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The onset of positive periodic solutions for a biochemical pest management model

Paul Georgescu · Hong Zhang

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Abstract In this paper, the bifurcation of nontrivial periodic solutions for an impulsively perturbed system of ordinary differential equations which models an integrated pest management strategy is studied by means of a fixed point approach. A biological control, consisting in the periodic release of infective pests, and a chemical control, consisting in pesticide spraying, are employed to maintain susceptible pests below an acceptable level. It is assumed that the biological and chemical control act with the same periodicity, but not in the same time. It is then shown that if the constant amount of infective pests released each time reaches a certain threshold value, then the trivial susceptible pest-eradication periodic solution loses its stability, which is transferred to a newly emerging nontrivial periodic solution.

Keywords Nontrivial periodic solutions · Bifurcation · Impulsive perturbation · Integrated pest management · Fixed point

Mathematics Subject Classification (2000) 37G15 · 34A37 · 92D25 · 92D40 · 93D40

P. Georgescu (✉)
Department of Mathematics, Gheorghe Asachi Technical University of Iaşi, Bd. Copou 11, 700506 Iaşi, Romania
e-mail: v.p.georgescu@gmail.com

H. Zhang
Department of Mathematics, Jiangsu University, ZhenJiang, Jiangsu 212013, China
e-mail: hongzhang@ujs.edu.cn

H. Zhang
e-mail: cnczzhanghong@163.com

1 Introduction

There are many problems associated with pesticide abuse. In high use areas, pesticides may contaminate the soil, the drinking and ground water, or the air. Also, the accumulation of synthetic pesticides such as DDT in the human food chain and their persistence in the environment may lead to chronic health problems or to loss of environmental biodiversity.

Pesticide resistance has become an issue in recent years, as many species of pests and mites became resistant to pesticides. This phenomenon creates a so-called treadmill syndrome: pests quickly develop resistance to chemicals, encouraging even greater pesticide use, with little useful effects, but causing yet more pest resistance, pest resurgence and secondary pest outbreaks. Comparatively, integrated pest management (IPM) strategies, relying on a combination of a wide array of preventive methods to control pests, including biological, mechanical and chemical controls, have been proven as being economically sustainable and environmental-friendly alternatives to pesticide use.

One of the most often mentioned IPM success stories is the Indonesian program against crop infestation with the rice brown plant hopper (Nilaparvata lugens) in the mid 80's. Progressively, the rice brown plant hopper became resistant to pesticides, while its natural enemies were killed as a result of repeated pesticide application. Consequently, massive infestations have dramatically reduced rice crops. The Indonesian program, based on farmer field schools which taught farmers new ways to control pests, was accompanied by a limitation of pesticide usage, many pesticides being banned by the Indonesian government from use on the rice fields. As a result, rice crops increased by 10%, while pesticide use decreased by 56% in the first five years after this program has been introduced (see Resosudarmo [19]).

Regarding the desired outcome of an IPM strategy, it is to be noted that IPM strategies are about controlling pests, not about eradicating them completely, as the latter may be unfeasible, counterproductive or may damage the ecosystem. In this regard, an IPM strategy is deemed successful (see Stern et al. [21]) if the pest population is stabilized under the so-called economic injury level (EIL), defined as the lowest population density of a pest that will cause economic damage.

Recently, many papers have been devoted to the analysis of mathematical models describing IPM strategies. See, for instance Liu, Zhi and Chen [12], Zhang, Liu and Chen [27], Tang and Chen [23], Liu, Chen and Zhang [13], Georgescu and Moroşanu [7].

An IPM strategy relying on impulsive biological and chemical controls has also been discussed in Zhang, Chen and Georgescu [25]. In [25], it has been assumed that the biological control consists in the periodic release of a constant amount of infective pests, on the grounds that infective pests do not reproduce or damage crops, and the chemical control consists in periodic pesticide spraying, which causes the instantaneous death of fixed proportions of the populations of infective pests and susceptible pests, respectively. The chemical and biological control were assumed to act with the same periodicity, but not in the same time.

Consequently, the existence of a threshold parameter has been established in [25], in the sense that if the amount μ of infective pests released each time is larger than a threshold value μ_C, then the trivial susceptible pest-eradication solution is globally asymptotically stable, while if $\mu < \mu_C$, then the stability of this solution is lost and the system becomes uniformly persistent. The behavior of the system in the critical case $\mu = \mu_C$ has not been discussed in [25].

Impulsively perturbed systems of ODEs have proved in many ways their usefulness in mathematical modeling, being used as models for impulsive vaccination (D'Onofrio [4], Shulgin, Stone and Agur [20]), birth pulses (Tang and Chen [22], Cao and Jin [2]), cancer therapy (Lakmeche and Arino [10], Lakmeche and Arino [11], Panetta [18]), the theory of the chemostat (Funasaki and Kot [5], Zhang, Xiu and Chen [28]), predator-prey interactions (Liu, Liu and Teng [14], Zhang and Chen [24]), management of exploitable resources (Gao, Chen and Sun [6], Zhang, Shuai and Wang [26]) and not only. A general overview of the theory of impulsive ordinary differential equations can be found in Bainov and Simeonov [1].

Most papers cited above use pulsed actions at prescribed moments, that is, it is assumed that the impulsive controls are used in a prescribed periodic fashion, irrespective of the amount of the infected prey (prey, infectives, tumor cells) present. A different approach, in the form of a state-dependent pest management model, in which biological and mechanical controls are employed as soon as the pest population reaches a certain economically significant threshold value has recently been formulated and studied in Meng, Song and Chen [17].

The bifurcation of nontrivial periodic solutions for a general class of two-dimensional impulsively perturbed systems of ordinary differential equations has been studied by Lakmeche and Arino in [10], who also indicated a concrete application of their theoretical framework to a model describing normal-tumor cell interaction originally introduced by Panetta in [18]. Their approach is to introduce an evolution operator which incorporates the effect of impulsive perturbations and to restate the problem of finding nontrivial periodic solutions for the impulsive system as a fixed point problem for this operator. The latter problem is then solved by using a certain projection method. Although this direction is not pursued in their concrete example, note that Lakmeche and Arino's robust approach allows the use of nonlinear impulsive controls.

Their method has been used, among others, by Lu, Chi and Chen in [16] for a predator-pest model controlled by means of pulsed insecticide use and by the same authors in [15] for a SIR epidemic model with horizontal and vertical transmission which is subject to pulsed vaccination. The case in which several impulsive controls are employed has been discussed by Lakmeche and Arino in [11] for a two-dimensional impulsive differential system arising from cancer therapy by means of several drugs with instantaneous effects and by Georgescu, Zhang and Chen in [8] for an IPM model subject to biological and chemical controls.

In this paper, we use the approach devised in [10] and [11], used also in [8], and prove that if the amount μ of infective pests released each time equals a threshold value μ_C, then the trivial periodic solution loses its stability, which is transferred to a nontrivial periodic solution arising via a supercritical bifurcation. Our notations are consistent to those used in [25], as the present paper complements the results therein with the treatment of the critical case $\mu = \mu_C$. Note that the model employed in this paper differs to the one used in [8], as the crowding term in the equation modeling the dynamics of the susceptible class depends on the size of the infective class as well and a different type of incidence rate has been used, of type $Ih(S)$ rather than $g(I)S$.

This paper is organized as follows: in Sect. 2, we formulate our impulsively perturbed pest management model and state a few useful facts about its well-posedness and stability which were previously deduced in [25]. In Sect. 3, a few notions and definitions are introduced and the problem of finding nontrivial periodic solutions is restated as a fixed point problem. The approach towards finding fixed points is given and pursued in Sect. 4. As a result, it is found that a nontrivial periodic solution appears via a supercritical bifurcation. Our theoretical results are then interpreted from an eco-epidemiological point of view in Sect. 5, while certain computations used in the above are deferred to an Appendix.

