# 1. CALCUL INTEGRAL

**1.1. Primitive.** Reamintim **tabloul primitivelor funcţiilor elementare** uzuale în paralel cu **tabloul primitivelor funcţiilor compuse**.

1. $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C,$ $x \in I \subset (0,\infty), \alpha \in \mathbb{R} \setminus \{-1\},$ $\qquad$ $\int u(x)^\alpha u'(x)dx = \frac{u(x)^{\alpha+1}}{\alpha+1} + C,$ $u(x) \in \mathbb{R}_+, \alpha \neq -1,$

2. $\int \frac{1}{x}dx = \ln|x| + C, x \in I \subset (0,\infty)$ sau $x \in I \subset (-\infty, 0),$ $\qquad$ $\int \frac{1}{u(x)}u'(x)dx = \ln|u(x)| + C,$ $u(x) \neq 0,$

3. $\int a^x dx = \frac{a^x}{\ln a}, x \in \mathbb{R}, a > 0, a \neq 1,$ $\qquad$ $\int a^{u(x)}u'(x)dx = \frac{a^{u(x)}}{\ln a}, a > 0, a \neq 1,$

4. $\int e^x dx = e^x + C, x \in \mathbb{R},$ $\qquad$ $\int e^{u(x)}u'(x)dx = e^{u(x)} + C, x \in \mathbb{I},$

5. $\int \sin x dx = -\cos x + C, x \in \mathbb{R},$ $\qquad$ $\int \sin u(x) \cdot u'(x)dx = -\cos u(x) + C,$

6. $\int \cos x dx = \sin x + C, x \in \mathbb{R},$ $\qquad$ $\int \cos u(x) \cdot u'(x)dx = \sin u(x) + C,$

7. $\int \frac{1}{\cos^2 x}dx = \operatorname{tg} x + C,$ $x \in I \subset \mathbb{R} \setminus \left\{(2k+1)\frac{\pi}{2}, k \in \mathbb{N}\right\},$ $\qquad$ $\int \frac{1}{\cos^2 u(x)}u'(x)dx = \operatorname{tg} u(x) + C,$ $u(x) \neq \left\{(2k+1)\frac{\pi}{2}, k \in \mathbb{N}\right\},$

8. $\int \frac{1}{\sin^2 x}dx = -\operatorname{ctg} x + C,$ $x \in I \subset \mathbb{R} \setminus \{k\pi, k \in \mathbb{N}\},$ $\qquad$ $\int \frac{1}{\sin^2 u(x)}u'(x)dx = -\operatorname{ctg} u(x) + C,$ $u(x) \neq \{k\pi, k \in \mathbb{N}\},$

9. $\displaystyle\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C,$ $x \in \mathbb{R}, a \neq 0,$ $\displaystyle\int \frac{u'(x)dx}{u^2(x)+a^2} = \frac{1}{a}\operatorname{arctg}\frac{u(x)}{a} + C,$ $a \neq 0,$

10. $\int \frac{1}{x^2-a^2}dx = \dfrac{1}{2a}\ln\left|\dfrac{x-a}{x+a}\right| + C,$ $x \in \mathbb{R}, a > 0, |x| \neq a,$ $\int \frac{u'(x)dx}{u^2(x)-a^2} = \dfrac{1}{2a}\ln\left|\dfrac{u(x)-a}{u(x)+a}\right| + C,$ $a > 0, |u(x)| \neq a,$

11. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\dfrac{x}{a} + C,$ $x \in I \subset (-a,a)\,, a > 0,$ $\int \frac{u'(x)dx}{\sqrt{a^2-u^2(x)}} = \arcsin\dfrac{u(x)}{a} + C,$ $u(x) \in (-a,a)\,, a > 0,$

12. $\displaystyle\int \operatorname{sh} x dx = \operatorname{ch} x + C, x \in \mathbb{R},$ $\displaystyle\int \operatorname{sh} u(x) \cdot u'(x)dx = \operatorname{ch} u(x) + C, x \in \mathbb{R},$

13. $\displaystyle\int \operatorname{ch} x dx = \operatorname{sh} x + C, x \in \mathbb{R},$ $\displaystyle\int \operatorname{ch} u(x) \cdot u'(x)dx = \operatorname{sh} u(x) + C, x \in \mathbb{R},$

