§ 2.3 SYMMETRIC BILINEAR FORMS AND QUADRATIC FORMS

The basic definitions and properties of the bilinear forms (abbreviated BLFs) were presented in the previous **§ 2.2** . Before giving the formal definition of the symmetric BLFs, let us mention that a symmetric BLF admits its arguments x, y to be interchanged without changing its value.

***Definition* 3.1.** Let V be a vector space over the field $\mathbf{K}\,(=\mathbb{R})$. A mapping $f: V \times V \longrightarrow \mathbf{K}$ is said to be a *symmetric bilinear functional* (or a *symmetric bilinear form*) if it satisfies both $(\mathbf{BLF}_1)$ and $(\mathbf{BLF}_2)$ of *Definition 2.1* plus the symmetry property

$$\text{(Symm)} \qquad \boxed{(\forall\, x, y \in V)\;\; f(y,x) = f(x,y).} \tag{3.1}$$

◊

***Remarks* 3.1.** In the case when the space V is defined over the complex field $\mathbb{C}$, it is possible (and appropriate) to define the so-called *Hermitian* BLFs which satisfy a specific version of the symmetry property (3.1), namely

$$(\forall\, x, y \in V)\;\; f(y,x) = \overline{f(x,y)}, \tag{3.2}$$

where $\bar{z}$ denotes the conjugate of the complex number $\bar{z}:\; z = x + i\,y \;\Rightarrow\; \bar{z} = x - i\,y.$

An immediate consequence of *Definition 3.1* regards the coefficient matrix of a symmetric BLF in any basis A of V: if f is a symmetric BLF on $V = \mathcal{L}(A)$ then its matrix in the (arbitrary) basis A is symmetric :

$$\boxed{F_A = F_A^{\mathrm{T}} \iff (\forall\, i,j)\;\; \alpha_{ji} = \alpha_{ij} \iff [\alpha] = [\alpha]^{\mathrm{T}}.} \tag{3.3}$$

This obvious property directly follows from *Def. 3.1*. Indeed,

$$(\forall\, i,j)\;\; \alpha_{ji} = f(a_j, a_i) = f(a_i, a_j) = \alpha_{ij} \;\;\Rightarrow\;\; [\alpha] = [\alpha]^{\mathrm{T}}.$$

***Definition* 3.2.** If $f: V \times V \longrightarrow \mathbf{K}$ is a symmetric BLF then the kernel of f is (the subset of V) defined by

$$\boxed{\operatorname{Ker} f \underset{\mathbf{def}}{=} \{x \in V: (\forall\, y \in V)\;\; f(x,y) = 0\}.} \tag{3.4}$$

***Remarks* 3.2.** This definition of the kernel of a symmetric BLF seem to be somehow asymmetric (as to the role of the two arguments). In fact, we could also consider the set

$$\{y \in V: (\forall\, x \in V)\;\; f(x,y) = 0\},$$

but it is identical to $\operatorname{Ker} f$ just because of the symmetry of f.

It is convenient to employ a simpler notation for defining the kernel of a symmetric BLF, and also for stating / proving other properties of such BLFs. It is grounded on o more general way to write the values of a function taken on a whole (sub)set : if $f: U \longrightarrow K$ and $A \subseteq U$ then

$$\{ f(x) : (\forall) x \in A \} \underset{\text{not}}{=} f(A) \subseteq K. \tag{3.5}$$

A similar notation may also be used in more general cases, when the argument taking all possible values in a certain (sub) set or (sub)space is one among several variables. For instance,

$$\{ f(x, y, \ldots) : (\forall\, x \in A) \} \underset{\text{not}}{=} f(A, y, \ldots). \tag{3.5'}$$

With such a notation, the definition of the kernel of a symmetric BLF - Eq. (3.4) - becomes

$$\boxed{\operatorname{Ker} f \underset{\text{def}}{=} \{ x \in V : f(x, V) = 0 \}.} \tag{3.4'}$$

As we saw for the kernel of a linear form (PROPOSITION 1.3 in **§ 2.1**), this subset is in fact more that a <u>subset</u> of the vector space: it is a <u>subspace</u>.

PROPOSITION 3.1. *If* $f: V \times V \longrightarrow \mathbf{K}$ *is a symmetric bilinear form, then*

$$\boxed{\operatorname{Ker} f \subseteq_{\text{subsp}} V.} \tag{3.6}$$

Proof. We have to check that the subset defined by (3.4) / (3.4') is a subspace since it is closed under arbitrary linear combinations of (two) vectors. Let us recall the property $(\mathbf{BLF}_{1'})$ from *Definition 2.1'* of **§ 2.1** :

$$\begin{cases} (\forall\, \lambda_1, \lambda_2 \in \mathbf{K})(\forall\, x_1, x_2, y \in V) \\ \qquad f(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 f(x_1, y) + \lambda_2 f(x_2, y). \end{cases} \tag{3.7}$$

This property with a notation of the form (3.5') becomes

$$(\forall\, \lambda_1, \lambda_2 \in \mathbf{K})(\forall\, x_1, x_2 \in V)\ f(\lambda_1 x_1 + \lambda_2 x_2, V) = \lambda_1 f(x_1, V) + \lambda_2 f(x_2, V). \tag{3.8}$$

But, by definition (3.4'),

$$x_1, x_2 \in \operatorname{Ker} f \subseteq V \Rightarrow [x_1, x_2 \in V \ \&\ f(x_1, V) = f(x_2, V) = 0] \underset{(3.8)}{\Rightarrow}$$

$$(\forall\, \lambda_1, \lambda_2 \in \mathbf{K})(\forall\, x_1, x_2 \in V)\ f(\lambda_1 x_1 + \lambda_2 x_2, V) = 0 + 0 = 0 \Rightarrow \lambda_1 x_1 + \lambda_2 x_2 \in \operatorname{Ker} f \Rightarrow$$

$$\Rightarrow (3.6).$$

■

Remark 3.3. Since the kernel of a symmetric BLF is a subspace, it necessarily contains the zero vector **0**, hence it is always nonempty. According to the inclusion relations (3.6) in *Remark 3.2* of **§ 1.3** , for any symmetric BLF f, $\{\mathbf{0}\} \subseteq \operatorname{Ker} f \subseteq V$.

The two extreme cases in the previous *Remark* correspond to the situations when *only the zero* *vector* $\mathbf{0}$ is mapped onto $0 \in \mathbf{K}$ by $f(x, V)$, and to the *constant zero BLF* $f(V, V) = 0$, respectively. A characterization of the kernel of a symmetric BLF (with respect to a certain basis in V) is given by

PROPOSITION 3.2. *If* $f : V \times V \longrightarrow \mathbf{K}$ *is a symmetric bilinear form and* $A = [a_i : i = \overline{1,n}]$ *is a basis of space* V, *then*

$$x \in \operatorname{Ker} \iff (\forall\, i \in \{1, 2, \ldots, n\})\ f(x, a_i) = 0. \tag{3.9}$$

Proof. $(\Rightarrow)$ This implication immediately follows from (3.4) in *Definition 3.2* since, for any $i \in \overline{1,n}$, $a_i \in V$. $(\Leftarrow)$ Let us now show that the n equations in (3.9) imply (3.4), that is the membership $x \in \operatorname{Ker} f$. **For any** $y \in V$, y is linearly expressible in the basis A :

$$(\exists^! \,[\eta_1\ \eta_2 \ldots \eta_n]^{\mathrm{T}} \in \mathbf{K}^n)\ y = A\,Y_A \iff y = \sum_{i=1}^{n} \eta_i\, a_i. \tag{3.10}$$

According to Eq. (2.4) of PROPOSITION 2.1 in **§ 2.2** ,

$$f(x, y) \underset{(3.10)}{=} f\Big(x, \sum_{i=1}^{n} \eta_i\, a_i\Big) = \sum_{i=1}^{n} \eta_i\, f(x, a_i) \underset{(3.9)}{=} \sum_{i=1}^{n} 0 = 0. \tag{3.11}$$

Hence $x \in \operatorname{Ker} f$ and the proof is complete. ■

Remark 3.4. Let us remark that the n equations in (3.9) are equivalent to a homogeneous system of matrix F_A. Indeed, these n equations can be expressed, using a matrix notation, as

$$f(x, A) = \mathbf{0}^{\mathrm{T}} \iff f(X_A^{\mathrm{T}} \cdot A^{\mathrm{T}}, A\ ) = \mathbf{0}^{\mathrm{T}} \iff$$

$$\iff X_A^{\mathrm{T}} \cdot f(A^{\mathrm{T}}, A\ ) = \mathbf{0}^{\mathrm{T}} \iff X_A^{\mathrm{T}} \cdot [\alpha] = \mathbf{0}^{\mathrm{T}} \iff [\alpha] \cdot X_A = \mathbf{0}. \tag{3.12}$$

The last equivalence in (3.12) follows by transposition and by symmetry of matrix $[\alpha]$: see (3.3) at page 101. Hence the coordinates of any vector in the kernel of the symmetric BLF f, with its matrix $f(A^{\mathrm{T}}, A) = [\alpha]$ in basis A, should satisfy the homogeneous system

$$\boxed{[\alpha] \cdot X_A = \mathbf{0}.} \tag{3.13}$$

Hence the column vector of the coordinates of $x \in \operatorname{Ker} f$ is given by the general solution of the homogeneous system in (3.13).

This remark gives the ground for stating and proving a property that involves the dimension of $\operatorname{Ker} f$. But its statement needs another notion to be previously introduced.

Definition 3.3. Let $f : V \times V \longrightarrow \mathbf{K}$ be a symmetric BLF. The rank of f is equal to the rank of its coefficient matrix F_A in any basis A of V :

$$\operatorname{rank} f \underset{\text{def}}{=} \operatorname{rank} F_A \, : \, A = \text{ a basis of } V. \tag{3.14}$$

◊

Certainly, the rank could be defined for more general BLFs of the form $f: U \times V \longrightarrow \mathbf{K}$ as the rank of any of its matrices in a pair of bases, $F_{A,B}$; but this notion is less relevant for such general bilinear forms. By the way, we briefly mentioned the rank of a BLF in an exercise of **§ 2.2-A , 2-A.3 .**

PROPOSITION 3.3. *If* $f: V \times V \longrightarrow \mathbf{K}$ *is a symmetric bilinear form* (*of rank* $= r$) *then*

$$\boxed{\operatorname{rank} f + \dim(\operatorname{Ker} f) = n = \dim V.} \tag{3.15}$$

Proof. According to *Remark 3.3* , the coordinates of any vector $x \in \operatorname{Ker} f$ should satisfy the homogeneous system (3.13), whose matrix $f(A^{\mathrm{T}}, A) = [\alpha]_{n,n}$ is of rank $= r$. As it is known from **§§ 1.2 - 1.3**, the subspace of the solutions S of such a system satisfies the equation

$$\dim S = n - r. \tag{3.16}$$

But S is just $\operatorname{Ker} f$, therefore $\dim(\operatorname{Ker} f) = n - r \Rightarrow$ (3.15). ■

Example 3.1. Let $f: V \times V \longrightarrow \mathbb{R}$ be a symmetric BLF whose matrix in a basis A of V is

$$f(A^{\mathrm{T}}, A) = [\alpha]_{n,n} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -1 & 1 & -3 \\ 3 & 1 & 4 & -2 \\ 1 & -3 & -2 & -4 \end{bmatrix}. \tag{3.17}$$

It is required to find $\operatorname{Ker} f$, $\operatorname{rank} f$ and to check Eq. (3.15) on this example.

We can look for the set of solutions of thee homogeneous system (3.13) as the nullspace of the coefficient matrix of f. By (3.17),

$$[\alpha] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -1 & 1 & -3 \\ 3 & 1 & 4 & -2 \\ 1 & -3 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -5 & -5 & -5 \\ 0 & -5 & -5 & -5 \\ 0 & -5 & -5 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.18}$$

Hence the general solution of this system, giving the coordinates (in basis A) of a vector in $\operatorname{Ker} f$, can be derived from the last matrix in the sequence of transformations (3.19) :

$$X_A(\lambda, \mu) = \begin{bmatrix} -\lambda + 2\mu \\ -\lambda - \mu \\ \lambda \\ \mu \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}. \tag{3.19}$$

It follows from (3.19) that the general form of a vector in $\mathrm{Ker}\, f$ is

$$x(\lambda,\mu) = (-\lambda+2\mu)\,a_1-(\lambda+\mu)\,a_2+\lambda\,a_3+\mu\,a_4 = \tag{3.20}$$

$$= \lambda(-a_1-a_2+a_3)+\mu(2a_1-a_2+a_4). \tag{3.21}$$

Thus, $\dim \mathrm{Ker}\, f = 2$ since the vectors between (...) and (...) in (3.21) are clearly independent, and $\mathrm{rank}\, f = 2$ as it follows from the quasi-triangular form of the last matrix in (3.18), which is rank-equivalent to $f(A^{\mathrm{T}},A) = [\alpha]$. Hence Eq. (3.15) is checked. □

Definition 3.4. Let $f: V\times V \longrightarrow \mathbf{K}$ be a symmetric BLF. Two vectors $x, y \in V$ are said to be *orthogonal with respect to* f if $f(x,y) = 0$. In this case we write $x \perp_f y$. Hence,

$$\boxed{x \perp_f y \underset{\text{def}}{\iff} f(x,y) = 0.} \tag{3.22}$$

♢

A couple of immediate properties of this binary relation of orthogonality (with respect to a given symmetric BLF) can be stated :

PROPOSITION 3.4. *If* $f: V\times V \longrightarrow \mathbf{K}$ *is a symmetric bilinear form on space* V *then* :

(i) $\quad x \perp_f y \iff y \perp_f x$;

(ii) $\quad (\forall\, x \in V)\ \mathbf{0} \perp_f x$;

(iii) $\quad x \perp_f u_1, u_2, \ldots, u_m \Rightarrow (\forall\, \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbf{K})$

$$x \perp_f \lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_m u_m.$$

Proof. (i) Immediately follows from *Def. 3.4* and from the symmetry of f.

