2 a_2 z_d^*(s) \varphi(0, z_d^*(s)) e^{\int_0^s r_2 - a_2 z_d^*(\tau) d\tau} e^{d(t-s)} ds.$$

It also follows that

$$u(t+) = (1 - \delta_1)u(t), \quad v(t+) = (1 - \delta_2)v(t), \quad w(t+) = (1 - \delta_3)w(t), \quad (12)$$

for $t = (n + l - 1)T+$, while

$$u(t+) = u(t), \quad v(t+) = v(t), \quad w(t+) = w(t), \quad (13)$$

for $t = nT$. Hence, if the eigenvalues of the monodromy matrix

$$\mathcal{M} = \begin{pmatrix} 1 - \delta_1 & 0 & 0 \\ 0 & 1 - \delta_2 & 0 \\ 0 & 0 & 1 - \delta_3 \end{pmatrix} \Phi_L(T)$$

have absolute values less than one, then E^* is locally stable. Since the eigenvalues of \mathcal{M} are

$$\begin{aligned} \lambda_1 &= (1 - \delta_1)e^{\int_0^T r_1 - a_1 z_d^*(t) \varphi(0, z_d^*(t)) dt} > 0; \\ \lambda_2 &= (1 - \delta_2)e^{\int_0^T r_2 - a_2 z_d^*(t) \varphi(0, z_d^*(t)) dt} > 0; \\ \lambda_3 &= (1 - \delta_3)e^{-dT} \in (0, 1), \end{aligned}$$

it follows that E^* is locally stable provided that (7) holds.

We shall now prove that E^* is globally asymptotically stable provided that (8) and (9) hold. Note also that

$$\begin{aligned} &\int_0^T z_d^*(t) \varphi(0, z_d^*(t)) dt \\ &> \max \left(\int_0^T z_d^*(t) \varphi \left(\frac{r_1}{a_{11}}, z_d^*(t) \right) dt, \int_0^T z_d^*(t) \varphi \left(\frac{r_2}{a_{22}}, z_d^*(t) \right) dt \right), \end{aligned}$$

due to (H1).

Let us choose $\varepsilon > 0$ such that

$$\begin{aligned} \xi_1 &= (1 - \delta_1)e^{\int_0^T r_1 - a_1(z_d^*(t) - \varepsilon) \varphi \left(\frac{r_1}{a_{11}} + \varepsilon, z_d^*(t) - \varepsilon \right) dt} < 1; \\ \xi_2 &= (1 - \delta_2)e^{\int_0^T r_2 - a_2(z_d^*(t) - \varepsilon) \varphi \left(\frac{r_2}{a_{22}} + \varepsilon, z_d^*(t) - \varepsilon \right) dt} < 1. \end{aligned}$$

We note that $z'(t) \geq -dz(t)$ and consequently, from the comparison lemma for systems of impulsively perturbed ordinary differential inequalities (see [2]), it follows that $z(t) \geq z_1(t)$ for all $t \geq 0$, where z_1 satisfies

$$\begin{cases} z_1'(t) = -dz_1(t), & t \neq nT, t \neq (n + l - 1)T; \\ \Delta z_1(t) = -\delta_3 z_1(t), & t = (n + l - 1)T; \\ \Delta z_1(t) = \mu, & t = nT; \\ z_1(0+) = z(0+). \end{cases} \quad (14)$$

From Lemma 3.5, it follows that $z(t) \geq z_d^*(t) - \varepsilon$ for t large enough; for the sake of simplicity, let us suppose that $z(t) \geq z_d^*(t) - \varepsilon$ for all $t \geq 0$. Similarly, since $x'(t) \leq x(t)(r_1 - a_{11}x(t))$, it follows that $x(t) \leq x_1(t)$ for all $t \geq 0$, where x_1 satisfies

$$\begin{cases} x_1'(t) = x_1(t)(r_1 - a_{11}x_1(t)), & t \neq (n+l-1)T; \\ \Delta x_1(t) = -\delta_1 x_1(t), & t = (n+l-1)T; \\ x_1(0+) = x(0+). \end{cases} \quad (15)$$

Since $\limsup_{t \rightarrow \infty} x_1(t) \leq \frac{r_1}{a_{11}}$, it then follows that $x(t) \leq \frac{r_1}{a_{11}} + \varepsilon$ for t large enough; for the sake of simplicity, let us suppose that $x(t) \leq \frac{r_1}{a_{11}} + \varepsilon$ for all $t > 0$. By the monotonicity assumptions on φ , it is seen that

$$\begin{cases} x'(t) \leq x(t) \left[r_1 - a_{11}(z_d^*(t) - \varepsilon) \varphi \left(\frac{r_1}{a_{11}} + \varepsilon, z_d^*(t) - \varepsilon \right) \right], & t \neq (n+l-1)T; \\ x(t+) = (1 - \delta_1)x(t) & t = (n+l-1)T. \end{cases}$$