2 The mathematical model

In what follows, we consider the integrated pest management model analyzed in [25] from the viewpoint of finding sufficient conditions for the global stability of the susceptible pest-eradication periodic solution and for the uniform persistence of the system, respectively. This time, of concern is the critical case, in which the constant amount of pests released each time equals the threshold value found in [25].

We suppose that each pest is either susceptible or infective and we divide the pest population into the infective class I and the susceptible class S. Having in mind the biological motivation indicated in [25], which we shall not reiterate here, we are led to consider the following impulsively controlled system

$$\begin{cases} I'(t) = \beta I(t)\dfrac{S(t)}{1+aS^l(t)} - wI(t), & t \neq (n+\tilde{l}-1)T, t \neq nT; \\ S'(t) = S(t)\left(1 - \dfrac{S(t)+I(t)}{K}\right) & \\ \qquad\qquad - \beta I(t)\dfrac{S(t)}{1+aS^l(t)}, & t \neq (n+\tilde{l}-1)T, t \neq nT; \\ \Delta I(t) = -p_2 I(t), & t = (n+\tilde{l}-1)T; \\ \Delta S(t) = -p_1 S(t), & t = (n+\tilde{l}-1)T; \\ \Delta I(t) = \mu, & t = nT; \\ \Delta S(t) = 0, & t = nT. \end{cases} \tag{1}$$

Here, $T > 0$ characterizes the periodicity of the impulsive controls, $\mu > 0$ represents the constant amount of infective pests released each time, $0 \leq p_2 < 1$ and $0 \leq p_1 < 1$ represent the fixed proportions of infective and susceptible pests, respectively, which are instantaneously killed each time the pesticides are sprayed, $K > 0$ represents the carrying capacity of the environment and $w > 0$ represents the death rate of the infective population. The term $\beta I(t)\frac{S(t)}{1+aS^l(t)}$, $0 < l < 1$, represents the (nonlinear) incidence rate of the infection. Also, $0 < \tilde{l} < 1$, $\Delta\varphi(t) = \varphi(t+) - \varphi(t)$ for $\varphi \in \{S, I\}$, $n \in \mathbb{N}^*$.

Of interest is also the following subsystem of (1) which describes the dynamics of the susceptible pest-eradication state

$$\begin{cases} I'(t) = -wI(t), & t \neq nT, (n+\tilde{l}-1)T; \\ \Delta I(t) = -p_2 I(t), & t = (n+\tilde{l}-1)T; \\ \Delta I(t) = \mu, & t = nT; \\ I(0+) = I_0. \end{cases} \tag{2}$$

It has been observed in [25] that the subsystem formed with the first three equations of (2) has a periodic solution I^* which is globally asymptotically stable. This periodic solution satisfies

$$\begin{cases} I^*(t) = e^{-wt} I^*(0+), & t \in (0, \tilde{l}T]; \\ I^*(t) = e^{-wt} I^*(0+)(1-p_2), & t \in (\tilde{l}T, T], \end{cases} \tag{3}$$

where

$$I^*(0+) = \frac{\mu}{1 - e^{-wT}(1-p_2)}. \tag{4}$$

Another stability results which have been proved in [25] (Theorems 4.2 and 4.4) assert that the susceptible pest-eradication periodic solution $(I^*, 0)$ of (1) is globally asymptotically stable provided that the inequality

$$\mu > \mu_C = \frac{w(T - \ln \frac{1}{1-p_1})(1 - (1-p_2)e^{-wT})}{(\frac{1}{K} + \beta)(1 - p_2 e^{-w\tilde{l}T} - (1-p_2)e^{-wT})} \tag{5}$$

is satisfied, while if $\mu < \mu_C$, then the susceptible pest-eradication solution becomes unstable and the system is uniformly persistent. In the following, we shall reformulate the above inequality in terms of expressions involving $\int_0^T I^*(s)ds$ and study the onset of a nontrivial periodic solution in the limiting case, that is, $\mu = \mu_C$, by means of bifurcation theory methods.

3 The fixed point problem

Our purpose is now to show the onset of a nontrivial periodic solution provided that $\mu = \mu_C$. To this goal, we first reformulate our problem of finding nontrivial periodic solutions into a fixed point problem for an evolution operator which will be defined in what follows. First, we introduce a few notations. We shall denote by $\Phi(t; X_0) = (\Phi_1(t; X_0), \Phi_2(t; X_0))$ the solution of the (unperturbed) system formed with the first two equations in (1) for the initial data $X_0 = (x_0^1, x_0^2)$. To account for the effects of the impulsive biological and chemical controls, we define the control operators $I_1, I_2 : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$I_1(x_1, x_2) = ((1-p_2)x_1, (1-p_1)x_2), \qquad I_2(x_1, x_2) = (x_1 + \mu, x_2).$$

We also define

$$F : [0, \infty) \times [0, \infty) \to \mathbb{R} \times \mathbb{R}, \qquad F(x_1, x_2) = (F_1(x_1, x_2), F_2(x_1, x_2)),$$

with

$$\begin{aligned} &F_1, F_2 : [0, \infty) \times [0, \infty) \to \mathbb{R}, \\ &F_1(x_1, x_2) = \beta x_1 \frac{x_2}{1 + a x_2^l} - w x_1, \\ &F_2(x_1, x_2) = x_2 \left(1 - \frac{x_1 + x_2}{K}\right) - \beta x_1 \frac{x_2}{1 + a x_2^l}. \end{aligned}$$

Let us now define the evolution operator $\Psi : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Psi(T, X_0) = I_2(\Phi((1 - \tilde{l})T; I_1(\Phi(\tilde{l}T; X_0))));$$

and

$$\Psi(T, X_0) = (\Psi_1(T; X_0), \Psi_2(T; X_0)).$$

Then X is a periodic solution of period T for (1) if and only if its initial data $X(0) = X_0$ verifies $\Psi(T, X_0) = X_0$, that is, X_0 is a fixed point for $\Psi(T, \cdot)$. As a result, it suffices to determine the fixed points of $\Psi(T, \cdot)$ in order to find the nontrivial periodic solutions of period T for (1).

We are interested in the bifurcation of nontrivial periodic solutions near the susceptible pest-eradication trivial periodic solution $(I^*, 0)$. To find a necessary condition for bifurcation (and to characterize the stability of the susceptible pest-eradication periodic solution yet again as a byproduct), we need to compute $D_X\Psi(T, X_0)$, where $X_0 = (x_0, 0)$ is the initial data of $(I^*, 0)$, that is, $x_0 = I^*(0+)$, $I^*(0+)$ being given by (4). By the chain rule, it is easy to see that

$$D_X\Psi(T, X) = D_X\Phi((1 - \tilde{l})T; I_1(\Phi(\tilde{l}T; X))) \begin{pmatrix} 1 - p_2 & 0 \\ 0 & 1 - p_1 \end{pmatrix} D_X\Phi(\tilde{l}T; X).$$

By formally deriving the first two equations in (1), we are able to compute $D_X\Phi(t; X_0)$ (see the Appendix) and consequently obtain that the Jacobi matrix of the evolution operator Ψ is given by

$$D_X\Psi(T, X_0) = \begin{pmatrix} d_{11} & d_{12} \\ 0 & d_{22} \end{pmatrix},$$

where

$$\begin{cases} d_{11} = (1-p_2)e^{-wT}, \quad d_{11} \in (0,1); \\ d_{12} = e^{-wT}\left[(1-p_2)\int_0^{\tilde{l}T} \beta I^*(s) e^{(1+w)s-\left(\beta+\frac{1}{K}\right)\int_0^s I^*(\tau)d\tau} ds \right. \\ \qquad \left. +(1-p_1)\int_{\tilde{l}T}^{T} \beta I^*(s) e^{(1+w)s-\left(\beta+\frac{1}{K}\right)\int_0^s I^*(\tau)d\tau} ds \right]; \\ d_{22} = (1-p_1)e^{T-\left(\beta+\frac{1}{K}\right)\int_0^T I^*(s)ds}, \quad d_{22} > 0. \end{cases}$$