14. $\displaystyle\int \frac{1}{\operatorname{ch}^2 x}dx = \operatorname{th} x + C, x \in \mathbb{R},$ $\displaystyle\int \frac{1}{\operatorname{ch}^2 u(x)}u'(x)dx = \operatorname{th} u(x) + C, x \in \mathbb{R},$

15. $\displaystyle\int \frac{1}{\operatorname{sh}^2 x}dx = \operatorname{cth} x + C, x \in \mathbb{R},$ $\displaystyle\int \frac{1}{\operatorname{sh}^2 u(x)}u'(x)dx = \operatorname{cth} u(x) + C, x \in \mathbb{R},$

16. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right) + C,$ $x \in \mathbb{R},$ $\int \frac{u'(x)dx}{\sqrt{u^2(x)+a^2}} = \ln\left(u(x)+\sqrt{u^2(x)+a^2}\right) + C,$ $x \in \mathbb{I} \subset \mathbb{R},$

17. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + C,$ $x \in I \subset (a,\infty)$ sau $x \in I \subset (-\infty,-a)\,, a > 0.$ $\int \frac{u'(x)dx}{\sqrt{u^2(x)-a^2}} = \ln\left|u(x)+\sqrt{u^2(x)-a^2}\right| + C,$ $u(x) \in (a,\infty)$ sau $u(x) \in (-\infty,-a)\,, a > 0.$

**Exerciţiul 1.** *Să se demonstreze că*

a) $\displaystyle\int \frac{1}{\cos^2 u(x)} u'(x)dx = \operatorname{tg} u(x) + C$

b) $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right) + C.$

c) $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin\frac{x}{a} + C$

d) $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + C$

e) $\displaystyle\int \frac{u'(x)dx}{\sqrt{u^2(x)-a^2}} = \ln\left|u(x)+\sqrt{u^2(x)-a^2}\right| + C.$

Indicaţie: b) $\left(\ln\left(x+\sqrt{x^2+a^2}\right)+C\right)' = \dfrac{1+\frac{2x}{2\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}} = \dfrac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}\left(x+\sqrt{x^2+a^2}\right)} = \dfrac{1}{\sqrt{x^2+a^2}}.$

**Exerciţiul 2.** *Să se verifice următoarele relaţii, calculând integralele nedefinite:*

**1.** $\int x^4 dx = \frac{1}{5}x^5 + C, \forall x \in \mathbb{R}, C \in \mathbb{R}.$

**2.** $\int \frac{1}{x^7} dx = \displaystyle\int x^{-7} dx = \frac{1}{-6}x^{-6} + C, x \neq 0.$

**3.** $\int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \dfrac{2}{3}x^{\frac{3}{2}} + C, x \geq 0.$

**4.** $\int \frac{1}{\sqrt[3]{x^7}} dx = \displaystyle\int x^{-\frac{7}{3}} dx = \frac{1}{\frac{-7}{3}+1} x^{\frac{-7}{3}+1} + C, \forall x \in \mathbb{R} \setminus \{0\}, C \in \mathbb{R}.$

**5.** $\int 3^x dx = \frac{1}{\ln 3} 3^x + C, \forall x \in \mathbb{R}, C \in \mathbb{R}.$

**6.** $\displaystyle\int \frac{1}{x^2+1} dx = \operatorname{arctg} x + C, x \in \mathbb{R},$

**7.** $\displaystyle\int \frac{1}{x^2+4} dx = \int \frac{1}{x^2+2^2} dx = \frac{1}{2}\operatorname{arctg}\frac{x}{2} + C, x \in \mathbb{R},$

**8.** $\int \frac{1}{x^2-3^2} dx = \frac{1}{2\cdot 3} \ln\left|\frac{x-3}{x+3}\right| + C, x \in \mathbb{R}, |x| \neq 3,$

**9.** $\int \frac{1}{4x^2+4x-2} dx = \frac{1}{2} \displaystyle\int \frac{1}{(2x+1)^2 - \left(\sqrt{3}\right)^2} \cdot (2x+1)' dx =$

$= \dfrac{1}{4\sqrt{3}} \ln\left|\dfrac{(2x+1)-\sqrt{3}}{(2x+1)+\sqrt{3}}\right| + C$

**10.** $\int \frac{1}{\sqrt{\frac{2}{3}-x^2}}dx = \arcsin \frac{x}{\sqrt{\frac{2}{3}}} + C, \forall x \in \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right), C \in \mathbb{R}.$