By integrating the above inequality on $((n+l-1)T, (n+l)T]$, it is seen that

$$x((n+l)T) \leq x((n+l-1)T)(1 - \delta_1) e^{\int_0^T r_1 - a_{11}(z_d^*(t) - \varepsilon) \varphi \left(\frac{r_1}{a_{11}} + \varepsilon, z_d^*(t) - \varepsilon \right) dt},$$

that is, $x((n+l)T) \leq x((n+l-1)T)\xi_1$. Then $x((n+l)T) \leq x(lT)\xi_1^n$ and consequently

$$x((n+l)T) \rightarrow 0 \quad \text{for } n \rightarrow \infty. \quad (16)$$

Since

$$0 < x(t) < x((n+l-1)T)e^{r_1 T} \quad \text{for } t \in ((n+l-1)T, (n+l)T]$$

it follows from (16) that

$$x(t) \rightarrow 0 \quad \text{for } t \rightarrow \infty.$$

In a similar fashion, one may prove that $y(t) \rightarrow 0$ as $t \rightarrow \infty$.

We now prove that $z(t) - z_d^*(t) \rightarrow 0$ as $t \rightarrow \infty$. To this purpose, let $\varepsilon' > 0$ be such that $\varepsilon' \varphi(\varepsilon', 0) < \frac{d}{e_1 a_1 + e_2 a_2}$. Then, using the convergence results above, it follows that there is $\tilde{T} > 0$ such that $0 < x(t), y(t) < \varepsilon'$ for $t \geq \tilde{T}$; without loss of generality, we may suppose that

$$0 < x(t), y(t) < \varepsilon' \quad \text{for } t \geq 0.$$

One then has

$$-dz(t) \leq z'(t) \leq -[d - e_1 a_1 \varepsilon' \varphi(\varepsilon', 0) - e_2 a_2 \varepsilon' \varphi(\varepsilon', 0)] z(t)$$

for $t \neq (n+l-1)T, t \neq nT$. Let us denote

$$\tilde{z}^* = z_{d - e_1 a_1 \varepsilon' \varphi(\varepsilon', 0) - e_2 a_2 \varepsilon' \varphi(\varepsilon', 0)}^*$$

Using a comparison argument and Lemma 3.5, it follows that $z_1(t) \leq z(t) \leq z_2(t)$ and $z_1(t) - z_d^*(t) \rightarrow 0, z_2(t) - \tilde{z}^*(t) \rightarrow 0$ as $t \rightarrow \infty$.

Let now $\varepsilon_1 > 0$. It follows that

$$z^*(t) - \varepsilon_1 \leq z(t) \leq \tilde{z}^*(t) + \varepsilon_1$$

for t large enough. Since $\sup_{t \in [0, T]} |z_d^*(t) - \tilde{z}^*(t)| \rightarrow 0$ for $\varepsilon' \rightarrow 0$, again from Lemma 3.5, and ε_1 is arbitrary, it follows that $z(t) - z_d^*(t) \rightarrow 0$ as $t \rightarrow \infty$, which ends the proof of the global stability result. \square

5. THE PERMANENCE OF THE SYSTEM

In this section, we shall study the permanence of (S). To this purpose, we introduce the following definition.

Definition 5.1. *The system (S) is said to be permanent (uniformly persistent) if there are $m, M > 0$ such that for each solution of (S) with strictly positive initial data $x(0), y(0), z(0)$, it follows that there is $T_0 > 0$ such that $m \leq x(t), y(t), z(t) \leq M$ for all $t \geq T_0$. Here, T_0 may depend upon the initial data, but m and M do not.*

In biological terms, if (S) is permanent, then the pests and the predator will coexist, none of them facing extinction or growing indefinitely. Of course, the permanence of (S) excludes any kind of stability of the prey-free periodic solution. We now address the permanence of (S).