Also, as it is known that $(I^*, 0)$ is exponentially stable if and only if the spectral radius $\rho\,(D_X\Psi(T, X_0))$ is less than 1 (see Iooss [9]) and $d_{11} \in (0,1)$, it follows that $(I^*, 0)$ is exponentially stable if and only if

$$(1-p_1)e^{T-\left(\beta+\frac{1}{K}\right)\int_0^T I^*(s)ds} < 1,$$

inequality which can be restated in the form (5) given in [25] since, using (4) and (3), one may easily see that

$$\int_0^T I^*(s)ds = \frac{1}{w}\left(1 - p_2 e^{-w\tilde{l}T} - (1-p_2)e^{-wT}\right)\frac{\mu}{1-e^{-wT}(1-p_2)}.$$

4 The existence of positive periodic solutions

As previously mentioned, to find a nontrivial periodic solution of (1) with period τ and initial data X, one needs to solve the fixed point problem $X = \Psi(\tau, X)$. Since we need to study the bifurcation of nontrivial periodic solutions near the trivial periodic solution $(I^*, 0)$, which has period T and initial data X_0 (where T is yet to be determined and will be characterized in what follows), we first change the variables τ and X by means of

$$\tau = T + \overline{\tau}, \qquad X = X_0 + \overline{X}.$$

Our fixed point problem can now be expressed using the new variables $\overline{\tau}$ and $\overline{X}$ as

$$X_0 + \overline{X} = \Psi(T + \overline{\tau}, X_0 + \overline{X}).$$

To solve this problem, we use the approach employed in [10, 11] and [8], which is based on the use of a projection method. Let us define

$$N(\overline{\tau}, \overline{X}) = X_0 + \overline{X} - \Psi(T + \overline{\tau}, X_0 + \overline{X});$$
$$N(\overline{\tau}, \overline{X}) = (N_1(\overline{\tau}, \overline{X}), N_2(\overline{\tau}, \overline{X})).$$

With this notation, we now have to solve the equation $N(\overline{\tau}, \overline{X}) = 0$. Let us denote

$$D_X N(0, (0,0)) = \begin{pmatrix} a_0' & b_0' \\ c_0' & d_0' \end{pmatrix}.$$

Since

$$D_X N(0,(0,0)) = I_2 - D_X \Psi(T, X_0),$$

one may see that

$$a'_0 = 1 - d_{11}, \qquad b'_0 = -d_{12}, \qquad c'_0 = -d_{21}, \qquad d'_0 = 1 - d_{22}$$

and consequently

$$a'_0 = 1 - (1-p_2)e^{-wT} \tag{6}$$

$$b'_0 = -e^{-wT}\Bigg[(1-p_2)\int_0^{\tilde{l}T} \beta I^*(s) e^{(1+w)s - \left(\beta + \frac{1}{K}\right)\int_0^s I^*(\tau)d\tau} ds$$
$$+ (1-p_1)\int_{\tilde{l}T}^T \beta I^*(s) e^{(1+w)s - \int_0^s \left(\beta + \frac{1}{K}\right) I^*(\tau)d\tau} ds\Bigg]; \tag{7}$$

$$c'_0 = 0; \tag{8}$$

$$d'_0 = 1 - (1-p_1)e^{T - \left(\beta + \frac{1}{K}\right)\int_0^T I^*(s)ds}. \tag{9}$$

A necessary condition for the bifurcation of nontrivial periodic solutions is then

$$\det[D_X N(0,(0,0))] = 0, \tag{10}$$

from which we obtain that $d'_0 = 0$. It then follows that a necessary condition for bifurcation is

$$(1-p_1)e^{T - \left(\beta + \frac{1}{K}\right)\int_0^T I^*(s)ds} = 1. \tag{11}$$

We shall show in what follows that this condition is sufficient as well. To find the nontrivial solutions of the equation $N(\overline{\tau}, \overline{X}) = 0$, we use a projection method. Of course, from all nontrivial solutions, only the positive ones are admissible. From (6)–(9) and (10), it is seen that

$$\dim(\mathrm{Ker}[D_X N(0,(0,0))]) = 1$$

and a basis in $\mathrm{Ker}[D_X N(0,(0,0))]$ is $B_{\mathrm{Ker}} = \{Y_0\}$, $Y_0 = (-\frac{b'_0}{a'_0}, 1)$, while a basis in $\mathrm{Im}[D_X N(0,(0,0))]$ is $B_{\mathrm{Im}} = \{Y_0\}$, $E_0 = (1,0)$. By using the associated direct sum decomposition, the equation $N(\overline{\tau}, \overline{X}) = 0$ can be expressed as

$$N_1(\overline{\tau}, \alpha Y_0 + z E_0) = 0; \tag{12}$$

$$N_2(\overline{\tau}, \alpha Y_0 + z E_0) = 0. \tag{13}$$

Let us denote

$$f_1(\overline{\tau}, \alpha, z) = N_1(\overline{\tau}, \alpha Y_0 + z E_0); \tag{14}$$

$$f_2(\overline{\tau}, \alpha, z) = N_2(\overline{\tau}, \alpha Y_0 + z E_0). \tag{15}$$

We are then led to solve the system

$$\begin{cases} f_1(\overline{\tau}, \alpha, z) = 0; \\ f_2(\overline{\tau}, \alpha, z) = 0. \end{cases}$$

First, since

$$\frac{\partial f_1}{\partial z}(0,0,0) = \frac{\partial N_1}{\partial x_1}(0,(0,0)) = a_0' \neq 0,$$

the equation $f_1(\overline{\tau}, \alpha, z) = 0$ can be solved locally near $(0,0,0)$ with respect to z as a function of α and $\overline{\tau}$, that is, one may find $z = z(\overline{\tau}, \alpha)$ defined in a small vicinity of $(0,0)$ such that $z(0,0) = 0$ and

$$f_1(\overline{\tau}, \alpha, z(\overline{\tau}, \alpha)) = N_1(\overline{\tau}, \alpha Y_0 + z(\overline{\tau}, \alpha) E_0) = 0$$

for $(\overline{\tau}, \alpha)$ in that vicinity. One may also compute the first order partial derivatives of z at $(0,0)$ and find that

$$\begin{cases} \dfrac{\partial z}{\partial \alpha}(0,0) = 0 \\ \dfrac{\partial z}{\partial \overline{\tau}}(0,0) = -\dfrac{w}{a_0'} I^*(T). \end{cases}$$

For details, see the Appendix. For this z, it now remains to study the solvability of the equation

$$f_2(\overline{\tau}, \alpha, z(\overline{\tau}, \alpha)) = 0, \tag{16}$$

or, equivalently, of

$$N_2(\overline{\tau}, \alpha Y_0 + z(\overline{\tau}, \alpha) E_0) = 0. \tag{17}$$

Equation (17) is called the determining equation (see Chow and Hale [3, Sect. 2.4] for details) and, as mentioned above, the number of its solutions equals the number of periodic solutions of (1). We now try to determine the number of solutions of (16) by using Taylor expansions. We denote

$$f(\overline{\tau}, \alpha) = f_2(\overline{\tau}, \alpha, z(\overline{\tau}, \alpha)). \tag{18}$$