(ii) is a consequence of *Remark 3.3* : if (at least) one of the arguments of a symmetric BLF is the zero vector $\mathbf{0}$ then $f(\mathbf{0},x) = f(x,\mathbf{0}) = 0$. This equality also follows from the linearity of any BLF (not necessarily symmetric) in each of its arguments : for an arbitrary vector $u \in V$,

$$f(\mathbf{0},x) = f(u-u,x) = f(u,x)-f(u,x) = 0.$$

(iii) also follows from the *Definition 3.4* and from the extended linearity of any BLF (in the second argument): see Eq. (2.4) in PROPOSITION 2.1 of **§ 2.2** (page 87). Indeed,

$$f(x,\ \lambda_1 u_1+\lambda_2 u_2+\ldots+\lambda_m u_m) = \sum_{i=1}^{m} \lambda_i\, f(u_i) \underset{(3.22)}{=} \sum_{i=1}^{m} 0 = 0 \Rightarrow (iii).$$

■

The relation of orthogonality may be extended *from vectors to subsets* or *subspaces* of vector spaces, as follows :

Definition 3.5. Let $f: V\times V \longrightarrow \mathbf{K}$ be a symmetric BLF and

$U, W \subseteq V$ or $U, W \subseteq_{\text{subsp}} V$ & $x \in U$.

Then

$$x \perp_f W \underset{\text{def}}{\iff} f(x, W) = 0 \iff (\forall\, w \in W)\; f(x, w) = 0\,; \tag{3.23}$$

$$U \perp_f W \underset{\text{def}}{\iff} f(U, W) = 0 \iff (\forall\, u \in U)(\forall\, w \in W)\; f(u, w) = 0. \tag{3.24}$$

If $U, W \subseteq_{\text{subsp}} V$ and (3.24) is satisfied then W is said to be the *orthogonal* (*subspace*) *onto* U and it is denoted $W = U^{\perp_f}$ or simply $W = U^{\perp}$ when f is understood. ◊

Properties of the orthogonal subset (subspace) onto another subset (subspace) are stated in

PROPOSITION 3.5. *If* $f : V \times V \longrightarrow \mathbf{K}$ *is a symmetric bilinear form on space* V *and if* $U \subseteq V$ *or* $U \subseteq_{\text{subsp}} V$ *then* :

(i) $U^{\perp_f} \subseteq_{\text{subsp}} V$;

(ii) *if* $\mathrm{A} = \{a_1, a_2, \ldots, a_m\}$ *spans* U *then* $y \in U^{\perp_f} \iff (\forall\, i : 1 \le i \le m)\; y \perp_f a_i$.

Proof. (i) will be proved using *Definition 3.1"* (in **§ 1.3**, Eq. (3.3)) of a subspace. Let us consider two vectors $y_1, y_2 \in W =_{\text{not}} U^{\perp_f}$ and two arbitrary scalars $\lambda_1, \lambda_2 \in \mathbf{K}$. According to (3.24) in *Def. 3.5*, $(\forall\, u \in U)\; f(u, y_1) = f(u, y_1) = 0$. It follows that

$$(\forall\, u \in U)\; f(u, \lambda_1 y_1 + \lambda_2 y_1) = \lambda_1 f(u, y_1) + \lambda_2 f(u, y_2) = \lambda_1 0 + \lambda_2 0 = 0. \tag{3.25}$$

But (3.25) shows that $W = U^{\perp_f}$ is closed under arbitrary linear combinations of two vectors, hence (i) holds.

(ii) Implication ($\Leftarrow$) follows from *Def. 3.5* , that is from (3.24) with $W = U^{\perp_f}$ C, $u = y$ and $w = a_i$, $i = \overline{1, m} \;\Rightarrow\; f(y, a_1) = f(y, a_2) = \ldots = f(y, a_m) = 0$. But

$$U = \mathcal{L}(\{a_1, a_2, \ldots, a_m\}) \;\Rightarrow\; (\forall\, u \in U)\; u = \sum_{i=1}^{m} \lambda_i a_i \;\Rightarrow$$

$$\Rightarrow\; (\forall\, u \in U)\; f(u, y) = f\Big(\sum_{i=1}^{m} \lambda_i a_i, y\Big) = \sum_{i=1}^{m} \lambda_i f(a_i, y) = \sum_{i=1}^{m} 0 = 0 \;\Rightarrow\; y \in U^{\perp_f}.$$

The converse implication ($\Rightarrow$) uses the hypothesis that $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$ spans U, hence $\mathcal{A} = \{a_1, a_2, \ldots, a_m\} \subset U \;\Rightarrow\; y \perp_f a_1, a_2, \ldots, a_m$.

■

Remarks 3.5. In the statement of PROPOSITION 3.5 we have admitted both variants, that is $U \subseteq V$ or $U \subseteq_{\text{subsp}} V$. Indeed, it is not essential for U to be a subspace of V: it may be a simple subset, but its orthogonal $W = U^{\perp_f}$ is a subspace. As regards the orthogonality relation between a vector and a subset or subspace - see (3.23) - or between two subsets /subspaces defined by

(3.24), the symmetric relations may also be considered :

$$U \perp_f y \underset{\text{def}}{\iff} f(U, y) = 0 \iff (\forall u \in U)\, f(u, y) = 0\ ; \tag{3.23'}$$

$$U \perp_f W \underset{\text{def}}{\iff} f(U, W) = 0 \iff (\forall u \in U)(\forall w \in W)\, f(u, w) = 0 \iff$$

$$\iff (\forall w \in W)(\forall u \in U)\, f(u, w) = 0 = f(w, u) \underset{\text{def}}{\iff} W \perp_f U. \tag{3.24'}$$

An example would be useful for illustrating this relation of orthogonality with respect to a symmetric (as well as the properties in PROPOSITION 3.5).

Example 3.2. Let $f: \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a symmetric BLF whose matrix in the standard basis E of $\mathbb{R}^3$ is

$$f(E^{\mathrm{T}}, E) = [\varepsilon]_{3,3} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 0 \end{bmatrix}. \tag{3.26}$$

It is required to check that $[2\ \ -1\ \ 2]^{\mathrm{T}} \perp_f [1\ \ 1\ \ 2]^{\mathrm{T}}$ and then to find $U^{\perp f}$ knowing that the subspace U is spanned by $a_1 = [3\ \ 4\ \ -1]^{\mathrm{T}}$ & $a_2 = [2\ \ 3\ \ -2]^{\mathrm{T}}$.

Using the analytical expression of a BLF $f: \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ in the standard basis, that is (2.13) in **§ 2.2** , we have

$$f\left(\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right) = [2\ \ -1\ \ 2] \cdot \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = [5\ \ -1\ \ -2] \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 0.$$

In order to find a general vector Y in the orthogonal $U^{\perp f}$ of the subspace $U = \mathcal{L}(\{a_1, a_2\})$, we have to impose the conditions stated in (ii) of PROPOSITION 3.5 for the two vectors spanning U:

$$f(a_1, Y) = f(a_2, Y) = 0 \iff \begin{bmatrix} 3 & 4 & -1 \\ 2 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 0 \end{bmatrix} \cdot Y = \mathbf{0}\,. \tag{3.27}$$

But the matrix equation (3.27) is, in fact, a homogeneous system whose matrix is

$$\begin{bmatrix} a_1^{\mathrm{T}} \\ a_1^{\mathrm{T}} \end{bmatrix} \cdot [\varepsilon]_{3,3} = \begin{bmatrix} 2 & 7 & 8 \\ 1 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 18 \\ 0 & 1 & -4 \end{bmatrix}. \tag{3.29}$$

The last matrix in (3.29) offers the general solution of the homogeneous system (3.27), hence the required vector $Y \in U^{\perp f}$:

$$Y(\lambda) = \begin{bmatrix} -18\lambda \\ 4\lambda \\ \lambda \end{bmatrix} \text{ with } y_3 = \lambda.$$

Hence the orthogonal subspace of $U^{\perp_f}$ is spanned by the vector

$$b = \begin{bmatrix} -18 \\ 4 \\ 1 \end{bmatrix}.$$

The example is over, but let us see a simple way to check this result. It suffices to compute

$f([a_1\ a_2]^{\mathrm{T}}, Y(\lambda))$ or

$$f([a_1\ a_2]^{\mathrm{T}}, b) = \begin{bmatrix} 3 & 4 & -1 \\ 2 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} -18 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & 8 \\ 1 & 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} -18 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

□

In what follows, a new type of mappings is defined and some of their properties are presented.

Quadratic Forms

Informally, a quadratic form (abbreviated Q-form) is also a scalar mapping but its argument is a single vector in a vector space and not an odered pair of vector (as it was the case with the bilinear forms). A Q-form is associated to a symmetric BLF. Its formal definition follows.

Definition 3.6. Let $f : V \times V \longrightarrow \mathbf{K}$ be a symmetric BLF. The mapping $\varphi : V \longrightarrow \mathbf{K}$ defined by

$$\boxed{(\forall\, x \in V)\ \ \varphi(x) =_{\mathrm{def}} f(x, x)} \tag{3.30}$$

is said to be the *quadratic form* associated to f (or induced by f).

Some of the notions and results on / regarding the (symmetric) bilinear forms are connected with the Q-forms. For instance, if $A = [a_1\ a_2\ \ldots\ a_n]$ is a (finite) basis spanning the space V, then the matrix of the Q-form $\varphi : V \longrightarrow \mathbf{K}$ in this basis is simply the matrix in of the symmetric BLF f it is associated to, by *Def. 3.6* - (3.30) :

$$f(A^{\mathrm{T}}, A) = [\alpha_{ij}]_{n,n} \underset{\mathrm{not}}{=} [\alpha] \text{ with } \alpha_{ij} = f(a_i, a_j). \tag{3.31}$$

According to the property (3.3) given in this section, the matrix of a Q-form in any basis of the space is symmetric :

$$[\alpha] = [\alpha]^{\mathrm{T}} \text{ since } \alpha_{ji} = f(a_j, a_i) = f(a_i, a_j) = \alpha_{ij}. \tag{3.32}$$

As regards the analytical expression of a quadratic form in a (fixed / given) basis A, it follows from definition (3.30) and from Eq. (2.17') in **§ 2.2** replacing Y_A by X_A and $F_{A,B}$ by $F_A = [\alpha]$. Thus, if the argument of the Q-form is $x = A X_A$ then

$$\boxed{\varphi(x) =_{\text{def}} f(x, x) = X_A^{\mathrm{T}}[\alpha] X_A.} \tag{3.33}$$

The explicit analytical expression of the ("value" of) a Q-form in terms of the coordinates $X_A = [\,\xi_1\ \xi_2\ \dots\ \xi_n\,]^{\mathrm{T}}$ of $x \in V$ follows from formula (2.23) in **§ 2.2** of ⟦𝒜.𝒞., 1999⟧, also by replacing Y_A for X_A :

$$\boxed{\varphi(x) =_{\text{def}} f(x, x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij}\, \xi_i\, \xi_j\,.} \tag{3.34}$$

Expression (3.34) gives the reason for using the term of "*quadratic forms*" : $\varphi(x)$ of Eq. (3.34) is a homogeneous polynomial or order two in the coordinates $\xi_i\ (i = \overline{1,n})$ of x in basis A, hence it is a **quadratic** function in n variables.