Theorem 5.1. *The system (S) is permanent provided that*

$$d > \max \left(e_1 a_1 \frac{r_1}{a_{11}} \varphi \left(\frac{r_1}{a_{11}}, 0 \right), e_2 a_2 \frac{r_2}{a_{22}} \varphi \left(\frac{r_2}{a_{22}}, 0 \right) \right), \tag{17}$$

$$\int_0^T z_d^*(t) \varphi(0, z_d^*(t)) dt < \min(A, B), \tag{18}$$

where

$$A = \frac{\ln(1 - \delta_1) + \left(r_1 - a_{12} \frac{r_2}{a_{22}} \right) T}{a_1}, \tag{19}$$

$$B = \frac{\ln(1 - \delta_2) + \left(r_2 - a_{21} \frac{r_1}{a_{11}} \right) T}{a_2} \tag{20}$$

Proof. Suppose that $(x(\cdot), y(\cdot), z(\cdot))$ is a solution of (S) which starts with strictly positive initial data $x(0), y(0), z(0)$. From Lemma 3.3, there is a constant $M > 0$ such that $x(t) \leq M, y(t) \leq M, z(t) \leq M$ for $t \geq 0$. Also, as done above, we note that $z(t) > z^*(t) - \varepsilon'$ for t large enough, where $\varepsilon' \in (0, \frac{\mu e^{-dT}(1-\delta_3)}{1-e^{-dT}(1-\delta_3)})$. Consequently,

$z(t) \geq m_3$ for t large enough, where

$$m_3 = \frac{\mu e^{-dT}(1 - \delta_3)}{1 - e^{-dT}(1 - \delta_3)} - \varepsilon'.$$

We then only need to find $\bar{m}_1 > 0$ and $\bar{m}_2 > 0$ such that $x(t) \geq \bar{m}_1$ and $y(t) \geq \bar{m}_2$ for t large enough.

First, let us choose $m_1 > 0$, $m_2 > 0$ and ε small enough, so that

$$m_1 < \frac{r_1}{a_{11}} + \varepsilon, \quad m_2 < \frac{r_2}{a_{22}} + \varepsilon \quad (21)$$

$$d > \max(E_1, E_2), \quad (22)$$

where

$$E_1 = e_1 a_1 \left(\frac{r_1}{a_{11}} + \varepsilon \right) \varphi \left(\frac{r_1}{a_{11}} + \varepsilon, 0 \right) + e_2 a_2 m_2 \varphi(m_2, 0)$$

$$E_2 = e_1 a_1 m_1 \varphi(m_1, 0) + e_2 a_2 \left(\frac{r_2}{a_{22}} + \varepsilon \right) \varphi \left(\frac{r_2}{a_{22}} + \varepsilon, 0 \right)$$

and

$$\theta_1 = (1 - \delta_1) e^{\int_0^T r_1 - a_{11} m_1 - a_{12} \left(\frac{r_2}{a_{22}} + \varepsilon \right) - a_1 z_1^*(t) \varphi(0, z_1^*(t)) dt} > 1$$

$$\theta_2 = (1 - \delta_2) e^{\int_0^T r_2 - a_{22} m_2 - a_{21} \left(\frac{r_1}{a_{11}} + \varepsilon \right) - a_2 z_2^*(t) \varphi(0, z_2^*(t)) dt} > 1$$

where

$$z_1^* = z^*_{d - e_1 a_1 m_1 \varphi(m_1, 0) - e_2 a_2 \left(\frac{r_2}{a_{22}} + \varepsilon \right) \varphi \left(\frac{r_2}{a_{22}} + \varepsilon, 0 \right)}$$

$$z_2^* = z^*_{d - e_1 a_1 \left(\frac{r_1}{a_{11}} + \varepsilon \right) \varphi \left(\frac{r_1}{a_{11}} + \varepsilon, 0 \right) - e_2 a_2 m_2 \varphi(m_2, 0)}.$$

We first prove that there are $t_1, t_2 > 0$ such that $x(t_1) \geq m_1$ and $x(t_2) \geq m_2$. Suppose that this is not the case. Then we are in one of the following situations.

- (1) $x(t) \leq m_1$ for all $t \geq 0$ but $y(t_2) > m_2$ for some $t_2 \geq 0$.
- (2) $y(t) \leq m_2$ for all $t \geq 0$ but $x(t_1) > m_1$ for some $t_1 \geq 0$.
- (3) $x(t) \leq m_1$ and $y(t) \leq m_2$ for all $t \geq 0$.