**11.** $\int \frac{1}{\sqrt{4-4x^2+4x}}dx = \frac{1}{2} \arcsin \frac{2x-1}{\sqrt{5}} + C, \forall x \in \left(\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right), C \in \mathbb{R}.$

**12.** $\int \frac{1}{\sqrt{x^2+2\pi}}dx = \ln\left(x+\sqrt{x^2+2\pi}\right) + C, \forall x \in \mathbb{R}, C \in \mathbb{R}.$

**13.** $\int \frac{1}{\sqrt{9x^2+6x+8}}dx = \frac{1}{3}\ln\left(3x+1+\sqrt{9x^2+6x+8}\right) + C, \forall x \in \mathbb{R}, C \in \mathbb{R},$

**14.** $\int \frac{1}{\sqrt{x^2-9}}dx = \ln\left|x+\sqrt{x^2-e}\right| + C, \forall x \in (-\infty,-3) \cup (3,\infty), C \in \mathbb{R},$

**15.** $\int \frac{1}{\sqrt{9x^2+6x-5}}dx = \frac{1}{3}\ln\left|3x+1+\sqrt{9x^2+6x-5}\right| + C, \forall x \in \left(-\infty, \frac{-1-\sqrt{6}}{3}\right) \cup \left(\frac{-1+\sqrt{6}}{3}, \infty\right), C \in \mathbb{R}.$

Indicaţii: **11.** $\int \frac{1}{\sqrt{4-4x^2+4x}}dx = \frac{1}{2}\int \frac{1}{\sqrt{\left(\sqrt{5}\right)^2-(2x-1)^2}} \cdot (2x-1)'\,dx,$

**13.** $\int \frac{1}{\sqrt{9x^2+6x+8}}dx = \frac{1}{3}\int \frac{1}{\sqrt{(3x+1)^2+\left(\sqrt{7}\right)^2}} \cdot (3x+1)'\,dx$

**Exerciţiul 3.** *Determinaţi primitivele următoarelor funcţii:*

*a)* $f(x) = (2x)\,7^{x^2+1}$, $R: F(x) = 7^{x^2+1} + C, \forall x \in \mathbb{R}, C \in \mathbb{R}.$
*b)* $f(x) = e^{2x+3}$, $R: F(x) = \frac{1}{2}e^{2x+3} + C, \forall x \in \mathbb{R}, C \in \mathbb{R}.$
*c)* $f(x) = e^{\sin x}\cos x$, $R: F(x) = e^{\sin x} + c, \forall x \in \mathbb{R}, c \in \mathbb{R}.$
*d)* $f(x) = \cos(3x+2)$, $R: F(x) = \frac{1}{3}\sin(3x+2) + c, \forall x \in \mathbb{R}, c \in \mathbb{R}.$
*e)* $f(x) = \cos^2 x$, $R: F(x) = \frac{1}{2}x + \frac{1}{4}\sin 2x + C, \forall x \in \mathbb{R}, C \in \mathbb{R}.$
*f)* $f(x) = \sin^2 x$, $R: F(x) = \frac{1}{2}x - \frac{1}{4}\sin 2x + C, \forall x \in \mathbb{R}, C \in \mathbb{R}.$
*g)* $f(x) = \cos^3 x$, $R: F(x) = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + C, \forall x \in \mathbb{R}, C \in \mathbb{R}.$
*h)* $f(x) = \sin^3 x$, $R: F(x) = \frac{1}{12}\cos 3x - \frac{3}{4}\cos x + C, \forall x \in \mathbb{R}, C \in \mathbb{R}.$

**Exerciţiul 4.** *Determinaţi primitivele următoarelor funcţii pe intervalele lor de definiţie:*

*a)* $f(x) = \frac{\cos(\ln x)}{x}$, $R: F(x) = \sin(\ln x) + c, \forall x \in (0,+\infty), c \in \mathbb{R}.$
*b)* $f(x) = \frac{\sin(\sqrt{x})}{2\sqrt{x}}$, $R: F(x) = -\cos\left(\sqrt{x}\right) + c, \forall x \in (0,+\infty), c \in \mathbb{R}.$
*c)* $f(x) = \operatorname{tg} x$, $R: F(x) = -\ln|\cos x| + c, \forall x \in \left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right), c \in \mathbb{R}.$