First, we observe that

$$f(0,0) = N(0,(0,0)) = 0$$

and

$$\frac{\partial f}{\partial \overline{\tau}}(0,0) = \frac{\partial f}{\partial \alpha}(0,0) = 0$$

(see the Appendix for details), so the first order expansion of f does not suffice. It now remains to compute the second order partial derivatives of f at $(0,0)$. It can be proved (see the Appendix) that

$$A = \frac{\partial^2 f}{\partial \alpha^2}(0,0) = 0$$

$$B = \frac{\partial^2 f}{\partial \alpha \partial \overline{\tau}}(0,0) < 0$$

$$C = \frac{\partial^2 f}{\partial \overline{\tau}^2}(0,0) > 0.$$

By constructing the second order Taylor expansion of f near $(0,0)$, one obtains

$$f(\overline{\tau},\alpha) = B\alpha\overline{\tau} + C\frac{\tau^2}{2} + o(\overline{\tau},\alpha)(\overline{\tau}^2 + \alpha^2).$$

Let us denote $\alpha = k\overline{\tau}$; $k = k(\overline{\tau})$. We see that

$$f(\overline{\tau},\alpha) = \overline{\tau}^2\left(Bk + \frac{C}{2} + o(\overline{\tau},k\overline{\tau})(k^2+1)\right).$$

Since $B < 0$ and $C > 0$, the equation

$$Bk + \frac{C}{2} + o(\overline{\tau},k\overline{\tau})(k^2+1) = 0,$$

has a nontrivial positive solution $k = k(\overline{\tau})$ provided that $\overline{\tau}$ is positive and small enough.

From the above considerations, it is seen that if the necessary condition for bifurcation (11) is satisfied, then a stable positive periodic solution arises via a supercritical bifurcation, so (11) is also sufficient. This discussion may be summarized in the following result, parallel to the one obtained in Georgescu, Zhang and Chen [8] for a model with a different incidence rate and crowding terms.

Theorem 1 *Suppose that condition* (11) *is satisfied. Then there is* $\varepsilon > 0$ *such that for all* $0 < \overline{\tau} < \varepsilon$ *there is a stable positive nontrivial periodic solution of* (1) *with initial data* $X_0 + \alpha(\overline{\tau})Y_0 + z(\overline{\tau},\alpha(\overline{\tau}))E_0$ *and period* $T + \overline{\tau}$.

Here, X_0, Y_0, E_0, z, $\overline{\tau}$ are as introduced above.

5 Concluding remarks

In this paper, the bifurcation of nontrivial solutions for an integrated pest management introduced in [25] is studied in the case in which the constant amount of infective pests μ released each time reaches a certain threshold value μ_C.

The problem of finding nontrivial periodic solutions is first reformulated as a fixed point problem for an evolution operator which incorporates the effect of the impulsive perturbations, the latter being then solved through a projection method. It is known from [25] that if $\mu > \mu_C$, then the susceptible pest-eradication solution is globally asymptotically stable, while if $\mu < \mu_C$, then the system becomes permanent and the stability of the susceptible pest-eradication solution is lost. The critical case $\mu = \mu_C$ is now discussed and it is shown that this loss of stability is due to the onset of a nontrivial periodic solution which appears via a supercritical bifurcation.

Note that, as observed in [8], the usual concept of a basic reproduction number does not carry out well to a pulsed model with constant supply of infectives. This is because what is at stake now is not the perpetuation of the infection, which is always realized due to the constant pulsed supply of infectives, but the perpetuation of the susceptibles. In this regard, condition $\mu = \mu_C$ can be rephrased as

$$T + \ln(1 - p_1) = \left(\beta + \frac{1}{K}\right) \int_0^T I^*(s)ds,$$

which is a balance condition for susceptibles near the susceptible-free periodic solution, rather than for the infectives near the infective-free equilibrium, as it is usual when threshold situations involving basic reproduction numbers are considered. To this purpose, let us observe that when the system approaches the susceptible pest-eradication trivial periodic solution $(I^*, 0)$, then T approximates the total production of newborn susceptible pests in a period, while $(\beta + \frac{1}{K}) \int_0^T I^*(s)ds$ approximates the total loss of susceptible pests due to their movement into the infective class $(\beta \int_0^T I^*(s)ds)$ or due to crowding effects $(\frac{1}{K} \int_0^T I^*(s)ds)$. Also, the correction term $\ln(1 - p_1)$ should be added in order to account for the loss of susceptibles due to pesticide spraying. Note that all quantities above are "per-susceptible" quantities. Note also the importance of the parameter μ, both from a conceptual and a computational point of view.

The balance equality can be termed as

$$\left(1 + \frac{\ln(1 - p_1)}{T}\right) \frac{1}{\beta + \frac{1}{K}} = \frac{1}{T} \int_0^T I^*(s)ds.$$

It is then seen, combining our result with those obtained in [25] and defining the epidemic threshold I_C as

$$I_C = \left(1 + \frac{\ln(1 - p_1)}{T}\right) \frac{1}{\beta + \frac{1}{K}},$$

that if the average of I^* over a period T is larger than I_C, then the susceptible pest-eradication periodic solution $(I^*, 0)$ is globally asymptotically stable, while if the average of I^* over a period T is smaller than I_C, then the stability of the susceptible pest-eradication periodic solution $(I^*, 0)$ is lost and the system becomes uniformly persistent. Finally, if the average of I^* over a period T equals I_C, it is seen that this loss of stability is due to the onset of a nontrivial periodic solution which appears via a supercritical bifurcation.

Appendix

A.1 The first order partial derivatives of Φ_1, Φ_2 and the Jacobi matrix of Ψ

To obtain the first order partial derivatives of Φ_1 and Φ_2, we start by formally deriving

$$\frac{d}{dt}\left(\Phi(t; X_0)\right) = F(\Phi(t; X_0)).$$

It is seen that

$$\frac{d}{dt}\left(D_X\Phi(t; X_0)\right) = D_X F(\Phi(t; X_0)) D_X\Phi(t; X_0). \tag{19}$$

Also, one may easily observe that

$$\Phi(t; X_0) = (\Phi_1(t; X_0), 0). \tag{20}$$

We may then deduce from (19) and (20) that

$$\frac{d}{dt}\begin{pmatrix} \frac{\partial \Phi_1}{\partial x_1} & \frac{\partial \Phi_1}{\partial x_2} \\ \frac{\partial \Phi_2}{\partial x_1} & \frac{\partial \Phi_2}{\partial x_2} \end{pmatrix}(t; X_0) = \begin{pmatrix} -w & \beta\Phi_1(t; X_0) \\ 0 & 1-\left(\beta+\frac{1}{K}\right)\Phi_1(t; X_0) \end{pmatrix}\begin{pmatrix} \frac{\partial \Phi_1}{\partial x_1} & \frac{\partial \Phi_1}{\partial x_2} \\ \frac{\partial \Phi_2}{\partial x_1} & \frac{\partial \Phi_2}{\partial x_2} \end{pmatrix}(t; X_0) \tag{21}$$

to which we associate the initial condition

$$D_X\Phi(0; X_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{22}$$

One easily deduces from (21) that

$$\frac{d}{dt}\left(\frac{\partial \Phi_2}{\partial x_1}(t; X_0)\right) = \left(1-\left(\beta+\frac{1}{K}\right)\Phi_1(t; X_0)\right)\frac{\partial \Phi_2}{\partial x_1}(t; X_0)$$

and consequently

$$\frac{\partial \Phi_2}{\partial x_1}(t; X_0) = e^{t-(\beta+\frac{1}{K})\int_0^t \Phi_1(s; X_0)ds}\frac{\partial \Phi_2}{\partial x_1}(0; X_0),$$

which implies, using (22), that

$$\frac{\partial \Phi_2}{\partial x_1}(t; X_0) = 0 \quad \text{for } t \geq 0. \tag{23}$$