Case (1). Let us choose $\eta_1 > 0$ such that

$$\theta'_1 = (1 - \delta_1) e^{\int_0^T r_1 - a_{11} m_1 - a_{12} \left(\frac{r_2}{a_{22}} + \varepsilon \right) - a_1 (z_1^*(t) + \eta_1) \varphi(0, z_1^*(t) + \eta_1) dt} > 1.$$

One then has

$$z'(t) \leq -z(t) \left[d - e_1 a_1 m_1 \varphi(m_1, 0) - e_2 a_2 \left(\frac{r_2}{a_{22}} + \varepsilon \right) \varphi \left(\frac{r_2}{a_{22}} + \varepsilon, 0 \right) \right]$$

for $t \neq (n + l - 1)T, t \neq nT$, while

$$z(t+) = (1 - \delta_3)z(t) \quad \text{for } t = (n + l - 1)T, \quad z(t+) = z(t) + \mu \quad \text{for } t = nT.$$

Consequently, there is $n_1 \in \mathbb{N}$ such that $z(t) \leq z_1^* + \eta_1$ for $t \geq n_1 T$. One then has

$$x'(t) \geq x(t) \left[r_1 - a_{11} m_1 - a_{12} \left(\frac{r_2}{a_{22}} + \varepsilon \right) - a_1 (z_1^*(t) + \eta_1) \varphi(0, z_1^*(t) + \eta_1) \right]$$

for $t \neq (n + l)T, t \geq n_1 T$, while

$$x(t+) = (1 - \delta_1)x(t) \quad \text{for } t = (n + l - 1)T.$$

By integrating the above on $((n + l - 1)T, (n + l)T]$, $n \geq n_1 + 1$, one sees that

$$x((n + l)T) \geq x((n + l - 1)T)\theta'_1. \quad (23)$$

Consequently, $x((n + k + l)T) \geq \theta_1^k x((n + l)T) \rightarrow \infty$ as $k \rightarrow \infty$, which is a contradiction, since $x(t) \leq m_1$ for all $t \geq 0$. The same argument can be applied for case (2).

Case (3). One then has

$$z'(t) \leq -z(t) [d - e_1 a_1 m_1 z(t) \varphi(m_1, 0) - e_2 a_2 m_2 \varphi(m_2, 0)]$$

for $t \neq (n + l - 1)T, t \neq nT$ and consequently there is $n_2 \in \mathbb{N}$ such that $z(t) \leq z_3^*(t) + \eta_1$ for $t \geq n_2 T$, where

$$z_3^* = z_{d - e_1 a_1 m_1 z(t) \varphi(m_1, 0) - e_2 a_2 m_2 \varphi(m_2, 0)}^*$$

and η_1 is chosen as in Case (1). Note that $z_3^*(t) \leq z_1^*(t)$, $z_3^*(t) \leq z_2^*(t)$ for all $t \geq 0$ and

$$\theta_3 = (1 - \delta_1) e^{\int_0^T r_1 - a_{11} m_1 - a_{12} m_2 - a_1 (z_3^*(t) + \eta_1) \varphi(0, z_3^*(t) + \eta_1) dt} \geq \theta'_1 > 1.$$

One then has

$$x'(t) \geq x(t) \left[r_1 - a_{11} m_1 - a_{12} m_2 - a_1 (z_3^*(t) + \eta_1) \varphi(0, z_3^*(t) + \eta_1) \right]$$

for $t \neq (n + l)T, t \geq n_2 T$, while

$$x(t+) = (1 - \delta_1)x(t) \quad \text{for } t = (n + l - 1)T.$$

By integrating the above on $((n + l - 1)T, (n + l)T]$, $n \geq n_2 + 1$, one sees that

$$x((n + l)T) \geq x((n + l - 1)T)\theta_3,$$

and consequently, as in Case (1), $x((n+k+l)T) \geq \theta_3^k x((n+l)T) \rightarrow \infty$ as $k \rightarrow \infty$, which is again a contradiction.

To summarize, there are $t_1, t_2 \geq 0$ such that $x(t_1) \geq m_1, y(t_2) \geq m_2$, which is a first step towards proving the persistence of (S). We now prove that the x -population is persistent.

If $x(t) \geq m_1$ for all $t \geq t_1$, there is nothing left to prove. Otherwise, $x(t) < m_1$ for some $t > t_1$. Let us denote $s_1 = \inf \{t > t_1; x(t) < m_1\}$. If $s_1 \neq (n+l-1)T$, then $x(s_1) = m_1$. If $s_1 = (n+l-1)T$, then not necessarily $x(s_1) = m_1$, but $x(s_1) \in [m_1, \frac{m_1}{1-\delta_1}]$. Note that it is not possible that $x(s) < m_1$ for all $s > s_1$, so necessary $x(t) \geq m_1$ for some $t \geq s_1$. By a similar argument, one may construct a sequence $(s_n)_{n \geq 2}$ such that

$$1 \quad x(s_{2k+1}) \in [m_1, \frac{m_1}{1-\delta_1}], x(s_{2k}) = m_1.$$

$$2 \quad x(s) \leq m_1 \text{ for } s \in (s_{2k+1}, s_{2k+2}).$$

$$3 \quad x(s) > m_1 \text{ for } s \in (s_{2k}, s_{2k+1}),$$

showcasing the fact that x oscillates about m_1 .