*d)* $f(x) = 1 + \operatorname{tg}^2 x dx$, $\quad R : F(x) = \operatorname{tg} x + c, \forall x \in \left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right), c \in \mathbb{R}$.
*e)* $f(x) = \operatorname{ctg} x$, $\quad R : F(x) = \ln|\sin x| + c, \forall x \in (k\pi, k\pi + \pi), c \in \mathbb{R}$.
*f)* $f(x) = 1 + \operatorname{ctg}^2 x dx$, $\quad R : F(x) = -\operatorname{ctg} x + c, \forall x \in (k\pi, k\pi + \pi), c \in \mathbb{R}$.
**g)** $f(x) = \dfrac{\cos(2x)}{\cos^2 x \cdot \sin^2 x}$, $\quad R : F(x) = -\operatorname{ctg} x - \operatorname{tg} x + c, \forall x \in \mathbb{I}, c \in \mathbb{R}$, $\mathbb{I} = \left(k\pi, k\pi + \frac{\pi}{2}\right)$ *sau* $\mathbb{I} = \left(k\pi - \frac{\pi}{2}, k\pi\right)$.
*h)* $f(x) = \left(\sqrt[5]{x} + \dfrac{1}{x\sqrt[5]{x}}\right)^2, x \neq 0.$ $\quad R : F(x) = \frac{5}{7}x^{\frac{7}{5}} + 2\ln x - \frac{5}{7}x^{-\frac{7}{5}} + C.$

**1.2. Integrarea prin părţi.** Reamintim formula de integrare prin părţi:
$\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx, \forall x \in \mathbb{I}.$

**Exerciţiul 5.** *Folosind integrarea prin părţi, să se calculeze următoarele integrale nedefinite:*
a) $\int x^2 \cos x dx, x \in \mathbb{R};$ $\quad R : F(x) = x^2 \sin x + 2x \cos x - 2 \sin x + c,$
b) $\int x^2 \sin(2x + 5) dx, x \in \mathbb{R};$
$\quad R : F(x) = \frac{1}{4}\cos(2x + 5) - \frac{1}{2}x^2 \cos(2x + 5) + \frac{1}{2}x \sin(2x + 5) + C,$
c) $\int x e^{-2x} dx, x \in \mathbb{R};$ $\quad F(x) = -\frac{1}{4}e^{-2x}(2x + 1) + C, x \in \mathbb{R},$
d) $\int e^{2x} \cos(3x) dx, x \in \mathbb{R};$ $\quad R : F(x) = \dfrac{3e^{2x}\sin(3x) + 2e^{2x}\cos(3x)}{3^2 + 2^2} + c$
e) $\int x^n \ln x dx, x \in (0, +\infty)$ *pentru* $n \in \mathbb{N}$ *fixat*;
$\quad R : F(x) = \dfrac{x^{n+1}}{n+1}\ln x - \dfrac{x^{n+1}}{(n+1)^2} + c, \forall x \in (0, +\infty).$
f) $\int x^3 e^{x^2} dx, \forall x \in \mathbb{R},$
$\quad R : F(x) = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + c,$
g) $\int \sqrt{9 - x^2} dx, x \in \mathbb{R}, x \in (-3, 3);$
$\quad R : F(x) = \dfrac{1}{2}\left(x\sqrt{9 - x^2} + 9 \arcsin \dfrac{x}{a}\right) + c, \forall x \in (-3, 3),$
h) $\int \sqrt{x^2 + 4} dx, x \in \mathbb{R};$
$\quad R : F(x) = \dfrac{1}{2}\left(x\sqrt{x^2 + 4} + 4 \ln\left(x + \sqrt{x^2 + 2^2}\right)\right) + c, \forall x \in \mathbb{R}$
i) $\displaystyle\int x\sqrt{x + 1} dx, x \in [-1, \infty);$
$\quad R : F(x) = \dfrac{2}{5}(x + 1)^{\frac{5}{2}} - \dfrac{2}{3}(x + 1)^{\frac{3}{2}} + C, \forall x \in [-1, \infty).$

Indicaţii.

1. Calculul primitivelor de tipul $\int e^{ax} \cdot \cos(bx) dx$ şi $\int e^{ax} \cdot \sin(bx) dx$ (Exerciţiul 5, exemplul d) se face cu formula integrării prin părţi, aplicată de două ori, alegând iniţial $f(x) = e^{ax}$; $g'(x) = \cos(bx)$ sau $\sin(bx)$.