By substituting (23) into (21), it follows that

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial \Phi_1}{\partial x_1}(t; X_0)\right) = -w\frac{\partial \Phi_1}{\partial x_1}(t; X_0) \\ \frac{d}{dt}\left(\frac{\partial \Phi_1}{\partial x_2}(t; X_0)\right) = -w\frac{\partial \Phi_1}{\partial x_2}(t; X_0) + \beta\Phi_1(t; X_0)\frac{\partial \Phi_2}{\partial x_2}(t; X_0) \\ \frac{d}{dt}\left(\frac{\partial \Phi_2}{\partial x_2}(t; X_0)\right) = \left(1-\left(\beta+\frac{1}{K}\right)\Phi_1(t; X_0)\right)\frac{\partial \Phi_2}{\partial x_2}(t; X_0) \end{cases} \tag{24}$$

From an easy integration of (24), one obtains using the initial conditions (22) that

$$\begin{cases} \dfrac{\partial \Phi_1}{\partial x_1}(t; X_0) = e^{-wt} \\ \dfrac{\partial \Phi_1}{\partial x_2}(t; X_0) = e^{-wt} \displaystyle\int_0^t \beta \Phi_1(s; X_0) e^{(1+w)s - \left(\beta + \frac{1}{K}\right) \int_0^s \Phi_1(\tau; X_0) d\tau} ds \\ \dfrac{\partial \Phi_2}{\partial x_2}(t; X_0) = e^{t - \left(\beta + \frac{1}{K}\right) \int_0^t \Phi_1(s; X_0) ds}. \end{cases}$$

We now compute the Jacobi matrix of the evolution operator Ψ. Since, by the chain rule,

$$D_X \Psi(T, X) = D_X \Phi((1 - \tilde{l})T; I_1(\Phi(\tilde{l}T; X))) \begin{pmatrix} 1 - p_2 & 0 \\ 0 & 1 - p_1 \end{pmatrix} D_X \Phi(\tilde{l}T; X),$$

it follows that

$$D_X \Psi(T, X_0) = \begin{pmatrix} d_{11} & d_{12} \\ 0 & d_{22} \end{pmatrix},$$

with

$$d_{11} = (1 - p_2) \frac{\partial \Phi_1}{\partial x_1}((1 - \tilde{l})T; I_1(\Phi(\tilde{l}T; X_0))) \frac{\partial \Phi_1}{\partial x_1}(\tilde{l}T; X_0); \tag{25}$$

$$\begin{aligned} d_{12} = {} & (1 - p_2) \frac{\partial \Phi_1}{\partial x_1}((1 - \tilde{l})T; I_1(\Phi(\tilde{l}T; X_0))) \frac{\partial \Phi_1}{\partial x_2}(\tilde{l}T; X_0) \\ & + (1 - p_1) \frac{\partial \Phi_1}{\partial x_2}((1 - \tilde{l})T; I_1(\Phi(\tilde{l}T; X_0))) \frac{\partial \Phi_2}{\partial x_2}(\tilde{l}T; X_0); \end{aligned} \tag{26}$$

$$d_{22} = (1 - p_1) \frac{\partial \Phi_2}{\partial x_2}((1 - \tilde{l})T; I_1(\Phi(\tilde{l}T; X_0))) \frac{\partial \Phi_2}{\partial x_2}(\tilde{l}T; X_0). \tag{27}$$

As a consequence, one may deduce

$$\begin{aligned} d_{11} = {} & (1 - p_2) e^{-wT}; \\ d_{12} = {} & (1 - p_2) e^{-w(1-\tilde{l})T} e^{-w\tilde{l}T} \int_0^{\tilde{l}T} \beta I^*(s) e^{(1+w)s - (\beta + \frac{1}{K}) \int_0^s I^*(\tau) d\tau} ds \\ & + (1 - p_1) e^{-w(1-\tilde{l})T} \int_0^{(1-\tilde{l})T} \beta \Phi_1(s; I_1(\Phi(\tilde{l}T; X_0))) \\ & \times e^{(1+w)s - (\beta + \frac{1}{K}) \int_0^s \Phi_1(\tau; I_1(\Phi(\tilde{l}T; X_0))) d\tau} ds \cdot e^{\tilde{l}T - (\beta + \frac{1}{K}) \int_0^{\tilde{l}T} I^*(s) ds} \\ = {} & e^{-wT} \Bigg[(1 - p_2) \int_0^T \beta I^*(s) e^{(1+w)s - (\beta + \frac{1}{K}) \int_0^s I^*(\tau) d\tau} ds \\ & + (1 - p_1) \int_0^{(1-\tilde{l})T} \beta I^*(s + \tilde{l}T) e^{(1+w)(s + \tilde{l}T) - (\beta + \frac{1}{K}) \int_0^{s + \tilde{l}T} I^*(\tau) d\tau} ds \Bigg] \end{aligned} \tag{28}$$

$$= e^{-wT}\left[(1-p_2)\int_0^{\tilde{l}T}\beta I^*(s)e^{(1+w)s-(\beta+\frac{1}{K})\int_0^s I^*(\tau)d\tau}ds\right.$$

$$\left.+(1-p_1)\int_{\tilde{l}T}^{T}\beta I^*(s)e^{(1+w)s-(\beta+\frac{1}{K})\int_0^s (I^*(\tau)d\tau}ds\right]; \tag{29}$$

$$d_{22} = (1-p_1)e^{(1-\tilde{l})T-(\beta+\frac{1}{K})\int_0^{(1-\tilde{l})T}\Phi_1(s;I_1(\Phi(\tilde{l}T;X_0)))ds}\cdot e^{\tilde{l}T-(\beta+\frac{1}{K})\int_0^{\tilde{l}T}I^*(s)ds}$$

$$= (1-p_1)e^{T-(\beta+\frac{1}{K})\int_0^{(1-\tilde{l})T}I^*(s+\tilde{l}T)ds-(\beta+\frac{1}{K})\int_0^{\tilde{l}T}I^*(s)ds}$$

$$= (1-p_1)e^{T-(\beta+\frac{1}{K})\int_0^T I^*(s)ds}. \tag{30}$$

A.2 The first order partial derivatives of z and f at $(0,0)$

The following computations are largely similar to those in the Sects. 6.2 and 6.3 of [8]. We give them here for the sake of completeness.

From the implicit function theorem applied to (12), one may deduce that

$$\frac{\partial N_1}{\partial x_1}(0,(0,0))\left(-\frac{b_0'}{a_0'}\right)+\frac{\partial N_1}{\partial x_2}(0,(0,0))+\frac{\partial N_1}{\partial x_1}(0,(0,0))\frac{\partial z}{\partial \alpha}(0,0)=0$$

and consequently

$$a_0'\left(-\frac{b_0'}{a_0'}\right)+b_0'+a_0'\frac{\partial z}{\partial \alpha}(0,0)=0,$$

from which we obtain that

$$\frac{\partial z}{\partial \alpha}(0,0)=0.$$

By the implicit function theorem applied to (13) and

$$\frac{\partial \Phi_2}{\partial x_1}(\tilde{l}T;X_0)=0, \tag{31}$$

$$\frac{\partial \Phi_2}{\partial \overline{\tau}}(\tilde{l}T;X_0)=0, \tag{32}$$

it follows that

$$\frac{\partial z}{\partial \overline{\tau}}(0,0)=\frac{\partial \Phi_1}{\partial \overline{\tau}}((1-\tilde{l})T;I_1(\Phi(\tilde{l}T;X_0)))(1-\tilde{l})$$
$$+\frac{\partial \Phi_1}{\partial x_1}((1-\tilde{l})T;I_1(\Phi(\tilde{l}T;X_0)))(1-p_2)$$
$$\times\left(\frac{\partial \Phi_1}{\partial \overline{\tau}}(\tilde{l}T;X_0)\cdot\tilde{l}+\frac{\partial \Phi_1}{\partial x_1}(\tilde{l}T;X_0)\frac{\partial z}{\partial \overline{\tau}}(0,0)\right).$$