In the particular case when $V = \mathbf{K}^n\ (=\mathbb{R}^n) \Rightarrow \varphi : \mathbf{K}^n \longrightarrow \mathbf{K}\ (\varphi : \mathbb{R}^n \longrightarrow \mathbb{R})$, the analytical expression of a Q-form with matrix notations as in (3.33) or under its explicit form of (3.34) are obtained from these formulas by replacing $X_A \rightarrow X\ \&\ \xi_i \rightarrow x_i\ (i = \overline{1,n})$:

$$\boxed{\varphi(X) =_{\text{def}} f(X, X) = X^{\mathrm{T}}[\varepsilon] X = \sum_{i=1}^{n} \sum_{j=1}^{n} \varepsilon_{ij}\, x_i x_j} \tag{3.35}$$

Kernels of Q-forms. The *kernel* of a quadratic form as defined by *Def. 3.6* - Eq. (3.30) is given by

$$\boxed{\operatorname{Ker} \varphi \underset{\text{def}}{=} \{x \in V :\ \varphi(x) = 0 \in \mathbf{K}\}.} \tag{3.36}$$

In fact, this subspace is just the kernel of the symmetric BLF f "staying behind" the quadratic form φ, according to (3.33).

Remarks 3.6. Other earlier defined notions (and properties – stated and proved) for the symmetric BLFs can be naturally transferred to the quadratic forms. Let us mention the practical way to find the kernel of a Q-form φ as the kernel of the BLF f that defines φ, as the solution subspace of the homogeneous system (3.13) :

$$\operatorname{Ker} \varphi \underset{(3.33)}{=} \operatorname{Ker} f \underset{(3.13)}{=} S = \{x \in V : x = A X_A\ \&\ [\alpha] X_A = \mathbf{0}\}. \tag{3.37}$$

If the quadratic form φ is defined on space $V = \mathbf{K}^n\ (=\mathbb{R}^n) \Rightarrow \varphi : \mathbf{K}^n \longrightarrow \mathbf{K}\ (\varphi : \mathbb{R}^n \longrightarrow \mathbb{R})$, and it is derived from the analytical expression of f or from the coefficient matrix thereof in the standard basis E, its expression is of the form (3.35) and the characterization (3.37) of its kernel becomes

$$\operatorname{Ker}\varphi \underset{(3.30)}{=} \operatorname{Ker} f = S \underset{(3.35)}{=} \{X \in V : X = EX \ \& \ [\varepsilon]X = \mathbf{0}\}. \tag{3.38}$$

As regards the rank of φ, it follows from (3.30) that

$$\operatorname{rank}\varphi = \operatorname{rank} f \underset{\text{def}}{=} \operatorname{rank} F_A : A = \text{ a basis of } V. \tag{3.39}$$

The definition in (3.39) includes the case when $\varphi : \mathbf{K}^n \longrightarrow \mathbf{K}$ ($\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$) and then $\operatorname{rank}\varphi$ will be determined by means of φ's matrix in the standard basis, $[\varepsilon] = f(E^{\mathrm{T}}, E)$. Let us close this discussion by the remark that the number of terms in the analytical expression of a Q-form φ, let it be either (3.34) or (3.35) theoretically equals n^2 but it can be reduced to

$$n + \frac{n^2 - n}{2} = \frac{n(n+1)}{2}$$

due to the symmetry of φ: if n^2 then

$$\alpha_{ij}\xi_i\xi_j = \alpha_{ji}\xi_j\xi_i \Rightarrow \alpha_{ij}\xi_i\xi_j + \alpha_{ji}\xi_j\xi_i = 2\alpha_{ij}\xi_i\xi_j. \tag{3.40}$$

Conversely, if φ is given by its analytical expression (3.34) or (3.35) involving only $n(n+1)/2$ terms, its matrix in the corresponding basis will be written by a kind of symmetrization: the coefficients $2\alpha_{ij}$ (or $2\varepsilon_{ij}$) will be written as $2\alpha_{ij} = \alpha_{ij} + \alpha_{ji}$ $(\alpha_{ij} = \alpha_{ji})$ and the two equal "halves" of every coefficient of a term $2\alpha_{ij}\xi_i\xi_j$ will be symmetrically introduced in φ's matrix $[\alpha]$ (and similarly for $[\varepsilon]$).

***Remark* 3.7.** Formula (3.33) gives the connection from a symmetric BLF to the Q-form it induces. But there also exists a converse connection, as stated in

PROPOSITION 3.6. *If* $\varphi : V \longrightarrow \mathbf{K}$ *is a quadratic form, the symmetric BLF* f *that induces* φ *by Eq.* (3.33) *is given by*

$$\boxed{f(x,y) = \tfrac{1}{4}[\varphi(x+y) - \varphi(x-y)].} \tag{3.41}$$

Proof.

$$\varphi(x+y) \underset{\text{def}}{=} f(x+y, x+y) \underset{\text{(LIN)}}{=} f(x,x) + 2f(x,y) + f(y,y); \tag{3.42}$$

$$\varphi(x-y) \underset{\text{def}}{=} f(x-y, x-y) \underset{\text{(LIN)}}{=} f(x,x) - 2f(x,y) + f(y,y). \tag{3.43}$$

Taking the difference (3.42) - (3.43) we get

$$\varphi(x+y) - \varphi(x-y) = \ldots = 4f(x,y) \Rightarrow \text{(3.41)}.$$

The proof is thus over. Let us however check connection (3.30) using expression (3.41) of f in terms of the Q-form φ :

$$f(x,x) \underset{(3.41)}{=} \tfrac{1}{4}[\varphi(x+x) - \varphi(x-x)] = \tfrac{1}{4}[\varphi(x+x) - \varphi(\mathbf{0})] = \tfrac{1}{4}4\,\varphi(x) = \varphi(x): \quad (3.33)$$

In deriving (3.41) from (3.42) & (3.43), not only the linearity of f has been involved but also its symmetry. As regards the way we have derived (3.33) from (3.41) – taking $y = x$ – two obvious properties of a Q-form have been taken into account :

$$\varphi(\mathbf{0}) = f(\mathbf{0},\mathbf{0}) = 0 \ \ \& \ \ \varphi(2x) = f(2x, 2x) = 4\,\varphi(x). \quad (3.44)$$

The last equality in (3.44) is quite natural for a <u>quadratic</u> function like a Q-form.

■

Example 3.3. Let $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a Q-form whose analytic expression (in the standard basis E of $\mathbb{R}^3$) is

$$\varphi(X) = 4x_1^2 - x_2^2 + 4x_3^2 + 6x_1x_2 - 8x_1x_3 + x_2x_3. \quad (3.45)$$

It is required to write the matrix of φ (in the standard basis E), to find its rank and kernel and to calculate $\varphi([-1\ \ 2\ \ 5]^{\mathrm{T}})$.

The matrix in the basis E of $\mathbb{R}^3$ follows from (3.45), also taking into account one of the *Remarks 3.6* (mainly relevant for the terms with odd coefficients) :

$$f(E^{\mathrm{T}}, E) = [\varepsilon]_{3,3} = \begin{bmatrix} 4 & 3 & -4 \\ 3 & -1 & 1/2 \\ -4 & 1/2 & 4 \end{bmatrix}. \quad (3.46)$$

$$(3.46) \Rightarrow \ \det[\varepsilon] = -49 \neq 0 \ \Rightarrow \ \operatorname{rank}\varphi = 3 \ \Rightarrow \ \operatorname{Ker}\varphi = \{\mathbf{0}\}.$$

Formula (3.35) with the vector in the statement and matrix of (3.46) gives

$$\varphi([-1\ \ 2\ \ 5]^{\mathrm{T}}) = [-1\ \ 2\ \ 5]\cdot\begin{bmatrix} 4 & 3 & -4 \\ 3 & -1 & 1/2 \\ -4 & 1/2 & 4 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = [-18\ \ -5/2\ \ 25]\cdot\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = 138.$$

□

Diagonalization of the Q - forms

An important problem concerning the quadratic forms regards the possibility to "simplify" their analytic expressions. In terms of the matrix-type formulations (3.33) and also (3.35), to obtain such reduced expressions means to bring the matrix of a Q-form to a *diagonal form* : the off-diagonal entries should vanish. Such simplifications can be accomplished by appropriate changes of basis, which transform the matrix of a symmetric BLF f into a diagonal one ; see **§ 2.2** - *Def. 2.1* at page 35. The effect of such transformations on analytical expressions like (3.34) - (3.35) should be a reduction of the n^2 terms in the respective sums to only n terms in the *squares* of the n coordinates (components). A preliminary result – a consequence of a formula presented in **§ 2.2**

– PROPOSITION 2.3' with Eq. (2.29) – should be re-stated for the Q-forms.

PROPOSITION 3.7. *If* $\varphi : V \longrightarrow \mathbf{K}$ *is a quadratic form with its matrix* $F_A = [\alpha]$ in a basis A of V *and if the basis* A *is changed for another basis* $\bar{A}$ *by the transformation*

$$\bar{A}^{\mathrm{T}} = T \cdot A^{\mathrm{T}} \iff \bar{A} = A \cdot T^{\mathrm{T}} \tag{3.47}$$

then the coefficient matrix of f *in the new basis* $\bar{A}$ *is given by*

$$\boxed{F_{\bar{A}} = f(\bar{A}^{\mathrm{T}}, \bar{A}) = T \cdot F_A \cdot T^{\mathrm{T}}} \quad \text{or} \quad \boxed{[\bar{\alpha}] = T \cdot [\alpha] \cdot T^{\mathrm{T}}.} \tag{3.48}$$

It follows to see what kind of transformations have to be applied to the basis A in order to obtain a diagonal expression / matrix for the Q-form φ. By the way, such a reduced analytical expression of a Q-form is usually called a *canonical* expression and it looks like

$$\varphi(x) = \sum_{i=1}^{n} c_i \bar{\xi}_i^2 \quad \text{or} \quad \varphi(X) =_{\text{def}} f(X, X) = \sum_{i=1}^{n} d_i \bar{x}_i^2. \tag{3.49}$$

Several methods exist for turning the analytical expression / the coefficient matrix of a Q-form into a "canonical" expression line the ones in (3.49), what is equivalent to bring the matrix $[\alpha]$ (and similarly for $[\varepsilon]$) to a diagonal form. The coefficients c_i / d_i that occur in (3.49) could be the diagonal entries in the transformed matrices in (3.48). That is,

$$c_i \underset{(3.48)}{=} \bar{\alpha}_{ii} \ / \ d_i = \beta_{ii} \text{ for } E \to B \text{ with } f(B^{\mathrm{T}}, B) = [\beta]. \tag{3.50}$$

The variables with bars denote the new coordinates of the argument vector after the change of basis : $\bar{\Xi} = [\bar{\xi}_1 \ \ldots \ \bar{\xi}_i \ldots \ \bar{\xi}_n]^{\mathrm{T}} = X_{\bar{A}}$, $\bar{X} = [\bar{x}_1 \ \ldots \ \bar{x}_i \ldots \ \bar{x}_n]^{\mathrm{T}} = X_B$.

Some of the methods we are going to present operate on the analytical expressions, other ones operate on the coefficient matrices.

I. Gauss's Method

This method operates on Q-form's analytical expression(s), that is (3.34) or (3.35). More precisely, the coordinates of x in basis A / the components of $X \in \mathbb{R}^n \vee X \in \mathbf{K}^n$ are transformed into new ones via a chain of linear transformations. We present the essentials of this method in the latter case, that is for a Q-form $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$; taking a more general field $\mathbf{K}$ instead of the real field $\mathbb{R}$ would be a fake generalization. Hence the transformations should change an expression like (3.35) into a canonical expression of the form (3.49), with the (second) notation in (3.50). This change of the coefficients ia achieved by linear transformations on the components of $X \in \mathbb{R}^n \vee X \in \mathbf{K}^n$ of the form

$$X \to Y = L^{(1)} X, \ Y \to Z = L^{(2)} Y, \ldots, \ U \to \bar{X} = L^{(m)} U. \tag{3.51}$$

The matrices $L^{(k)} = [\lambda_{ij}^k]$ $(1 \leq k \leq m)$ are square matrices, whose entries λ_{ij}^k have to be determined. The idea behind this method is to eliminate the terms in expression (3.34) or (3.35) containing products of the form $x_i x_j$ $(i \neq j)$. In other words, the coefficients a_{ij}^k with $i \neq j$ should be turned to zero. A rather intuitive description of the method follows. Since it is applied to either of the two kinds of analytical expression (3.34) or (3.35) and also to the intermediate expression obtained during the application of this algorithm, we denote the coefficients by a_{ij} instead of ε_{ij} or α_{ij} $(1 \leq i, j \leq n)$, while $\bar{a}_{ij}$ will denote the coefficients got after a step in a chain of transformations as in (3.51).