2. Calculul primitivelor de tipul $\int P(x) \cdot \ln(ax)\, dx$ (Exerciţiul 5, exemplul e)se face cu formula integrării prin părţi, alegând $f(x) = \ln(ax)$; $g'(x) = P(x)$.

## 2. Primitivele funcţiilor raţionale

**Exerciţiul 6.** *Să se calculeze următoarele integrale nedefinite precizându-se intervalul pe care există primitivele:*

a) $\int \frac{x}{x+1}dx$, b) $\int \frac{x^2}{x^2+1}dx$, c) $\int \frac{x}{x^2+1}dx$, d) $\int \frac{3x+5}{x^2+1}dx$,
e) $\int \frac{x+1}{x^2-3x+2}dx$, f) $\int \frac{2x-5}{x^2-5x+6}dx$, g) $\int \frac{2x+1}{x^2+2x+2}dx$.

**Exerciţiul 7.** *Să se calculeze următoarele integrale nedefinite pe intervalul precizat:*

a) $\int \frac{x^2+4}{3x^3+4x^2-4x}dx, x \in [2,\infty)$, b) $\int \frac{x^2-3x+2}{x^3+2x^2+x}dx, x \in (0,\infty)$,
c) $\int \frac{x^2-29x+5}{(x-4)^2(x^2+3)}dx, x \in (-\infty,4)$, d) $\int \frac{x+1}{x^5+4x^3+4x}dx, x \in (0,+\infty)$,
e) $\int \frac{x^3+10x^2+3x+36}{(x^2+4)^2(x-1)}dx, x \in (1,\infty)$, f) $\int \frac{1}{(x+1)(x^2+x+1)}dx, x \in (-1,\infty)$,
g) $\int \frac{3x+5}{(x^2+2x+2)^2}dx$, h) $\int \frac{x^3}{x+1}dx, x \in (0,+\infty)$.
i) $\int \frac{x^2}{x^2-1}dx, x \in (1,\infty)$. j) $\int \frac{x}{x^4-x^2-2}dx$.

*Rezolvări şi răspunsuri:*

**a)** $\int \frac{x^2+4}{3x^3+4x^2-4x}dx, x \in [2,\infty)$.
$3x^3+4x^2-4x = x(x+2)(3x-2)$.

Observăm că $f(x) = \frac{x^2+4}{3x^3+4x^2-4x}$ este continuă pe $[2,\infty)$, deci admite primitive pe acest interval. Pentru a calcula integrala nedefinită descompunem funcţia în fracţii simple.

$\frac{x^2+4}{x(x+2)(3x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{3x-2} \Leftrightarrow$
$x^2+4 = A(x+2)(3x-2) + Bx(3x-2) + Cx(x+2)$.
Pentru $x=0 \Rightarrow 4 = A(2)(-2) \Rightarrow A = -1$,
Pentru $x=-2 \Rightarrow 8 = B(-2)(-8) \Rightarrow B = \frac{1}{2}$,
Pentru $x = \frac{2}{3} \Rightarrow \frac{4}{9} + 4 = C\frac{2}{3}\left(\frac{2}{3}+2\right)$, $C = \frac{5}{2}$.

$\int \frac{x^2+4}{3x^3+4x^2-4x}dx = \int \frac{-1}{x}dx + \int \frac{\frac{1}{2}}{x+2}dx + \int \frac{\frac{5}{2}}{3x-2}dx = -\ln x + \frac{1}{2}\ln(x+2) + \frac{5}{6}\ln\left(x-\frac{2}{3}\right)+C$

**c)** $\int \frac{x^2-29x+5}{(x-4)^2(x^2+3)}dx = \ln(4-x) + \frac{5}{x-4} - \frac{1}{2}\ln\left(x^2+3\right) + \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{x}{\sqrt{3}} + C.$

**e)** $\int \frac{x^3+10x^2+3x+36}{(x^2+4)^2(x-1)}dx = 2\ln(x-1) - \ln\left(x^2+4\right) - \frac{1}{2}\operatorname{arctg}\frac{x}{2} - \frac{1}{4}\frac{1}{x^2+4} + C.$

**f)** $\int \frac{1}{(x+1)(x^2+x+1)}dx = \ln\frac{|x+1|}{\sqrt{x^2+X+1}} + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + c.$