Consequently, from (6), it follows that

$$\frac{\partial z}{\partial \overline{\tau}}(0,0)=\frac{1}{a_0'}\left[\frac{\partial \Phi_1}{\partial \overline{\tau}}((1-\tilde{l})T;I_1(\Phi(\tilde{l}T;X_0)))(1-\tilde{l})\right.$$

$$+\frac{\partial \Phi_1}{\partial x_1}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0)))(1-p_2)\frac{\partial \Phi_1}{\partial \overline{\tau}}(\tilde{l}T; X_0)\cdot \tilde{l}\Bigg],$$

which implies that

$$\frac{\partial z}{\partial \overline{\tau}}(0,0) = -\frac{w}{a_0'}I^*(T).$$

We now compute the first order partial derivatives of f at $(0,0)$. It is easy to see that

$$\begin{aligned}
&\frac{\partial f}{\partial \alpha}(\overline{\tau},\alpha)\\
&\quad = 1-\frac{\partial \Phi_2}{\partial x_1}((1-\tilde{l})(T+\overline{\tau}); I_1(\Phi(\tilde{l}(T+\overline{\tau}); X_0+\alpha Y_0+z(\overline{\tau},\alpha)E_0)))\\
&\qquad \times (1-p_2)\left(\frac{\partial \Phi_1}{\partial x_1}(\tilde{l}(T+\overline{\tau}); X_0+\alpha Y_0+z(\overline{\tau},\alpha)E_0)\left(-\frac{b_0'}{a_0'}+\frac{\partial z}{\partial \alpha}(\overline{\tau},\alpha)\right)\right.\\
&\qquad \left.+\frac{\partial \Phi_1}{\partial x_2}(\tilde{l}(T+\overline{\tau}); X_0+\alpha Y_0+z(\overline{\tau},\alpha)E_0)\right)\\
&\qquad -\frac{\partial \Phi_2}{\partial x_2}((1-\tilde{l})(T+\overline{\tau}); I_1(\Phi(\tilde{l}(T+\overline{\tau}); X_0+\alpha Y_0+z(\overline{\tau},\alpha)E_0)))\\
&\qquad \times (1-p_1)\left(\frac{\partial \Phi_2}{\partial x_1}(\tilde{l}(T+\overline{\tau}); X_0+\alpha Y_0+z(\overline{\tau},\alpha)E_0)\left(-\frac{b_0'}{a_0'}+\frac{\partial z}{\partial \alpha}(\overline{\tau},\alpha)\right)\right.\\
&\qquad \left.+\frac{\partial \Phi_2}{\partial x_2}(\tilde{l}(T+\overline{\tau}); X_0+\alpha Y_0+z(\overline{\tau},\alpha)E_0)\right).
\end{aligned}$$

It then follows that

$$\begin{aligned}
\frac{\partial f}{\partial \alpha}(0,0) = 1 &- \frac{\partial \Phi_2}{\partial x_1}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0)))(1-p_2)\\
&\times \left(\frac{\partial \Phi_1}{\partial x_1}(\tilde{l}T; X_0)\left(-\frac{b_0'}{a_0'}+\frac{\partial z}{\partial \alpha}(0,0)\right)+\frac{\partial \Phi_1}{\partial x_2}(\tilde{l}T; X_0)\right)\\
&- \frac{\partial \Phi_2}{\partial x_2}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0)))(1-p_1)\\
&\times \left(\frac{\partial \Phi_2}{\partial x_1}(\tilde{l}T; X_0)\left(-\frac{b_0'}{a_0'}+\frac{\partial z}{\partial \alpha}(0,0)\right)+\frac{\partial \Phi_2}{\partial x_2}(\tilde{l}T; X_0)\right).
\end{aligned}$$

From (31) and

$$\frac{\partial \Phi_2}{\partial x_1}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0))) = 0, \tag{33}$$

it is seen that

$$\frac{\partial f}{\partial \alpha}(0,0) = 1-\frac{\partial \Phi_2}{\partial x_2}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0)))(1-p_1)\frac{\partial \Phi_2}{\partial x_2}(\tilde{l}T; X_0) = 0.$$

Similarly, one may see that

$$\begin{aligned}
&\frac{\partial f}{\partial \overline{\tau}}(\overline{\tau}, \alpha)\\
&\quad = -\frac{\partial \Phi_2}{\partial \overline{\tau}}((1-\tilde{l})(T+\overline{\tau}); I_1(\Phi(\tilde{l}(T+\overline{\tau}); X_0+\alpha Y_0+z(\overline{\tau},\alpha)E_0)))(1-\tilde{l})\\
&\qquad - \frac{\partial \Phi_2}{\partial x_1}((1-\tilde{l})(T+\overline{\tau}); I_1(\Phi(\tilde{l}(T+\overline{\tau}); X_0+\alpha Y_0+z(\overline{\tau},\alpha)E_0)))\\
&\qquad \times (1-p_2)\left(\frac{\partial \Phi_1}{\partial \overline{\tau}}(\tilde{l}(T+\overline{\tau}); X_0+\alpha Y_0+z(\overline{\tau},\alpha)E_0)\cdot \tilde{l}\right.\\
&\qquad \left. + \frac{\partial \Phi_1}{\partial x_1}(\tilde{l}(T+\overline{\tau}); X_0+\alpha Y_0+z(\overline{\tau},\alpha)E_0)\frac{\partial z}{\partial \overline{\tau}}(\overline{\tau},\alpha)\right)\\
&\qquad - \frac{\partial \Phi_2}{\partial x_2}((1-\tilde{l})(T+\overline{\tau}); I_1(\Phi(\tilde{l}(T+\overline{\tau}); X_0+\alpha Y_0+z(\overline{\tau},\alpha)E_0)))\\
&\qquad \times (1-p_1)\left(\frac{\partial \Phi_2}{\partial \overline{\tau}}(\tilde{l}(T+\overline{\tau}); X_0+\alpha Y_0+z(\overline{\tau},\alpha)E_0)\cdot \tilde{l}\right.\\
&\qquad \left. + \frac{\partial \Phi_2}{\partial x_1}(\tilde{l}(T+\overline{\tau}); X_0+\alpha Y_0+z(\overline{\tau},\alpha)E_0)\frac{\partial z}{\partial \overline{\tau}}(\overline{\tau},\alpha)\right).
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{\partial f}{\partial \overline{\tau}}(0,0) = &-\frac{\partial \Phi_2}{\partial \overline{\tau}}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0)))(1-\tilde{l})\\
&- \frac{\partial \Phi_2}{\partial x_1}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0)))\\
&\times (1-p_2)\left(\frac{\partial \Phi_1}{\partial \overline{\tau}}(\tilde{l}T; X_0)\cdot \tilde{l} + \frac{\partial \Phi_1}{\partial x_1}(\tilde{l}T; X_0)\frac{\partial z}{\partial \overline{\tau}}(0,0)\right)\\
&- \frac{\partial \Phi_2}{\partial x_2}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0)))\\
&\times (1-p_1)\left(\frac{\partial \Phi_2}{\partial \overline{\tau}}(\tilde{l}T; X_0)\cdot \tilde{l} + \frac{\partial \Phi_2}{\partial x_1}(\tilde{l}T; X_0)\frac{\partial z}{\partial \overline{\tau}}(0,0)\right).
\end{aligned}$$