G.1 If $a_{11} \neq 0$, a_{11} is taken out as a forced factor from the group of terms containing x_1 :

$$a_{11}\left(x_1^2 + 2\,\frac{a_{12}}{a_{11}}\,x_1 x_2 + \ldots + 2\,\frac{a_{1n}}{a_{11}}\,x_1 x_n\right). \tag{3.52}$$

G.2 Next, the terms inside the parentheses of (3.52) are written as the square of a linear function, minus the terms artificially added :

$$x_1^2 + 2\,\frac{a_{12}}{a_{11}}\,x_1 x_2 + \ldots + 2\,\frac{a_{1n}}{a_{11}}\,x_1 x_n = (x_1 + \bar{a}_{12}\,x_2 + \ldots + \bar{a}_{1n}\,x_n)^2 - S_2 \tag{3.53}$$

where $\bar{a}_{1j} = a_{1j}/a_{11}$, $j = \overline{2,n}$ and S_2 contains just the artificially added terms. If we denote

$$-\,a_{11}\,S_2 + a_{22}x_2^2 + 2a_{23}\,x_2 x_3 + \ldots + a_{nn}x_n^2 = T_2, \tag{3.54}$$

this T_2 is a quadratic form in variables $x_2, x_3, \ldots, x_n$ and let us denote it as

$$\varphi_2(x_2, x_3, \ldots, x_n) = T_2.$$

G.3 The first variable change to be applied is

$$(\mathrm{T}_1): \begin{cases} y_1 = x_1 + \bar{a}_{12}x_2 + \ldots + \bar{a}_{1n}x_n, \\ y_2 = x_2, \\ \quad\vdots \\ y_n = x_n \end{cases} \quad \Rightarrow \quad (\mathrm{T}_1^{-1}): \begin{cases} x_1 = y_1 - \bar{a}_{12}y_2 - \ldots - \bar{a}_{1n}y_n, \\ x_2 = y_2, \\ \quad\vdots \\ x_n = y_n. \end{cases}$$

This transformation $(\mathrm{T}_1): X \to Y$ changes the initial Q-form

$$\varphi(X) = f(X,X) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\,x_{ij} = a_{11}\,x_1^2 + \varphi_2(x_2, x_3, \ldots, x_n) \text{ into}$$

$$\varphi^{(1)}(Y) = a_{11}\,y_1^2 + \varphi_2(y_2, y_3, \ldots, y_n). \tag{3.55}$$

The procedure is continued with the Q-form $\varphi_2(y_2, y_3, \ldots, y_n)$ from step **G.1** thru **G.3** and each such cycle creates another "perfect square" term like $a_{11}\,y_1^2$ in (3.55) until all the "mixed terms" are eliminated. If the last Q-form obtained after the n - th transformation is

$$\varphi^{(n)}(U) = \sum_{i=1}^{n} d_i u_i^2$$

then the canonical expression of the Q-form is

$$\varphi^{(n)}(U) = \bar{\varphi}(\bar{X}) = \sum_{i=1}^{n} d_i \bar{x}_i^2. \tag{3.56}$$

The last transformation leading to the canonical expression (3.56) is a simple change of letters / symbols : $u_i =_{\text{not}} \bar{x}_i$, $i = \overline{1,n}$. The matrix of this pseudo-transformation is just I_n.

A special case may occur, namely the one when the first coefficient in the initial analytic expression (or the north-western entry in the initial matrix) is zero : $a_{11} = 0$. In this situation, a preliminary transformation can be applied before the first step $\boxed{\mathbf{G.1}}$:

$\boxed{\mathbf{G.0}}$ $\quad x_1 = u_1 - u_2 \;\&\; x_1 = u_1 + u_2 \Rightarrow x_1 x_2 = u_1^2 - u_2^2$ (3.57)

and te other coordinates / components are identically transformed (re-denoted). This transformation (3.57) introduces a perfect square $2a_{12} u_1^2$ and the algorithm is continued from step $\boxed{\mathbf{G.1}}$ with U instead of X.

The way this GAUSS's METHOD "works" will become more explicit through the following

Example 3.4. It is required to bring to a canonical (or diagonal) expression the Q-form

$$\varphi(X) = x_1^2 + x_2^2 + x_3^2 - 2x_4^2 - 2x_1x_2 + 2x_1x_3 - 2x_1x_4 + 2x_2x_3 - 2x_2x_4. \tag{3.58}$$

The first step in Gauss's method starts by grouping the terms that contain x_1 and then (according to step **G.2**) by taking together the similar terms.

$$\begin{aligned}\varphi(X) &= (x_1^2 - 2x_1x_2 + 2x_1x_3 - 2x_1x_4) + x_2^2 + x_3^2 - 2x_4^2 + 2x_2x_3 - 2x_2x_4 = \\ &= [(x_1 - x_2 + x_3 - x_4)^2 - x_2^2 - x_3^2 - x_4^2 + 2x_2x_3 - 2x_2x_4 + 2x_3x_4] + \\ &\qquad + x_2^2 + x_3^2 - 2x_4^2 + 2x_2x_3 - 2x_2x_4 = \\ &= (x_1 - x_2 + x_3 - x_4)^2 - 3x_4^2 + 4x_2x_3 - 4x_2x_4 + 2x_3x_4.\end{aligned} \tag{3.59}$$

The expression in (3.59) determines the first transformation (T_1) of step **G.3** :

$$(\mathrm{T}_1): \begin{cases} y_1 = x_1 - x_2 + x_3 - x_4, \\ y_2 = x_2, \\ y_3 = x_3, \\ y_4 = x_4. \end{cases} \tag{3.60}$$

$$(3.59)\ \&\ (3.60)\ \Rightarrow\quad \varphi^{(1)}(Y) = y_1^2 + \varphi_2(y_2, y_3, y_4) =$$
$$= y_1^2 - 3y_4^2 + 4y_2y_3 - 4y_2y_4 + 2y_3y_4. \tag{3.61}$$

Proceeding as earlier, the terms of $\varphi^{(1)}(Y)$ can be grouped and rewritten as

$$\begin{aligned}\varphi^{(1)}(Y) &= y_1^2 - 3y_4^2 + 4y_2y_3 - 4y_2y_4 + 2y_3y_4 = \\ &= y_1^2 + 4(y_2y_3 - y_2y_4) + 2y_3y_4 - 3y_4^2.\end{aligned} \tag{3.62}$$

The two terms inside (. . .) of (3.62) would have to produce the square of a linear form, but no term in y_2^2 exists. Consequently, a transformation of type (3.57) should be applied :

$$y_2 = u_2 - u_3\ \ \&\ \ y_3 = u_2 + u_3\ \Rightarrow\ y_2y_3 = u_2^2 - u_3^2. \tag{3.63}$$

Variables y_1 & y_4 are simply re-denoted, $y_1 = u_1$ & $y_4 = u_4$ and, with (3.62) & (3.63) we get

$$\begin{aligned}\varphi^{(2)}(U) &= u_1^2 + 4(u_2^2 - u_3^2) - 4(u_2 - u_3)u_4 + 2(u_2 + u_3)u_4 - 3u_4^2 = \\ &= u_1^2 + 4u_2^2 - 4u_3^2 - 4u_2u_4 + 2u_2u_4 + 4u_3u_4 + 2u_3u_4 - 3u_4^2 = \\ &= u_1^2 + (4u_2^2 - 2u_2u_4) - 4u_3^2 + 6u_3u_4 - 3u_4^2 = \\ &= u_1^2 + 4(u_2^2 - \tfrac{1}{2}u_2u_4) - 4u_3^2 + 6u_3u_4 - 3u_4^2 = \\ &= u_1^2 + 4[(u_2 - \tfrac{1}{4}u_4)^2 - \tfrac{1}{16}u_4^2] - 4u_3^2 + 6u_3u_4 - 3u_4^2 = \\ &= u_1^2 + 4(u_2 - \tfrac{1}{4}u_4)^2 - \tfrac{13}{4}u_4^2 - 4u_3^2 + 6u_3u_4 = \\ &= u_1^2 + 4(u_2 - \tfrac{1}{4}u_4)^2 - 4(u_3^2 - \tfrac{3}{2}u_3u_4) - \tfrac{13}{4}u_4^2 = \\ &= u_1^2 + 4(u_2 - \tfrac{1}{4}u_4)^2 - 4[(u_3 - \tfrac{3}{4}u_4)^2 - \tfrac{9}{16}u_4^2 - \tfrac{13}{4}u_4^2 = \\ &= u_1^2 + 4(u_2 - \tfrac{1}{4}u_4)^2 - 4(u_3 - \tfrac{3}{4}u_4)^2 - \tfrac{61}{16}u_4^2.\end{aligned} \tag{3.64}$$

A final transformation can turn expression (3.64) into a canonical expression.

$$(\mathrm{T}_2):\begin{cases}\bar{x}_1 = u_1, \\ \bar{x}_2 = u_2 - \tfrac{1}{4}u_4, \\ \bar{x}_3 = u_3 - \tfrac{3}{4}u_4, \\ \bar{x}_4 = u_4.\end{cases} \tag{3.65}$$

$$(3.64)\ \&\ (3.65)\ \Rightarrow\quad \varphi^{(2)}(\bar{X}) = \bar{\varphi}(\bar{X}) = \bar{x}_1^2 + 4\,\bar{x}_2^2 - 4\,\bar{x}_3^2 - \tfrac{61}{16}\,\bar{x}_4^2\,. \tag{3.66}$$

Comment. The transformation in (3.62) was applied for illustrating how a square can be formed when it does not exist: this was the case with the group of terms containing y_2 in (3.61). However, it was possible to operate another grouping of the terms containing y_4 since y_4^2 existed in expression (3.61). We go along this idea below.

$$\begin{aligned}\varphi^{(1)}(Y) &= y_1^2 - 3y_4^2 + 4\,y_2\,y_3 - 4\,y_2\,y_4 + 2\,y_3\,y_4 = \\ &= y_1^2 - 3\,(y_4^2 + \tfrac{4}{3}y_2\,y_4 - \tfrac{2}{3}y_3\,y_4) + 4\,y_2\,y_3 = \\ &= y_1^2 - 3\,[\,(y_4 + \tfrac{2}{3}y_2 - \tfrac{1}{3}y_3)^2 - \tfrac{4}{9}\,y_2^2 - \tfrac{1}{9}\,y_3^2 + \tfrac{4}{9}\,y_2 y_3\,] + 4\,y_2\,y_3 = \\ &= y_1^2 - 3\,(y_4 + \tfrac{2}{3}\,y_2 - \tfrac{1}{3}y_3)^2 + \tfrac{4}{3}\,y_2^2 + \tfrac{1}{3}\,y_3^2 + \tfrac{8}{3}\,y_2\,y_3\,.\end{aligned} \tag{3.67}$$

The last three terms of (3.67), in y_2 & y_3, must be further grouped for turning them into the square of an LF in y_2 & y_3 $\pm$ a term in y_2^2 or y_3^2. Let us denote this Q-form by

$$\begin{aligned}\varphi_3(y_2,y_3,y_4) &= \tfrac{4}{3}\,y_2^2 + \tfrac{1}{3}\,y_3^2 + \tfrac{8}{3}\,y_2\,y_3 = \tfrac{4}{3}\,(y_2^2 + 2\,y_2\,y_3 + \tfrac{1}{4}\,y_3^2) = \\ &= \tfrac{4}{3}\,(y_2^2 + 2\,y_2\,y_3) + \tfrac{1}{3}\,y_3^2 = \tfrac{4}{3}\,[\,(y_2 + y_3)^2 - y_3^2\,] + \tfrac{1}{3}\,y_3^2 = \\ &= \tfrac{4}{3}\,(y_2 + y_3)^2 - y_3^2\,.\end{aligned} \tag{3.68}$$

It now follows to apply another change of variables, namely

$$(\tilde{\mathrm{T}}_2):\begin{cases}\tilde{x}_1 = y_1\,,\\ \tilde{x}_2 = y_2 + y_3\,,\\ \tilde{x}_3 = y_3\,,\\ \tilde{x}_4 = \tfrac{2}{3}\,y_2 - \tfrac{1}{3}\,y_3 + y_4\,.\end{cases} \tag{3.69}$$

$$(3.67),\ (3.68)\ \&\ (3.69)\ \Rightarrow\quad \tilde{\varphi}^{(2)}(\tilde{X}) = \tilde{x}_1^2 + \tfrac{4}{3}\,\tilde{x}_2^2 - \tilde{x}_3^2 - 3\,\tilde{x}_4^2\,. \tag{3.70}$$

□

As a final remark to this example, let us see that Gauss's method is not very easy to be applied, especially to Q-forms defined on many-dimensional spaces; in this case, on $\mathbb{R}^4$. But this method is applicable to *any* Q-form, yielding a canonical expression. On another hand, at certain steps of the transformations, several ways to group the terms can be selected. As a consequence, several final canonical expressions will result. In our case, the canonical expressions (3.66) and (3.70) are different, but the numbers of positive, respectively negative terms *are the same*. This illustrates an important property of the Q-forms that will be approached a little later.

Another method to diagonalize quadratic forms is

II. Jacobi's Method

This method operates *on the matrix* of a Q-form (in a certain basis). However, the effective transformations of basis (or on the coordinates / components of the vector argument) that lead to the respective canonical / diagonal expression can be identified.