We now show that $\bar{T} = \sup \{s_{2k} - s_{2k-1}; k \in \mathbb{N}^*\} < \infty$. Suppose that this is not the case. Then there is $(k_j)_{j \geq 1}$ such that $s_{2k_j} - s_{2k_j-1} > j$. Consequently, in a way similar to the derivation of (23), it follows that

$$x(s_{2k}) \geq x(s_{2k-1})\theta_3^j e^{-2r_1 T},$$

which is a contradiction, as $\theta_3^j \rightarrow \infty$ for $j \rightarrow \infty$ and $x(s_{2k}) = m_1$. It then follows that

$$x'(t) \geq x(t) \left[r_1 - a_{11}m_1 - a_{12} \left(\frac{r_2}{a_{22}} + \varepsilon \right) - a_1 M \varphi(0, M) \right]$$

for $t \in (s_{2k+1}, s_{2k+2})$, and consequently

$$x(s) \geq \tilde{m}_1 \quad \text{for } s \in (s_{2k+1}, s_{2k+2}),$$

where

$$\tilde{m}_1 = m_1 e^{\left[r_1 - a_{11}m_1 - a_{12} \left(\frac{r_2}{a_{22}} + \varepsilon \right) - a_1 M \varphi(0, M) \right] T}.$$

Putting $\bar{m}_1 = \min(\tilde{m}_1, m_1)$, it follows that $x(t) \geq \bar{m}_1$ for t large enough, so the x -population is persistent. By a similar argument, one may prove that the y -population is persistent, which ends the proof of the persistence result. \square

From the proofs of Theorems 4.1 and 5.1, it can also be seen that the following result holds.

Corolar 5.1. Let $(x(\cdot), y(\cdot), z(\cdot))$ be a solution of (S) starting with strictly positive initial data. Then x and z are permanent and $y(t) \rightarrow 0$ as $t \rightarrow \infty$ provided that

$$\int_0^T z_d^*(t) \varphi\left(\frac{r_2}{a_{22}}, z_d^*(t)\right) dt > \frac{\ln(1 - \delta_2) + r_2 T}{a_2}$$

$$d > e_1 a_1 \frac{r_1}{a_{11}} \varphi\left(\frac{r_1}{a_{11}}, 0\right)$$

$$\int_0^T z_d^*(t) \varphi(0, z_d^*(t)) dt < \frac{(r_1 - a_{12} \frac{r_2}{a_{22}}) T + \ln(1 - \delta_1)}{a_1}.$$

A similar result proving the persistence of the y -population and the extinction of the x -population can be derived in the same manner. \square

6. BIOLOGICAL SIGNIFICANCE AND CONCLUDING REMARKS

First, let us rewrite the equation for the prey population x as

$$x'(t) = x(t)[r_1 - a_{11}x(t) - a_{12}y(t) - a_1z(t)\varphi(x(t), z(t))].$$

It is seen that the integral $a_1 \int_0^T z_d^*(t) \varphi(0, z_d^*(t)) dt$ approximates the per capita loss of x -prey in a period T due to predation when the size of the x -population approaches 0 and rT approximates the per capita production of newborn x -prey individuals in the same conditions and in the same amount of time, while $\ln(1 - \delta_1)$ is a correction term which accounts for the x -prey loss due to pesticide spraying. A similar interpretation holds for y -population.

Consequently, when the sizes of the prey populations approach 0 and the local stability condition (7) holds, the losses of both prey populations in a period exceeds their growths in the same amount of time and the prey populations cannot escape extinction. The meanings of (8) and (9) are similar, but this time $\int_0^T z_d^*(t) \varphi\left(\frac{r_1}{a_{11}}, z_d^*(t)\right) dt$ represents the minimal (at carrying capacity) per capita loss of x -pests in a period T due to predation, the other integral term in (9) having a similar meaning, so the controls are able to keep the pests in check even at higher densities and the global stability of the prey-free periodic solution follows as a consequence.

Regarding the permanence property, conditions (17), (18) express the fact that per capita predation of one prey is weak even at high densities of the other prey (one should remember that there is competition between the prey populations, so high density of the other prey is an unfavorable circumstance for a given prey). Also, note that the predator population always escapes extinction due to the pulsed supply of predator individuals which occurs at $t = nT$ and only the survival the prey populations is actually of concern. In this regard, condition (17) asserts that the death rate of the predator population is large enough and consequently the size of the predator population does not grow too large to affect the survival of the prey populations.

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