From (31), (32), (33) and

$$\frac{\partial \Phi_2}{\partial \overline{\tau}}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0))) = 0, \tag{34}$$

it consequently follows that

$$\frac{\partial f}{\partial \overline{\tau}}(0,0) = 0.$$

A.3 The second order partial derivatives of Φ_2

To obtain the second order partial derivatives of Φ_2, we again derive

$$\frac{d}{dt}\left(\Phi(t;X_0)\right)=F(\Phi(t;X_0)).$$

One obtains

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_2}{\partial x_1^2}(t;X_0)\right)=\left[1-\left(\beta+\frac{1}{K}\right)\Phi_1(t;X_0)\right]\frac{\partial^2\Phi_2}{\partial x_1^2}(t;X_0)$$

and consequently

$$\frac{\partial^2\Phi_2}{\partial x_1^2}(t;X_0)=e^{t-\left(\beta+\frac{1}{K}\right)\int_0^t\Phi_1(s;X_0)ds}\frac{\partial^2\Phi_2}{\partial x_1^2}(0;X_0).$$

Since $\frac{\partial^2\Phi_2}{\partial x_1^2}(0;X_0)=0$, this implies

$$\frac{\partial^2\Phi_2}{\partial x_1^2}(t;X_0)=0\quad\text{for } t\geq 0.$$

Also,

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_2}{\partial x_2^2}(t;X_0)\right)=\left[1-\left(\beta+\frac{1}{K}\right)\Phi_1(t;X_0)\right]\frac{\partial^2\Phi_2}{\partial x_2^2}(t;X_0)$$
$$-\left(\beta+\frac{1}{K}\right)\frac{\partial\Phi_1}{\partial x_2}(t;X_0)\frac{\partial\Phi_2}{\partial x_2}(t;X_0)$$

and since

$$\frac{\partial^2\Phi_2}{\partial x_2^2}(0;X_0)=0,$$

it follows that

$$\frac{\partial^2\Phi_2}{\partial x_2^2}(t;X_0)=-e^{t-\left(\beta+\frac{1}{K}\right)\int_0^t\Phi_1(s;X_0)ds}\left(\beta+\frac{1}{K}\right)\int_0^t\frac{\partial\Phi_1}{\partial x_2}(s;X_0)ds. \tag{35}$$

In a similar manner, one deduces that

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_2}{\partial x_1\partial x_2}(t;X_0)\right)=\left[1-\left(\beta+\frac{1}{K}\right)\Phi_1(t;X_0)\right]\frac{\partial^2\Phi_2}{\partial x_1\partial x_2}(t;X_0)$$
$$-\left(\beta+\frac{1}{K}\right)\frac{\partial\Phi_1}{\partial x_1}(t;X_0)\frac{\partial\Phi_2}{\partial x_2}(t;X_0)$$

and since

$$\frac{\partial^2\Phi_2}{\partial x_1\partial x_2}(0;X_0)=0,$$

one sees that

$$\frac{\partial^2 \Phi_2}{\partial x_1 \partial x_2}(t; X_0) = -e^{t-\left(\beta+\frac{1}{K}\right)\int_0^t \Phi_1(s;X_0)ds} \int_0^t \left(\beta + \frac{1}{K}\right) \frac{\partial \Phi_1}{\partial x_1}(s; X_0)ds. \tag{36}$$

A.4 The second order partial derivatives of f at $(0,0)$

First, as done in [8], one remarks that

$$\frac{\partial^2 \Phi_2}{\partial x_1 \partial \overline{\tau}}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0)) = 0 \tag{37}$$

$$\frac{\partial^2 \Phi_2}{\partial x_1^2}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0)) = 0 \tag{38}$$

$$\frac{\partial^2 \Phi_2}{\partial x_1^2}(\tilde{l}T; X_0) = 0. \tag{39}$$

By (37)–(39), combined with (31)–(34), it follows that

$$\frac{\partial^2 f}{\partial \overline{\tau}^2}(0,0) = -\frac{\partial^2 \Phi_2}{\partial \overline{\tau}^2}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0)))(1-\tilde{l})^2.$$

Since

$$\frac{\partial^2 \Phi}{\partial \overline{\tau}^2}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0))) = 0, \tag{40}$$

it is then concluded that

$$\frac{\partial^2 f}{\partial \overline{\tau}^2}(0,0) = 0.$$

Also, by (31), (33) and (38), it follows that

$$\begin{aligned}
\frac{\partial^2 f}{\partial \alpha^2}(0,0) &= -2\frac{\partial^2 \Phi_2}{\partial x_1 \partial x_2}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0)))(1-p_1)(1-p_2) \\
&\quad \times \left(\frac{\partial \Phi_1}{\partial x_1}(\tilde{l}T; X_0)\left(-\frac{b_0'}{a_0'}\right) + \frac{\partial \Phi_1}{\partial x_2}(\tilde{l}T; X_0)\right)\frac{\partial \Phi_2}{\partial x_2}(\tilde{l}T; X_0) \\
&\quad - \frac{\partial^2 \Phi_2}{\partial x_2^2}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0)))(1-p_1)^2\left(\frac{\partial \Phi_2}{\partial x_2}(\tilde{l}T; X_0)\right)^2 \\
&\quad - \frac{\partial \Phi_2}{\partial x_2}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0)))(1-p_1) \\
&\quad \times \left[2\frac{\partial^2 \Phi_2}{\partial x_2 \partial x_1}(\tilde{l}T; X_0)\left(-\frac{b_0'}{a_0'}\right) + \frac{\partial^2 \Phi_2}{\partial x_2^2}(\tilde{l}T; X_0)\right].
\end{aligned}$$

Consequently, from (35), (36), (6) and (7) one easily deduces that

$$\frac{\partial^2 f}{\partial \alpha^2}(0,0) > 0.$$

It is also seen that

$$\begin{aligned}
\frac{\partial^2 f}{\partial \alpha \partial \overline{\tau}}(0,0) = &-\frac{\partial^2 \Phi_2}{\partial x_2 \partial \overline{\tau}}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0))))(1-p_1)\frac{\partial \Phi_2}{\partial x_2}(\tilde{l}T; X_0)(1-\tilde{l}) \\
&- \frac{\partial^2 \Phi_2}{\partial x_2 \partial x_1}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0))))(1-p_1)\frac{\partial \Phi_2}{\partial x_2}(\tilde{l}T; X_0) \\
&\times (1-p_2)\left(\frac{\partial \Phi_1}{\partial \overline{\tau}}(\tilde{l}T; X_0)\cdot \tilde{l} + \frac{\partial \Phi_1}{\partial x_1}(\tilde{l}T; X_0)\frac{\partial z}{\partial \overline{\tau}}(0,0)\right) \\
&- \frac{\partial \Phi_2}{\partial x_2}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0))) \\
&\times (1-p_1)\left(\frac{\partial^2 \Phi_2}{\partial x_2 \partial \overline{\tau}}(\tilde{l}T; X_0)\cdot \tilde{l} + \frac{\partial^2 \Phi_2}{\partial x_2 \partial x_1}(\tilde{l}T; X_0)\frac{\partial z}{\partial \overline{\tau}}(0,0)\right).
\end{aligned}$$

We now compute the right-hand side of the equation above. It follows that

$$\begin{aligned}
&-\frac{\partial^2 \Phi_2}{\partial x_2 \partial \overline{\tau}}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0))))(1-p_1)\frac{\partial \Phi_2}{\partial x_2}(\tilde{l}T; X_0)(1-\tilde{l}) \\
&\quad = -\left(1 - \left(\beta + \frac{1}{K}\right)\Phi_1((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0)))\right) \\
&\qquad \times \frac{\partial \Phi_2}{\partial x_2}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0)))(1-p_1)\frac{\partial \Phi_2}{\partial x_2}(\tilde{l}T; X_0)(1-\tilde{l}) \\
&\quad = -\left(1 - \left(\beta + \frac{1}{K}\right) I^*(T)\right)(1-\tilde{l}).
\end{aligned}$$