J.0 The matrix of φ in a certain basis should be available. If the Q-form φ is given by its analytical expression (in the coordinates $X_A = [\xi_1 \ \xi_2 \ \dots \ \xi_n]^{\mathrm{T}}$ or in the components of $X = [x_1 \ x_2 \ \dots \ x_n]^{\mathrm{T}}$), the matrix $[\alpha] = f(A^{\mathrm{T}}, A)$ or $[\varepsilon] = f(E^{\mathrm{T}}, E)$ should be written.

J.1 The "north-western" minors of matrix $[\alpha]$ or $[\varepsilon]$ are calculated :

$$\Delta_0 =_{\mathrm{def}} 1 \,,\ \Delta_1 = \alpha_{11} \,,\ \Delta_2 = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}, \dots ,\ \Delta_n = \det[\alpha] \,. \tag{3.71}$$

J.2 If all the determinants in the chain (3.71) are $\neq 0$, then φ admits a canonical (or diagonal) expression, namely

$$\bar{\varphi}(\bar{X}) = \sum_{i=1}^{n} d_i \bar{x}_i \ \text{ where } \ d_i = \frac{\Delta_{i-1}}{\Delta_i} \,,\ i = 1,2,\dots,n. \tag{3.72}$$

Remarks 3.8. A special situation occurs when not all the determinants in (3.71) are $\neq 0$; for instance,

$$\operatorname{rank}[\alpha] = r < n = \dim V \ \Rightarrow\ \Delta_{r+1} = \Delta_{r+2} = \dots \Delta_n = 0. \tag{3.73}$$

However, in the hypothesis of (3.73) on the rank of the matrix, a reordering of the variables (by simply reordering them) would be theoretically possible leading to a chain of "north-western" *nonzero* minors of the form

$$\Delta_0 =_{\mathrm{def}} 1 \,,\ \Delta_1 = \alpha_{11} \,,\ \Delta_2 = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}, \dots ,\ \Delta_r = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1r} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2r} \\ \vdots & \vdots & & \vdots \\ \alpha_{r1} & \alpha_{r2} & \dots & \alpha_{rr} \end{vmatrix} \tag{3.74}$$

such that a canonical expression of the form (3.72) (with r instead of n) exists :

$$\bar{\varphi}(\bar{X}) = \sum_{i=1}^{r} d_i \bar{x}_i \ \text{ where } \ d_i = \frac{\Delta_{i-1}}{\Delta_i} \,,\ i = 1,2,\dots,r.$$

Example 3.4'. Let us illustrate this remark on the Q-form of **Example 3.4**. Let us recall the analytical expression of that Q-form φ - Eq. (3.58) at page 90 :

$$\varphi(X) = x_1^2 + x_2^2 + x_3^2 - 2x_4^2 - 2x_1x_2 + 2x_1x_3 - 2x_1x_4 + 2x_2x_3 - 2x_2x_4 \,. \tag{3.58}$$

Since the matrix of φ in the standard basis corresponds to the analytical expression (3.58) in the components of vector φ we denote it as

$$[\varepsilon] \underset{\text{not}}{=} A_X = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & -2 \end{bmatrix}. \tag{3.75}$$

It can be easily seen that the chain (3.71) of *nonzero* minors is broken since

$$\Delta_2 = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0.$$

It is possible to interchange the variables x_2 & x_4 of (3.58) by the simple transformation

$$x_2 = y_4 \ \& \ x_4 = y_2$$

while the other two variables keep their subscripts : $x_1 = y_1$ & $x_3 = y_3$. This transformation changes both the analytical expression on (3.58) and the corresponding matrix :

$$\varphi(X) \to \psi(Y) = y_1^2 + y_4^2 + y_3^2 - 2y_2^2 - 2y_1y_4 + 2y_1y_3 - 2y_1y_2 + 2y_4y_3 - 2y_4y_2.$$

The matrix in (3.75) becomes

$$A_Y = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & -2 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}. \tag{3.76}$$

The chain of determinants (3.71) assigned to matrix (3.76) is

$$\Delta_0 =_{\text{def}} 1, \ \Delta_1 = 1, \ \Delta_2 = -3, \ \Delta_3 = -1, \ \Delta_4 = 4.$$

The corresponding canonical expression in the components of Y, by Jacobi's formula (3.72), is

$$\bar{\psi}(Y) = y_1^2 - \tfrac{1}{3} y_2^2 + 3y_3^2 - \tfrac{1}{4} y_4^2. \tag{3.77}$$

Since we used to denote the coordinates in the final (canonical) expression of a Q-form by $\bar{x}_i$, $i = \overline{1,4}$, we may put

$$y_1 = x_1 = \bar{x}_1, \ y_2 = x_4 = \bar{x}_2, \ y_3 = x_3 = \bar{x}_3, \ y_4 = x_2 = \bar{x}_4 \tag{3.78}$$

and the final canonical expression is

$$\bar{\varphi}(\bar{X}) = \bar{x}_1^2 - \tfrac{1}{3} \bar{x}_2^2 + 3\bar{x}_3^2 - \tfrac{1}{4} \bar{x}_4^2. \tag{3.79}$$

It can be seen that expression in (3.79) is somehow similar to the one in (3.69), and the number of positive / negative terms in all the three expressions (3.66), (3.70), (3.79) is the same. The transformation in (3.78) could be written in a similar way as (3.69), but this is not an essential point. □

However, a composition of transformations like the ones in Gauss's method, in terms of corresponding (triangular) matrices whose product is a matrix giving the connection

$$X = T^{\mathrm{T}}\overline{X} \iff \overline{X} = T^{-\mathrm{T}}X \tag{3.80}$$

would make possible to identify the basis in which the initial analytical expression (3.34) / (3.35) takes the canonical form (3.56). The transformations applied in the cycles of steps **G.1 , G.2 , G.3** of Gauss's method result in a triangular transformation matrix : this results from the transformation (T_1) in step **G.3** at page 89, follow ed by the other transformations that eliminate, one by one, the squared variable from each intermediate Q-form. The matrix of the transformation (T_1) in **Example 3.4** follows from its explicit form :

$$(\mathrm{T}_1):\begin{cases} y_1 = x_1 + \overline{a}_{12}x_2 + \ldots + \overline{a}_{1n}x_n, \\ y_2 = x_2, \\ \quad\vdots \\ y_n = x_n \end{cases} \Rightarrow\ Y = T^{(1)}X \ \text{ with } \ T^{(1)} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For the general transformation (T_1) (at page 89) this matrix is

$$T^{(1)} = \begin{bmatrix} 1 & \overline{a}_{12} & \ldots & \overline{a}_{1n} \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}. \tag{3.81}$$

If the next quadratic form in Y is (see Eq. (3.55) at page 89)

$$\begin{aligned} \varphi^{(1)}(Y) &= a_{11}y_1^2 + \varphi_2(y_2, y_3, \ldots, y_n) = a_{11}y_1^2 + \sum_{i=2}^{n}\sum_{j=2}^{n} b_{ij}y_iy_j = \\ &= a_{11}y_1^2 + b_{22}y_2^2 + 2b_{23}y_2y_3 + \ldots + b_{nn}y_n^2, \end{aligned} \tag{3.82}$$

the next transformation matrix (in $Z = T^{(2)}Y$) will be of the form

$$T^{(2)} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & \overline{b}_{23} & \ldots & \overline{b}_{2n} \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}. \tag{3.83}$$

It can be seen that the matrices $T^{(1)}, T^{(2)}, \ldots$ are upper-triangular and each of them contains an identity submatrix of decreasing order (in south-eastern / lower-right position) :

$$I_{n-1} \text{ in } T^{(1)}), I_{n-2} \text{ in } T^{(2)}, \ldots, I_1 \text{ in } T^{(n-1)}. \tag{3.84}$$

But the last (pseudo)transformation matrix is just I_n, corresponding to $U = \bar{X}$; see the mention at the end of step **G.3** in GAUSS's METHOD - page 89.

The compound effect of these transformations, from the initial variables (= the components of X, re-denoted as $w_1, w_2 \ldots, w_n$) to the last ones of U may be described by

$$(\mathrm{T}_{n-1}): \begin{cases} u_1 = w_1 + \bar{a}_{12} w_2 + \bar{a}_{13} w_3 + \ldots + \bar{a}_{1n} u_n, \\ u_2 = \qquad w_2 + \bar{b}_{23} w_3 + \ldots + \bar{b}_{2n} w_n, \\ \vdots \qquad\qquad \ddots \qquad\qquad \vdots \\ u_n = \qquad\qquad\qquad\qquad w_n. \end{cases} \tag{3.85}$$

The last basis resulting from the initial base (in our case E or A) can be looked for in terms of a triangular transformation, corresponding to the triangular change of coordinates of (3.85), under the form

$$C^{\mathrm{T}} = TA^{\mathrm{T}} \text{ where } T = \begin{bmatrix} \tau_{11} & 0 & 0 & \ldots & 0 \\ \tau_{21} & \tau_{22} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \tau_{n1} & \tau_{n2} & \ldots & \ldots & \tau_{nn} \end{bmatrix}. \tag{3.86}$$

If, in basis A, the analytical expression of the Q-form was (according to Eq. (3.34))

$$\varphi(X) = f(X,X) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} x_i x_j \quad \text{with} \quad \alpha_{ij} = f(a_i, a_j) \tag{3.87}$$

then, in order to obtain the canonical form of φ in basis $C = [c_1\ c_2\ \ldots\ c_n]$, the symmetric BLF f should satisfy the equations

$$f(c_i, c_j) = 0 \text{ for } i \neq j,\ 1 \leq i, j \leq n. \tag{3.88}$$

It is rather easy to see that, if $f(c_i, a_j) = 0$ then $f(c_i, c_j) = 0$ for $i \neq j$, $1 \leq i, j \leq n$. Indeed, it follows from (3.86) that $f(c_1, c_j) = \tau_{11} f(a_1, c_j)$, $j \neq 1$; hence

$$f(a_1, c_j) = 0 \ \&\ \tau_{11} \neq 0 \Rightarrow f(c_1, c_j) = 0,\ j > 1. \tag{3.89}$$

The expression of c_2 comes from the second row in matrix (3.86) :

$$c_2 = \tau_{21} a_1 + \tau_{22} a_2 \Rightarrow f(c_2, c_j) = \tau_{21} f(a_1, c_j) + \tau_{22} f(a_2, c_j) \Rightarrow$$

$$\Rightarrow f(c_2, c_j) \underset{(3.89)}{=} \tau_{22} f(a_2, c_j) = 0 \text{ if } f(a_2, c_j) = 0. \tag{3.90}$$

The implications of (3.89) & (3.90) hold for the next subscripts on c_j, until $j = n$. It follows that the conditions to be imposed on the transformation coefficients – the entries of the matrix T in (3.86) – are

$$f(a_i, c_j) = 0 \text{ for } i = 1, 2, \ldots, j-1 \;\&\; j = 1, 2, \ldots, n \text{ plus } f(a_j, c_j) = 1. \tag{3.91}$$

The explicit expressions of the vectors of basis C in terms of the vectors of $A = [a_1 \; a_2 \; \ldots \; a_n]$ follow from (3.86) :

$$\begin{cases} c_1 = \tau_{11} a_1, \\ c_2 = \tau_{21} a_1 + \tau_{22} a_2, \\ \vdots \qquad \vdots \qquad\qquad \ddots \\ c_n = \tau_{n1} a_1 + \tau_{n2} a_2 + \ldots + \tau_{nn} a_n. \end{cases} \tag{3.92}$$

Conditions (3.91) plus expressions (3.92), up to j, lead to the linear system

$$\begin{cases} \alpha_{11} \tau_{1j} + \alpha_{21} \tau_{2j} + \ldots + \alpha_{j-1,1} \tau_{j-1,1} + \alpha_{j1} \tau_{jj} = 0, \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \alpha_{1,j-1} \tau_{1j} + \alpha_{2,j-1} \tau_{2j} + \ldots + \alpha_{j-1,j-1} \tau_{j-1,j} + \alpha_{j,j-1} \tau_{jj} = 0, \\ \alpha_{1j} \tau_{1j} + \alpha_{2j} \tau_{2j} + \ldots + \alpha_{j-1,j} \tau_{j-1,j} + \alpha_{jj} \tau_{jj} = 1. \end{cases} \tag{3.93}$$

The matrix of this system is the square submatrix of $[\alpha]$, of order j, whose determinant is $\Delta_j \neq 0$. According to Cramer's rule, this system admits a unique solution, namely

$$\tau_{jj} = \frac{1}{\Delta_j} \Delta_{j-1}. \tag{3.94}$$

Let now $\Gamma = f(C^{\mathrm{T}}, C) = [\gamma_{ij}]$ be the matrix of the Q-form in basis C. Then, for any $i = \overline{1, j-1}$ we have

$$\gamma_{ij} = f(c_i, c_j) = f\Big(\sum_{k=1}^{i} \tau_{ki} a_k, c_j \Big) = \sum_{k=1}^{i} \tau_{ki} f(a_k, c_j) = 0 \Rightarrow \gamma_{ji} = 0 \tag{3.95}$$

for any $j \neq i$ since the matrix is symmetric. Next,

$$\gamma_{jj} = f(c_j, c_j) = f\Big(\sum_{k=1}^{j} \tau_{ki} a_k, c_j \Big) = \tau_{jj} f(a_j, c_j) = \tau_{jj} = \frac{1}{\Delta_j} \Delta_{j-1}. \tag{3.96}$$

In (3.95), $f(a_k, c_j) = 0$ since $k \leq i < j$ and the first $j-1$ equations in (3.91) hold; in (3.96), $f(a_j, c_j) = 1$ by the last equation in (3.91).