**g)** $x+1=t$, $\int \frac{3x+5}{(x^2+2x+2)^2}dx = -\frac{3}{2}\frac{1}{x^2+2x+2} - \frac{7}{24}\left(\operatorname{arctg}(x+1) + \frac{x+1}{(x+1)^2+1}\right) + C,$

**h)** $\int \frac{x^3}{x+1}dx = \frac{x^3}{3} - \frac{x^2}{2} + x - \ln(x+1) + c, \forall x \in (-1,+\infty), c \in \mathbb{R}$

**j)** $\int \frac{x^2}{x^2-1}dx = x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.$

## 3. Primitivele unor funcţii iraţionale

**Exerciţiul 8.** *Să se calculeze următoarele integrale nedefinite pe intervalul precizat:*

a) $\int \frac{x+\sqrt{x^2+x+1}}{x-\sqrt{x^2+x+1}}dx, x \in \mathbb{R},$ b) $\int \frac{1}{(1+x)\sqrt{x^2+2x+2}}dx, x \in (-1,+\infty),$

c) $\int \frac{1}{(x-1)\sqrt{x^2-2x+3}}dx, x \in (1,2)$

d) $\int \frac{dx}{1+\sqrt{1-2x-x^2}}, x \in \left(-1-\sqrt{2}, -1+\sqrt{2}\right)$

e) $\int \frac{xdx}{(x-1)\sqrt{1+x-x^2}}, x \in \left(1, \frac{1+\sqrt{5}}{2}\right).$

f) $\int \frac{1}{(2x-3)\sqrt{4x-x^2}}dx, x \in \left(0, \frac{3}{2}\right).$

g) $\int \frac{8x-3}{\sqrt{12x-4x^2-5}}dx, x \in \left(\frac{1}{2}, \frac{5}{2}\right).$

h) $\int \frac{xdx}{(1+x^2)\sqrt{x^4+2x^2+2}}, x \in \mathbb{R}.$

*Rezolvări şi răspunsuri:*

**a)** $f:\mathbb{R}\to\mathbb{R}, f(x)=\dfrac{x+\sqrt{x^2+x+1}}{x-\sqrt{x^2+x+1}}$

$f$ este continuă pe $\mathbb{R} \Rightarrow f$ admite primitive pe $\mathbb{R}$

Facem schimbarea de variabilă de integrare $(a>0)$

$$\begin{cases} \sqrt{x^2+x+1}=x+t, \text{exprimăm } x \text{ în funcţie de } t \\ x^2+x+1=x^2+2xt+t^2 \Rightarrow x=\dfrac{t^2-1}{1-2t}, \\ dx=\dfrac{2t(1-2t)-(t^2-1)(-2)}{(1-2t)^2}dt \Rightarrow dx=\dfrac{-2t^2+2t-2}{(1-2t)^2}dt \end{cases}$$

Calculăm $\sqrt{x^2+x+1}=\dfrac{t^2-1}{1-2t}+t=\dfrac{1}{2t-1}\left(t^2-t+1\right)$

Înlocuim

$$\int \frac{\frac{t^2-1}{1-2t}+\frac{1}{2t-1}\left(t^2-t+1\right)}{\frac{t^2-1}{1-2t}-\frac{1}{2t-1}\left(t^2-t+1\right)}\frac{-2t^2+2t-2}{(1-2t)^2}dt=\int\frac{-\frac{t-2}{2t-1}}{-t}\frac{-2t^2+2t-2}{(1-2t)^2}dt=$$

$$=2\int\frac{(t-2)(t^2-t+1)}{t(1-2t)^3}dt$$

$$=2\int\left(-\frac{2}{t}-\frac{15}{4(1-2t)}-\frac{3}{(1-2t)^2}-\frac{9}{4(1-2t)^3}\right)dt$$

$$=-4\ln t+\frac{15}{4}\ln|1-2t|-3(1-2t)^{-1}-\frac{9}{8}(1-2t)^{-2}+C.$$

Revenim la substituţie$\Rightarrow$

$$\int\frac{x+\sqrt{x^2+x+1}}{x-\sqrt{x^2+x+1}}dx=-4\ln\left(\sqrt{x^2+x+1}-x\right)+\frac{15}{4}\ln\left|1-2\left(\sqrt{x^2+x+1}-x\right)\right|$$

$$-3\left(1-2\left(\sqrt{x^2+x+1}-x\right)\right)^{-1}-\frac{9}{8}\left(1-2\left(\sqrt{x^2+x+1}-x\right)\right)^{-2}+c, \forall c\in\mathbb{R}.$$