It is also seen that

$$\begin{aligned}
&- \frac{\partial^2 \Phi_2}{\partial x_2 \partial x_1}((1-\tilde{l})T; I_1(\Phi(\tilde{l}T; X_0))))(1-p_1)\frac{\partial \Phi_2}{\partial x_2}(\tilde{l}T; X_0) \\
&= e^{(1-\tilde{l})T - \left(\beta + \frac{1}{K}\right)\int_0^{(1-\tilde{l})T} \Phi_1(s; I_1(\Phi(\tilde{l}T; X_0)))ds} \\
&\quad \times \left(\int_0^{(1-\tilde{l})T} \left(\beta + \frac{1}{K}\right)\frac{\partial \Phi_1}{\partial x_1}(s; I_1(\Phi(\tilde{l}T; X_0)))e^{-ws}ds\right) \\
&\quad \times (1-p_1)e^{\tilde{l}T - \left(\beta + \frac{1}{K}\right)\int_0^{\tilde{l}T} \Phi_1(s; X_0)ds} \\
&= e^{T - \left(\beta + \frac{1}{K}\right)\int_0^{(1-\tilde{l})T} I^*(s+\tilde{l}T)ds - \left(\beta + \frac{1}{K}\right)\int_0^{\tilde{l}T} I^*(s)ds}
\end{aligned}$$

$$\times (1-p_1)\left(\int_0^{(1-\tilde{l})T}\left(\beta+\frac{1}{K}\right)e^{-ws}ds\right)$$
$$=e^{T-\left(\beta+\frac{1}{K}\right)\int_0^T I^*(s)ds}(1-p_1)\left(\beta+\frac{1}{K}\right)\frac{1}{w}\left(1-e^{-w(1-\tilde{l})T}\right).$$

Using (11), it follows that

$$-\frac{\partial^2\Phi_2}{\partial x_2\partial x_1}((1-\tilde{l})T;I_1(\Phi(\tilde{l}T;X_0))))(1-p_1)\frac{\partial\Phi_2}{\partial x_2}(\tilde{l}T;X_0)$$
$$=\left(\beta+\frac{1}{K}\right)\frac{1}{w}\left(1-e^{-w(1-\tilde{l})T}\right).$$

Also,

$$(1-p_2)\left(\frac{\partial\Phi_1}{\partial\overline{\tau}}(\tilde{l}T;X_0)\cdot\tilde{l}+\frac{\partial\Phi_1}{\partial x_1}(\tilde{l}T;X_0)\frac{\partial z}{\partial\overline{\tau}}(0,0)\right)$$
$$=-w(1-p_2)e^{-w\tilde{l}T}\left(I^*(0+)\cdot\tilde{l}+\frac{1}{a_0'}I^*(T)\right).$$

One may see that

$$-\frac{\partial\Phi_2}{\partial x_2}((1-\tilde{l})T;I_1(\Phi(\tilde{l}T;X_0)))(1-p_1)$$
$$\times\left[\frac{\partial^2\Phi_2}{\partial x_2\partial\overline{\tau}}(\tilde{l}T;X_0)\cdot\tilde{l}+\frac{\partial^2\Phi_2}{\partial x_2\partial x_1}(\tilde{l}T;X_0)\frac{\partial z}{\partial\overline{\tau}}(0,0)\right]$$
$$=-\frac{\partial\Phi_2}{\partial x_2}((1-\tilde{l})T;I_1(\Phi(\tilde{l}T;X_0)))(1-p_1)$$
$$\times\left[(1-\left(\beta+\frac{1}{K}\right)\Phi_1(\tilde{l}T;X_0))\frac{\partial\Phi_2}{\partial x_2}(\tilde{l}T;X_0)\cdot\tilde{l}\right.$$
$$\left.-\left(\frac{\partial\Phi_2}{\partial x_2}(\tilde{l}T;X_0)\int_0^{\tilde{l}T}\left(\beta+\frac{1}{K}\right)\frac{\partial\Phi_1}{\partial x_1}(s;X_0)ds\right)\frac{\partial z}{\partial\overline{\tau}}(0,0)\right].$$

Since $d_0'=0$, it follows that

$$-\frac{\partial\Phi_2}{\partial x_2}((1-\tilde{l})T;I_1(\Phi(\tilde{l}T;X_0)))(1-p_1)$$
$$\times\left[\frac{\partial^2\Phi_2}{\partial x_2\partial\overline{\tau}}(\tilde{l}T;X_0)\cdot\tilde{l}+\frac{\partial^2\Phi_2}{\partial x_2\partial x_1}(\tilde{l}T;X_0)\frac{\partial z}{\partial\overline{\tau}}(0,0)\right]$$
$$=-\left[\left(1-\left(\beta+\frac{1}{K}\right)I^*(\tilde{l}T)\right)\cdot\tilde{l}+\frac{w}{a_0'}\left(\beta+\frac{1}{K}\right)\left(\int_0^{\tilde{l}T}e^{-ws}ds\right)I^*(T)\right]$$
$$=-\left[\left(1-\left(\beta+\frac{1}{K}\right)I^*(\tilde{l}T)\right)\cdot\tilde{l}+\frac{1}{a_0'}\left(\beta+\frac{1}{K}\right)\left(1-e^{-w\tilde{l}T}\right)I^*(T)\right].$$

One then obtains that

$$\begin{aligned}
&\frac{\partial^2 f}{\partial\alpha\partial\overline{\tau}}(0,0)\\
&\quad=-\left(1-\left(\beta+\frac{1}{K}\right)I^*(T)\right)(1-\tilde{l})\\
&\qquad-\left(\beta+\frac{1}{K}\right)\frac{1}{w}\left(1-e^{-w(1-\tilde{l})T}\right)w(1-p_2)e^{-w\tilde{l}T}\left(I^*(0+)\cdot\tilde{l}+\frac{1}{a_0'}I^*(T)\right)\\
&\qquad-\left[\left(1-\left(\beta+\frac{1}{K}\right)I^*(\tilde{l}T)\right)\cdot\tilde{l}+\frac{1}{a_0'}\left(\beta+\frac{1}{K}\right)\left(1-e^{-w\tilde{l}T}\right)I^*(T)\right]. \qquad (41)
\end{aligned}$$

Note that

$$\begin{aligned}
&-\left(1-\left(\beta+\frac{1}{K}\right)I^*(T)\right)(1-\tilde{l})-\left(1-\left(\beta+\frac{1}{K}\right)I^*(\tilde{l}T)\right)\tilde{l}\\
&\quad=-1+\left(\beta+\frac{1}{K}\right)\left[(1-\tilde{l})I^*(T)+\tilde{l}I^*(\tilde{l}T)\right], \qquad (42)
\end{aligned}$$

and, since I^* is decreasing on $(0,T]$,

$$\begin{aligned}
(1-\tilde{l})I^*(T)+\tilde{l}I^*(\tilde{l}T)&<\frac{1}{T}\left[\int_0^{\tilde{l}T}I^*(s)ds+\int_{\tilde{l}T}^{T}I^*(s)ds\right]\\
&=\frac{1}{T}\int_0^T I^*(s)ds. \qquad (43)
\end{aligned}$$

Since

$$\left(\beta+\frac{1}{K}\right)\int_0^T I^*(s)ds=T-\ln\frac{1}{1-p_1}<T,$$

it follows from (41), (42) and (43) that

$$\frac{\partial^2 f}{\partial\alpha\partial\tau}(0,0)<0.$$

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