Hence, the analytical expression of the Q-form in basis C is

$$\boxed{\bar{\varphi}(\bar{X}) = \sum_{i=1}^{n} \gamma_{ii} \bar{x}_i^2 = \sum_{i=1}^{n} \frac{\Delta_{j-1}}{\Delta_j} \bar{x}_i^2 .} \tag{3.97}$$

The previous discussion is more than that. We may now state a result which has just been, practically, proved.

PROPOSITION 3.8. (***Jacobi's Method***) *If* $\varphi : V \longrightarrow \mathbf{K}$ *is a quadratic form with its matrix* $F_A = [\alpha]$ *in a basis* A of V *with the chain of principal minors of* (3.71) *being nonzero, and if* $\overline{A} = C = [c_1 \ c_2 \ \dots \ c_n]$ *is a new basis obtained from* A *by the transformation* (3.92), *with the entries of the transformation matrix* $T = [\tau_{ij}]$ *satisfying system* (3.93) *and the matrix of* φ *in basis* $C = [\gamma_{ij}]$ *given by expressions* (3.95) & (3.96), *then the analytic expression of* φ *is* (3.97).

We illustrate Jacobi's method for diagonalizing Q-forms by the next two examples.

Example 3.5. Let us consider the quadratic form $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}$ defined by

$$\varphi(X) = x_1^2 + x_2^2 - 3x_3^2 + x_4^2 - x_1 x_2 + 3x_2 x_3 + 5x_3 x_4. \tag{3.98}$$

$$(3.98) \Rightarrow \ [\varepsilon] =_{\text{not}} A_X = \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ -1/2 & 1 & 3/2 & 0 \\ 0 & 3/2 & -3 & 5/2 \\ 0 & 0 & 5/2 & 1 \end{bmatrix}. \tag{3.99}$$

Jacobi's chain of minors (3.74) follows from (3.99) :

$$\Delta_0 =_{\text{def}} 1, \ \Delta_1 = 1, \ \Delta_2 = 3/4, \ \Delta_3 = -9/2, \ \Delta_4 = -147/16.$$

By formula (3.97), the canonical expression of φ is

$$\overline{\varphi}(\overline{X}) = \overline{x}_1^2 + \frac{4}{3}\overline{x}_2^2 - \frac{1}{6}\overline{x}_3^2 + \frac{24}{49}\overline{x}_4^2. \tag{3.100}$$

This example was found in the textbook [M. Rosculet, 1987], where Gauss's method is also applied to the Q-form in (3.98). □

Example 3.6. Another quadratic form $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is defined by

$$\varphi(X) = 5x_1^2 - 4x_1 x_2 - 4x_1 x_3 + 6x_2^2 + 4x_3^2. \tag{3.101}$$

$$(3.101) \Rightarrow \ [\varepsilon] =_{\text{not}} A_X = \begin{bmatrix} 5 & -2 & -2 \\ -2 & 6 & 0 \\ -2 & 0 & 4 \end{bmatrix} \Rightarrow \ \Delta_0 = 1, \ \Delta_1 = 5, \ \Delta_2 = 26, \ \Delta_3 = 80 \Rightarrow$$

$$\Rightarrow \ \overline{\varphi}(\overline{X}) = \frac{1}{5}\overline{x}_1^2 + \frac{5}{26}\overline{x}_2^2 + \frac{13}{40}\overline{x}_3^2. \tag{3.102}$$

The "canonical" basis $C = [c_1 \ c_2 \ c_3]$ in which the analytical expression of the Q-form is (3.102) can be found by writing and solving the systems (3.93) that are implied by Eqs. (3.91) :

$$f(a_i, c_j) = 0 \text{ for } i = 1, 2, \ldots, j-1 \ \& \ j = \overline{1,3} \text{ plus } f(a_j, c_j) = 1. \tag{3.103}$$

We keep the notation for Q-form's matrix in a general basis, that is $[\alpha] = f(A^{\mathrm{T}}, A) = [\alpha_{ij}]$ instead of the specific notation for the standard basis of E of $\mathbb{R}^3$.

$$(3.92) \ \Rightarrow \begin{cases} c_1 = \tau_{11}\, a_1, \\ c_2 = \tau_{21}\, a_1 + \tau_{22}\, a_2, \\ c_3 = \tau_{31}\, a_1 + \tau_{32}\, a_2 + \tau_{33}\, a_3 \end{cases} \Rightarrow \begin{cases} f(a_1, c_1) = \alpha_{11}\,\tau_{11}, \\ f(a_1, c_2) = \alpha_{11}\,\tau_{21} + \alpha_{12}\,\tau_{22}, \\ f(a_2, c_2) = \alpha_{21}\,\tau_{21} + \alpha_{22}\,\tau_{22}, \\ f(a_1, c_3) = \alpha_{11}\,\tau_{31} + \alpha_{12}\,\tau_{32} + \alpha_{13}\,\tau_{33}, \\ f(a_2, c_3) = \alpha_{21}\,\tau_{31} + \alpha_{22}\,\tau_{32} + \alpha_{23}\,\tau_{33}, \\ f(a_3, c_3) = \alpha_{31}\,\tau_{31} + \alpha_{32}\,\tau_{32} + \alpha_{33}\,\tau_{33}. \end{cases} \tag{3.104}$$

Three systems of the form (3.93) are respectively obtained for $j = 1$, $j = 2$, and $j = 3$:

$$j = 1 \ \Rightarrow \ \alpha_{11}\,\tau_{11} = 1 \ \Rightarrow \ \tau_{11} = 1/\alpha_{11}\ ; \tag{3.105}$$

For the next two systems we write the respective equations and their augmented matrices.

$$j = 2 \ \Rightarrow \begin{cases} \alpha_{11}\,\tau_{21} + \alpha_{12}\,\tau_{22} = 0, \\ \alpha_{21}\,\tau_{21} + \alpha_{22}\,\tau_{22} = 1 \end{cases} \rightarrow \left[\begin{array}{cc|c} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 1 \end{array}\right]; \tag{3.106}$$

$$j = 3 \ \Rightarrow \begin{cases} \alpha_{11}\,\tau_{31} + \alpha_{12}\,\tau_{32} + \alpha_{13}\,\tau_{33} = 0, \\ \alpha_{21}\,\tau_{31} + \alpha_{22}\,\tau_{32} + \alpha_{23}\,\tau_{33} = 0, \\ \alpha_{31}\,\tau_{31} + \alpha_{32}\,\tau_{32} + \alpha_{33}\,\tau_{33} = 1 \end{cases} \rightarrow \left[\begin{array}{ccc|c} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 1 \end{array}\right]. \tag{3.107}$$

Jacobi's condition on the north-western minors imply that the systems (3.106) & (3.107) are of Cramer type and can be solved by Cramer's method.

$$(3.106) \ \Rightarrow \tau_{21} = \frac{1}{\Delta_2}\begin{vmatrix} 0 & \alpha_{12} \\ 1 & \alpha_{22} \end{vmatrix} = -\frac{\alpha_{12}}{\Delta_2}, \ \ \tau_{22} = \frac{1}{\Delta_2}\begin{vmatrix} \alpha_{11} & 0 \\ \alpha_{22} & 1 \end{vmatrix} = \frac{\alpha_{11}}{\Delta_2}\ ; \tag{3.108}$$

$$(3.107) \ \Rightarrow \tau_{31} = \frac{1}{\Delta_3}\begin{vmatrix} 0 & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23} \\ 1 & \alpha_{32} & \alpha_{33} \end{vmatrix} = \frac{1}{\Delta_3}\begin{vmatrix} \alpha_{12} & \alpha_{13} \\ \alpha_{22} & \alpha_{23} \end{vmatrix}, \tag{3.109}$$

and – similarly –

$$\tau_{32} = \frac{1}{\Delta_3}\begin{vmatrix} \alpha_{11} & 0 & \alpha_{13} \\ \alpha_{21} & 0 & \alpha_{23} \\ \alpha_{31} & 1 & \alpha_{33} \end{vmatrix} = -\frac{1}{\Delta_3}\begin{vmatrix} \alpha_{11} & \alpha_{13} \\ \alpha_{21} & \alpha_{23} \end{vmatrix}, \ \ \tau_{33} = \frac{1}{\Delta_3}\begin{vmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ \alpha_{31} & \alpha_{32} & 1 \end{vmatrix} = \frac{1}{\Delta_3}\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}.$$

(3.110)

For the numerical data in the statement of this example, leading to $\varphi's$ matrix in the standard basis

$$[\varepsilon] =_{\text{not}} A_X = \begin{bmatrix} 5 & -2 & -2 \\ -2 & 6 & 0 \\ -2 & 0 & 4 \end{bmatrix}, \tag{3.111}$$

the entries of transformation matrix in (3.105) & (3.108) - (3.110) are

$$\tau_{11} = 1/5\ ;$$

$$\tau_{21} = -\frac{1}{26}\begin{vmatrix} 0 & -2 \\ 1 & 6 \end{vmatrix} = \frac{2}{26} = \frac{1}{13}\,, \quad \tau_{22} = \frac{1}{26}\begin{vmatrix} 5 & 0 \\ -2 & 1 \end{vmatrix} = \frac{5}{26}\ ;$$

$$\tau_{31} = \frac{1}{80}\begin{vmatrix} 0 & -2 & -2 \\ 0 & 6 & 0 \\ 1 & 0 & 4 \end{vmatrix} = \frac{12}{80} = \frac{3}{20}\,, \quad \tau_{32} = \frac{1}{80}\begin{vmatrix} 5 & 0 & -2 \\ -2 & 0 & 0 \\ -2 & 1 & 4 \end{vmatrix} = \frac{4}{80} = \frac{1}{20}\,,$$

$$\tau_{33} = \frac{1}{80}\begin{vmatrix} 5 & -2 & 0 \\ -2 & 6 & 0 \\ -2 & 0 & 1 \end{vmatrix} = \frac{26}{80} = \frac{13}{40}\,.$$

It follows that

$$T = \begin{bmatrix} 1/5 & 0 & 0 \\ 2/26 & 5/26 & 0 \\ 12/80 & 4/80 & 26/80 \end{bmatrix}. \tag{3.112}$$

According to the formula that gives the transformation matrix from the standard basis E of space $\mathbb{R}^n$ to another basis, let it be our $C = [\,c_1\ c_2\ c_3\,]$ in $\mathbb{R}^3$, the matrix of this basis is just the transpose of T; see $T = B^{\mathrm{T}} \iff T^{\mathrm{T}} = B$ at page 23 in **§ 1.1.** Hence

$$C = T^{\mathrm{T}} = \begin{bmatrix} 1/5 & 2/26 & 12/80 \\ 0 & 5/26 & 4/80 \\ 0 & 0 & 26/80 \end{bmatrix} \Rightarrow c_1 = \begin{bmatrix} 1/5 \\ 0 \\ 0 \end{bmatrix},\ c_2 = \begin{bmatrix} 2/26 \\ 5/26 \\ 0 \end{bmatrix},\ c_3 = \begin{bmatrix} 12/80 \\ 4/80 \\ 26/80 \end{bmatrix}. \tag{3.113}$$

Certainly, certain entries of (3.112) & (3.113) could be simplified but we leave them as such in view of a checking that follows.