**b)** $\displaystyle\int\frac{1}{(1+x)\sqrt{x^2+2x+2}}dx=\ln\frac{\sqrt{x^2+2x+2}-x-2}{\sqrt{x^2+2x+2}-x}+C,\ \forall x\in(-1,+\infty),$ $\forall C\in\mathbb{R}.$

**d)** $\displaystyle\int\frac{dx}{1+\sqrt{1-2x-x^2}}.$

$1-2x-x^2\geq 0.$

Facem schimbarea de variabilă $(a<0, c>0)$: $\sqrt{1-2x-x^2}=xt-1.$

**f)** $\displaystyle\int\frac{1}{(2x-3)\sqrt{4x-x^2}}dx, x\in\left(0,\frac{3}{2}\right).$

$f:\left(0,\frac{3}{2}\right)\to\mathbb{R}, f(x)=\dfrac{1}{(2x-3)\sqrt{4x-x^2}}$

$f$ este continuă pe $\left(0,\frac{3}{2}\right)\Rightarrow f$ admite primitive pe $\left(0,\frac{3}{2}\right)$

Calculăm integrala.

Deoarece $a < 0$, $c = 0$ şi $4x - x^2 = 0 \Rightarrow x_1 = 0$ şi $x_2 = 4$ facem schimbarea de variabilă de integrare $\sqrt{4x - x^2} = t\,(x - 0)$, exprimăm $x$ în funcţie de $t$

$4x - x^2 = t^2x^2 \Rightarrow x = \dfrac{4}{t^2+1} \Rightarrow x = \dfrac{4}{t^2+1} \Rightarrow$

$dx = -4\dfrac{2t}{(t^2+1)^2}dt \Rightarrow dx = \dfrac{-8t}{(t^2+1)^2}dt.$

Calculăm $2x - 3\sqrt{4x - x^2} = tx \Rightarrow \sqrt{4x - x^2} = \dfrac{4t}{t^2+1}$

$2x - 3 = 2\dfrac{4}{t^2+1} - 3 = -\dfrac{1}{t^2+1}\left(3t^2 - 5\right)$

Înlocuim

$$\int \frac{1}{-\frac{1}{t^2+1}\left(3t^2-5\right)\cdot\dfrac{4t}{t^2+1}}\frac{-8t}{(t^2+1)^2}dt = 2\int\frac{1}{3t^2-5}dt$$

$$= \frac{2}{3}\int\frac{1}{t^2 - \left(\sqrt{\frac{5}{3}}\right)^2}dt = \frac{2}{3}\cdot\frac{1}{2\sqrt{\frac{5}{3}}}\cdot\ln\left(-\frac{t-\sqrt{\frac{5}{3}}}{t+\sqrt{\frac{5}{3}}}\right) + C\ .$$

Revenim la substituţie$\Rightarrow$

$$\int\frac{1}{(2x-3)\sqrt{4x-x^2}}dx = \frac{2}{3}\cdot\frac{1}{2\sqrt{\frac{5}{3}}}\cdot\ln\left(-\frac{\dfrac{\sqrt{4x-x^2}}{x}-\sqrt{\dfrac{5}{3}}}{\dfrac{\sqrt{4x-x^2}}{x}+\sqrt{\dfrac{5}{3}}}\right)+c,\ \forall x\in\left(0,\tfrac{3}{2}\right),\forall c\in\mathbb{R}.$$

**h)** Indicaţie. Se face schimbarea de variabilă $x^2 = t$ şi se analizează integrala de la punctul b).

## 3.1. Tema seminar I.

**Exerciţiul 9.** *Să se calculeze integralele*

*a)* $\displaystyle\int \left(x^2 \operatorname{arctg} x\right) dx,$

*b)* $\displaystyle\int \left(\operatorname{tg}^3 x\right) dx,$

*c)* $\displaystyle\int \frac{x}{\sqrt{1-x^2}}e^{\arcsin x}dx$

*d)* $\displaystyle\int \frac{x^2+x+1}{x^2+1}dx$

*e)* $\displaystyle\int \frac{x^2}{(x+2)^2(x+4)^2}dx$