When a basis is changed for another, the matrix of a BLF (defined on the same space) changes by Eq. (1.65) in **§ 3.1**, page 68 or Eq. (3.40) in **§ 2.3** at page 98 of ⟦𝒜.𝒞., 1999⟧. In particular, when the standard basis $E_n \rightarrow A : \mathbb{R}^n = \mathcal{L}(A)$ then

$$F_A = f(A^{\mathrm{T}}, A) \underset{\text{not}}{=} [\alpha] = A^{\mathrm{T}} \cdot f(E_n^{\mathrm{T}}, E_n) \cdot A = A^{\mathrm{T}} \cdot [\varepsilon] \cdot A. \tag{1.65}$$

Taking $A = C\ \&\ E_n = E_3$ in (1.65), the transformation matrix of (3.113) should bring the matrix A_X of (3.111) to a diagonal form.

$$T \cdot A_X \cdot T^{\mathrm{T}} = C^{\mathrm{T}} \cdot A_X \cdot C \underset{(3.111,113)}{=}$$

$$= \begin{bmatrix} 1/5 & 0 & 0 \\ 2/26 & 5/26 & 0 \\ 12/80 & 4/80 & 26/80 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2 & -2 \\ -2 & 6 & 0 \\ -2 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1/5 & 2/26 & 12/80 \\ 0 & 5/26 & 4/80 \\ 0 & 0 & 26/80 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & -2/5 & -2/5 \\ 0 & 1 & -4/26 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/5 & 2/26 & 12/80 \\ 0 & 5/26 & 4/80 \\ 0 & 0 & 26/80 \end{bmatrix} = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 5/26 & 0 \\ 0 & 0 & 26/80 \end{bmatrix}.$$

Hence, the diagonal matrix corresponding to the canonical expression (3.102) has been retrieved. The canonical basis is just C of (3.113). □

The next method for diagonalizing quadratic forms is based upon certain properties of square matrices. The matrix of the Q-form in a starting basis A (or E) of space V is changed by a special type of transformations resulting in a diagonal matrix,

$$\boxed{D = \lceil d_1 \ d_2 \ \dots \ d_n \rfloor.} \tag{3.114}$$

But the transformation matrix T turning the initial matrix

$$f(A^{\mathrm{T}}, A) = [\alpha_{ij}]_{n,n} \underset{\text{not}}{=} [\alpha] \quad \text{or} \quad f(E^{\mathrm{T}}, A) = [\varepsilon_{ij}]_{n,n} \underset{\text{not}}{=} [\varepsilon]$$

of the symmetric BLF f that determines φ into a matrix of the form (3.114) can be obtained in a special way. More precisely, T has to be an *orthogonal matrix*, according to the definition that follows. Hence, a couple of preliminaries are necessary.

Orthogonal Matrices

Definition 3.7. A square matrix $A \in \mathcal{M}_n(\mathbb{R})$ is said to be *orthogonal* if

$$\boxed{A A^{\mathrm{T}} = A^{\mathrm{T}} A = I_n.} \tag{3.115}$$

The (Euclidean) inner product of two vectors $X, Y \in \mathbb{R}^n$ is defined by

$$X = [x_1 \ x_2 \ \dots \ x_n]^{\mathrm{T}} \ \& \ Y = [y_1 \ y_2 \ \dots \ y_n]^{\mathrm{T}} \Rightarrow X \cdot Y \underset{\text{def}}{=} \sum_{i=1}^{n} x_i y_i. \tag{3.116}$$

The vectors $X, Y \in \mathbb{R}^n$ of (3.116) are said to be *orthogonal* if $X \cdot Y = 0 \underset{\text{not}}{\Longleftrightarrow} X \perp Y$. The (*Euclidean*) *norm* of a vector $X = [x_1 \ x_2 \ \dots \ x_n]^{\mathrm{T}} \in \mathbb{R}^n$ is defined by

$$\| X \| \underset{\text{def}}{=} \sqrt{X \cdot X} \underset{(3.116)}{=} \sqrt{\sum_{i=1}^{n} x_i^2} \,. \tag{3.117}$$

A vector $X = [x_1 \ x_2 \ \ldots \ x_n]^{\mathrm{T}} \in \mathbb{R}^n$ is a *unit-norm* (or *unit*) *vector* if $\| X \| = 1$. Two (or more) vectors are said to be *orthonormal* if they are orthogonal unit vectors : X & Y are orthonormal if $X \cdot Y = 0 \underset{\text{not}}{\Longleftrightarrow} X \perp Y$ and $\| X \| = \| Y \| = 1$. ◊

Comment. The relation of orthogonality between two vectors (in a space with a symmetric BLF f defined on it) was earlier met in this Section **3.1**, but that relation of orthogonality was dependent on the considered BLF f. In fact, the operation of inner product in (3.116) is just a symmetric BLF whose matrix (in the standard basis) is the identity matrix I_n. But these notions introduced by *Def. 3.7* will be studied, in more detail, in **CHAPTER 4**. We have presented them here since they are going to be involved in the following result, on the properties pof the orthogonal matrices.

PROPOSITION 3.9. (**Properties of Orthogonal Matrices**) *If* $A \in \mathrm{M}_n(\mathbb{R})$ *is an orthogonal matrix then* :

(i) A *is nonsingular and* (*hence*) *invertible, with* $A^{-1} = A^{\mathrm{T}}$; (3.118)

(ii) *The rows and the columns of* A *are pairwise orthogonal and, moreover,*

$$(\forall\, i, k \in \{1, 2, \ldots, n\}) \ A_i \cdot A_k = \delta_{ik} \,, \tag{3.119}$$

$$(\forall\, j, \ell \in \{1, 2, \ldots, n\}) \ A^j \cdot A^\ell = \delta_{j\ell} \,; \tag{3.120}$$

(iii) *The rows and the columns of an orthogonal matrix* A *are orthonormal vectors* $\in \mathbb{R}^n$;

(iv) *If* $A, B \in \mathrm{M}_n(\mathbb{R})$ *are orthogonal matrices then their product is orthogonal, too.*

Proofs. (i) Immediately follows from *Def. 3.7* , Eq. (3.115) and from the definition of an (the) inverse of a square matrix (see **§ 1.2** and PROPOSITION 2.5 for inverse's uniqueness in [A. Carausu, 1999] - page 30). The invertibility of A follows from the same P. 2.5 but also from (3.115) and the property of the determinant of a matrix product AB : see (iii) in PROPOSITION 2.3 - page 28 in the same textbook.

$$(3.115) \Rightarrow \ \det(A A^{\mathrm{T}}) = (\det A)(\det A^{\mathrm{T}}) = 1 \ \Rightarrow \ \det A \neq 0 \ \& \ \det A^{\mathrm{T}} \neq 0 \,. \tag{3.121}$$

In fact, the determinant of the transpose of a matrix equals the determinant of that matrix, hence (3.121) may be rewritten as

$$\det(A A^{\mathrm{T}}) = (\det A)(\det A^{\mathrm{T}}) = (\det A)^2 = 1 \ \Rightarrow \ \det A = \pm 1 \,. \tag{3.122}$$

The equation $A^{-1} = A^{\mathrm{T}}$ of (3.118) immediately follows from the uniqueness of the inverse.

(ii) follows from the definition of the matrix product (see **§ 1.2** in ⟦𝒜.𝒞., 1999 ⟧). Indeed, the current entry in the product of (3.115) is the inner product of i-th row of A by the k-th column of A^{T}:

$$A_i \cdot A_k \underset{(3.115)}{=} A_i \cdot (A^{\mathrm{T}})^k = \delta_{ik} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases} : \quad (3.119)$$

since Kronecker's δ_{ik} is the current entry of the identity matrix I_n. The proof of (3.120), involving two columns of the matrix , is similar.

(iii) is an obvious consequence of property (ii) : it follows from (3.119) - (3.120) that any two distinct rows / columns are orthogonal, and the norm of any row / column is - according to definition in (3.117) -

$$\| A_i \| = \sqrt{A_i \cdot A_i} \underset{(3.119)}{=} \sqrt{1} = 1 .$$

Similarly, the columns of matrix A are unit vectors, too.

(iv) We have to check the definition (3.115) for the matrix product $A\,B$: by PROPOSITION 2.7 in **§ 1.2** (Eq. (2.22) at page 33 of ⟦𝒜. 𝒞., 1999 ⟧),

$$(A\,B)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}} \Rightarrow (A\,B) \cdot (A\,B)^{\mathrm{T}} = (A\,B) \cdot (B^{\mathrm{T}} A^{\mathrm{T}}) = A \cdot (B \cdot B^{\mathrm{T}}) \cdot A^{\mathrm{T}} \underset{(3.115)}{=}$$

$$= A \cdot I_n \cdot A^{\mathrm{T}} = A \cdot A^{\mathrm{T}} \underset{(3.115)}{=} I_n .$$

■

Another pair of notions connected with a square matrix A needs to be introduced by

Eigenvectors and Eigenvalues

Definition 3.8. Let $A \in \mathcal{M}_n(\mathbb{R})$ be an arbitrary square matrix of order n. A column vector $U \in \mathbb{R}^n$, $U \neq \mathbf{0}$ is said to be an *eigenvector* of matrix A if there exists a scalar $\lambda \in \mathbb{R}$ such that

$$\boxed{A \cdot U = \lambda\, U.} \quad (3.123)$$

The scalar $\lambda \in \mathbb{R}$ that occurs in (3.123) is called an *eigenvalue* of matrix A. ◇

Certainly, the "C dot" that we have used in the proof of the above property (iv) as well as in (3.123) could be omitted. It stands for the product of matrices, an operation that is often denoted by simply juxtaposing the two matrix factors: $A\,B$. In the left-hand side of (3.123) a square matrix times a column vector gives a column vector ; in the r.h.s. of the same equation, the column vector U multiplied by the scalar λ remains a column vector in $\mathbb{R}^n$.

We do not here insist on the properties of the eigenvalues and eigenvectors of a square matrix. They will be met later, in **CHAPTER 4** (on linear morphisms). It should however be noticed that an eigenvector U is connected with the eigenvalue λ through Eq. (3.123). This defining equation can be equivalently written as

$$AU - \lambda U = \mathbf{0} \iff AU - \lambda(I_n U) = \mathbf{0} \iff AU - (\lambda I_n)U = \mathbf{0} \iff$$

$$\iff \boxed{(A - \lambda I_n)U = \mathbf{0}.} \tag{3.124}$$

This last equation (3.124) is a matrix equation which is equivalent to a *homogeneus system* of matrix $A - \lambda I_n$ and we have to look for *nonzero* (or nontrivial) solutions U of its. As it is known from the highschool Algebra (and recalled in **§ 1.2** of ⟦𝒜. 𝒞., 1999 ⟧), a homogeneous system admits nontrivial solutions if and only if the rank of its matrix *is strictly less than the numbers of unknowns*. If the system is square of matrix *M*, this condition is equivalent to $\det M = 0$. If we now take $M = A - \lambda I_n$, the condition for the existence of nontrivial solutions to this system becomes

$$\boxed{\det(A - \lambda I_n) = 0.} \tag{3.125}$$

In fact, this is an algebraic equation of order n over the field $\mathbb{R}$ but – in a more general approach – even *Definition 3.8* can be reformulated by simply replacing the real field with a more general field $\mathbb{F}$ (or $\mathbb{K}$) since the l.h.s. of Eq. (3.125) is just a polynomial of order *n*. It is called the characteristic polynomial of matrix A and it is (usually) denoted as

$$P_A(\lambda) =_{\text{not}} \det(A - \lambda I_n). \tag{3.126}$$

An important problem concerns the *roots* of Eq. (3.125). If the field $\mathbb{K}$ is *algebraically closed*, then an equation of the form $P(\lambda) = 0$ with $P \in \mathrm{POL}_n(\mathbf{K})$ admits exactly n roots in that field, let them be $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{K}$, that can be distinct or not. The *complex field* $\mathbb{C}$ is algebraically closed (Galois's Theorem). But if the field is not algebraically closed – and *this is the case for the real field* – the number of roots in that field can be less than n and even = 0. We give (without proof) a property of the symmetric matrices over $\mathbb{R}$ which is relevant for the diagonalization of the Q-forms :

PROPOSITION 3.10. *If* $A \in \mathcal{M}_n(\mathbb{R})$ *is a symmetric matrix then all its eigenvalues are real* :

$$A \in \mathcal{M}_n(\mathbb{R}) \ \& \ A = A^{\mathrm{T}} \Rightarrow \{\lambda_i : (\exists\, U \in \mathbb{R}^n)\, AU = \lambda_i U\} \subset \mathbb{R}. \tag{3.127}$$

The set of eigenvalues of a square matrix is called its spectrum. Hence, the latest PROPOSITION states that the *spectrum of a symmetric matrix over* $\mathbb{R}$ *consists of real elements only*. The spectrum of a square matrix *If* $A \in \mathcal{M}$ is denoted $\sigma(A)$.

The next definition introduces a binary relation among the square matrices.

Definition 3.9. If $A, B \in \mathcal{M}_n(\mathbb{R})$ are two matrices, then they are said to be *similar* if

$$(\exists\, S \in \mathcal{M}_n(\mathbb{R})) : \det S \neq 0 \ \& \ B = S^{-1} A S. \tag{3.128}$$

A matrix $A \in \mathcal{M}_n(\mathbb{R})$ is said to be *diagonalizable* if there exists a diagonal matrix D that is similar to A.

In other words, a matrix A is diagonalizable iff there exists a diagonal matrix $D = \lceil d_1 \; d_2 \; \ldots \; d_n \rfloor$ and a nonsingular matrix S such that $D = S^{-1} A S$. But the notion of orthogonal matrices makes possible to state an alternative to *Definition 3.9*, in fact a result of equivalence (that is, a characterization).

PROPOSITION 3.11. *If* $A, B \in \mathcal{M}_n(\mathbb{R})$ *are two matrices and P is an orthogonal matrix such that*

$$\boxed{B = P^{\mathrm{T}} A P} \tag{3.129}$$

then A & B are similar matrices. If B is a diagonal matrix then A is diagonalizable.

Proof. This result immediately follows from *Definition 3.9* and from the remark following *Definition 3.7* : for an orthogonal matrix P, $P^{-1} = P^{\mathrm{T}}$.

■

Let us now consider the case when matrix A admits n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{K}$, that can be distinct or not. At least one eigenvector U_i corresponds to each eigenvalue λ_i and they are connected by a relation of the form (3.123). We write down these n equations (but without the C-dot for the matrix product) :

$$\boxed{A U_i = \lambda_i U_i, \;\; i = \overline{1,n}.} \tag{3.130}$$

These n equations, whose left and right sides are column vectors, may be written together as a single matrix equation, as follows. If we denote $\lceil \lambda_1 \; \lambda_2 \; \ldots \; \lambda_n \rfloor$ the diagonal matrix whose entries on the (main) diagonal are just the n eigenvalues of matrix A then the n equations of (3.130) can be written together as the following matrix equation :

$$A[U_1 \; U_2 \; \ldots \; U_n] = [\lambda_1 U_1 \; \lambda_2 U_2 \; \ldots \; \lambda_n U_n] = [U_1 \; U_2 \; \ldots \; U_n] \lceil \lambda_1 \; \lambda_2 \; \ldots \; \lambda_n \rfloor. \tag{3.131}$$

Indeed, the first column in the product of the rightmost side in (3.131) comes by multiplying the whole matrix $[U_1 \; U_2 \; \ldots \; U_n]$ by the column $[\lambda_1 \; 0 \; \ldots \; 0]^{\mathrm{T}}$, what results in $\lambda_1 U_1$, and similarly for the next $n-1$ columns. Let us now denote

$$\boxed{[U_1 \; U_2 \; \ldots \; U_n] \underset{\text{not}}{=} S.} \tag{3.132}$$

$$(3.131) \;\&\; (3.132) \;\Rightarrow\; \boxed{A S = S \lceil \lambda_1 \; \lambda_2 \; \ldots \; \lambda_n \rfloor.} \tag{3.133}$$

Let us now compare this Eq. (3.133) with the equation defining the similarity relation between two square matrices, that is (denoting the similarity relation by ~)

$$A \sim B \iff (\exists\, S : \det S \neq 0) \;\; B = S^{-1} A S. \tag{3.134}$$

This equivalence of (3.134) holds *only if the matrix* S *is nonsingular*, hence invertible, in an equation of the form

$$[SB = AS \text{ or } AS = SB] \Rightarrow B = S^{-1}AS. \tag{3.135}$$

The left-hand side of Eq. (3.133) coincides with the l.h.s. in the second equation between [. . .] of (3.135). Hence we could draw the conclusion that

$$(3.133) \Rightarrow \qquad \boxed{S^{-1}AS = \lceil \lambda_1 \ \lambda_2 \ \ldots \ \lambda_n \rfloor,} \tag{3.136}$$

but only *provided the matrix* S *be invertible*.

We have denoted by S, in (3.132), the matrix whose columns are n eigenvectors $U_i \ (i = \overline{1,n})$ corresponding to the n eigenvalues of matrix A. Hence a (first) condition on the vectors $U_i \ (i = \overline{1,n})$ for the non-singularity of matrix S consists in their *linear independence*. See the second definition of the rank of a matrix in **§ 1.2** , proof of PROPOSITION 2.9 at page 39 in ⟦$\mathcal{A}$. $\mathcal{C}$., 1999 ⟧ ; certainly, the eigenvectors should be mutually distinct.

Coming now back to *the problem of diagonalizing a quadratic form* φ, it follows from this discussion that – if n independent eigenvectors corresponding to n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ can be found – then a matrix built with these eigenvectors as its columns can be used for getting a diagonal form of the matrix of φ in an "initial" basis, let it be $[\alpha] = f(A^{\mathrm{T}}, A)$ or $[\varepsilon] = f(E^{\mathrm{T}}, E)$, or A_X like in **Example 3.6** at pages 98-99, when the Q-form is given by its analytic expression. Moreover, if such and invertible matrix $S = [U_1 \ U_2 \ \ldots \ U_n]$ is available, then the similarity transformation of (3.134) – (3.136) will provide a diagonal form of φ's matrix, with the n eigenvalues of matrix A on its diagonal, as it follows from (3.136).

But if we require more from S, namely to be not only non-singular (hence invertible) but even ***orthogonal***, we could use the similarity relation and transformation by orthogonal matrices, as in previous PROPOSITION 3.11. The conclusions of this discussion can now be formulated in terms of the next result :

PROPOSITION 3.12. *If* $\varphi : V \longrightarrow \mathbf{K}$ *is a quadratic form with its matrix* $F_A = [\alpha]$ *in a basis* A *of* V, *if* $U_i \ (i = \overline{1,n})$ *are n orthonormal eigenvectors of* $[\alpha]$ *corresponding to the eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{K}$, *and we denote* $[U_1 \ U_2 \ \ldots \ U_n] \underset{\text{not}}{=} P$ *then* P *is an orthogonal matrix and*

$$\boxed{P^{\mathrm{T}}[\alpha]P = \lceil \lambda_1 \ \lambda_2 \ \ldots \ \lambda_n \rfloor.} \tag{3.137}$$

Proof. Let us represent the relationship eigenvalues - eigenvectors as follows :

$$\begin{array}{ccc} \lambda_1, & \lambda_2, \ldots, & \lambda_n \\ \downarrow & \downarrow & \downarrow \\ U_1, & U_2, \ldots, & U_n. \end{array} \tag{3.138}$$

The vertical arrows in this diagram do not represent a one-to-one correspondence. Each distinct eigenvalue $\lambda_j\ (1 \le j \le n)$ may be either a simple root of the characteristic equation (3.125), or (possibly) a multiple root, of multiplicity $k_j > 1$. This latter situation occurs then and only then when the characteristic polynomial $P_A(\lambda) =_{\text{not}} \det(A - \lambda I_n)$, with $A \to [\alpha]$, contains a factor of the form $(\lambda - \lambda_j)^{k_j}$ and this eigenvalue λ_j does not appear elsewhere in the factorization of $P_{[\alpha]}(\lambda)$. If eigenvalue λ_j's multiplicity is $k_j > 1$ then it occurs exactly k_j times on the first row of (3.138) and also in the diagonal matrix of (3.136). But exactly k_j distinct and linearly independent eigenvectors should be found and they will appear on the second row of (3.138), under the repeated k_j occurrences of λ_j. In (3.138), the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ may be written in any order, but it would be convenient to write them according to the $\le$ order relation, *when they are real* : $\lambda_j \le \lambda_{j+1}\ (1 \le j \le n-1)$. This discussion on the eigenvalues and eigenvectors of a square matrix will be extended in **Chapter 4 - § 4.3**, dealing with linear endomorphisms and their diagonalization.

Coming back to the proof of the assertions in the statement, the hypothesis that $U_i\ (i = \overline{1,n})$ are *n* orthonormal eigenvectors of $[\alpha]$ obviously implies the orthogonality of the matrix $P = [U_1\ U_2\ \ldots\ U_n]$. The current entry in the product PP^{T} is

$$U_i \cdot U_j = \delta_{ij} \Rightarrow PP^{\mathrm{T}} = I_n. \tag{3.139}$$

The first equation in (3.138) follows from *Definition 3.7* - page 101. Equation (3.137) readily follows from (3.136) with $S \to P$ and the property of any orthogonal matrix that *its transpose is just its inverse* : Eq. (3.40) in PROPOSITION 3.9 at page 98 of . As a matter of notation, we are going to change the notation for unit eigenvectors $(U_i \to u_i,\ i = \overline{1,n})$ in order to distinguish between general eigenvectors and unit eigenvectors – a little later. ■

Before presenting the algorithm for diagonalizing Q-forms by means of eigenvalues, eigenvectors and orthogonal matrices, let us see that the left-hand side of Eq. (3.137) is quite similar to the r.h.s. of Eq. (2.29) in **§ 2.3** - page 94 of ⟦$\mathcal{A}$. $\mathcal{C}$., 1999⟧ or to Eq. (3.78) at page 108 of the same reference ; certainly, the eigenvectors should be mutually distinct. In fact, $P^{\mathrm{T}}[\alpha]P$ is obtained from $T[\alpha]T^{\mathrm{T}}$ by simply taking $T = P^{\mathrm{T}}$. If the Q-form is defined on a space like $\mathbb{R}^n$ then the transformation matrix from the standard basis E_n to any other basis of this space, let it be the "canonical" basis *C*, is just

$$T = C^{\mathrm{T}} \iff C = T^{\mathrm{T}} = P. \tag{3.140}$$

Remark 3.9. This last equation in (3.140) has both a theoretical significance and a practical importance, as well. If n independent and mutually orthonormal eigenvectors corresponding to the n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ have been found, then *they form just the canonical basis* in which the matrix of the Q-form takes the diagonal form of (3.137). But if the vectors on the second row are only independent, they *should be checked for mutual orthogonality* and they have to be "reduced" (or normalized) to unit vectors ; if they are not orthogonal, some vectors may be replaced for obtaining the condition $1 \le i, j \le n \;\&\; i \ne j \;\Rightarrow\; U_i \perp U_j$. Next, we should turn

$$[U_1\ U_2\ \ldots\ U_n] \rightarrow P = [u_1\ u_2\ \ldots\ u_n] : u_i \cdot u_j = \delta_{ij}. \tag{3.141}$$

The C-dot in (3.141) is the (Euclidean) inner product of Eq. (3.116) - page 126.

The practical ways to perform these obtain a canonical basis and the corresponding diagonal matrix of (3.137) are presented in what follows.

III. Diagonalization of Q-forms by Orthogonal Transformations (The EVV Method)

OT.1 Given a Q-form $\varphi(X) = X^{\mathrm{T}}[\alpha]X$, the n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of matrix $[\alpha]$ are determined from the characteristic equation (3.125).

OT.2 For each distinct eigenvalue $\lambda_j\ (1 \le j \le m \le n)$, the homogeneous system of matrix

$$[\alpha] - \lambda_j I_n, \text{ that is } \boxed{\left([\alpha] - \lambda_j I_n\right)U = \mathbf{0},} \tag{3.142}$$

is solved. If $S(\lambda_j)$ is the set of nontrivial solutions to the system (3.142), a basis

$$B_j = \{U_{j_1}, U_{j_2}, \ldots, U_{j_{h_j}}\} \text{ with } S(\lambda_j) = \mathcal{L}(B_j) \tag{3.143}$$

has to be found. The natural number h_j that occurs in (3.143) equals the dimension of the subspace $S(\lambda_j) \cup \{\mathbf{0}\}$, that is $h_j = \dim W(\lambda_j)$. Recall from **§ 1.2** that the solution set S of any homogeneous linear system is a subspace.

OT.3 If, in step **OT.2**, it has been found an eigenvalue λ_j such that $h_j < k_j =$ the (algberaic) multiplicity of that root of Eq. (3.125), the algorithm has to be STOPped : the method of orthogonal transformation cannot be applied since a diagonalization of matrix $[\alpha]$ in the sense of Eq. (3.136) - by a similarity transformation - *is not possible*. If $(\forall j : 1 \le j \le m)\ h_j = k_j$ then the **EVV**-based **Method** is applicable and the algorithm goes on.

OT.4 The subsets (sub-bases) of eigenvectors $B_j = \{U_{j_1}, U_{j_2}, \ldots, U_{j_{h_j}}\}$ are joined resulting in a basis of n eigenvectors,

$$B = \bigcup_{j=1}^{m} B_j,\ B = [U_1\ U_2\ \ldots\ U_n]. \tag{3.144}$$

OT.5 This basis of (3.144) is turned into an orthonormal basis as follows :

(i) The vectors $U_i \ (i = \overline{1,n})$ are checked for orthogonality :

$$1 \le i, j \le n \ \ \& \ \ i \ne j \ \Rightarrow \ U_i \perp U_j .$$

If a pair of vectors is found to be not orthogonal, one of them is replaced by another vector taken from the solution set $S(\lambda_j)$ of the same homogeneous system whose matrix is the one of (3.142) and the procedure is continued until all the n vectors are mutually orthogonal.

(ii) Each vector $U_i \ (i = \overline{1,n})$ *is replaced by its corresponding unit vector* :

$$(\forall \ i \in \overline{1,n}) \ U_i \ \rightarrow \ u_i = \frac{1}{\| U_i \|} U_i. \tag{3.145}$$

OT.6 The resulting basis is the canonical basis $C = [u_1 \ u_2 \ \dots \ u_n] : u_i \cdot u_j = \delta_{ij}$.

The orthogonal matrix $P = [u_1 \ u_2 \ \dots \ u_n]$ is written and transformation (3.137) is applied, turning the matrix $[\alpha]$ of φ (in the initial basis) into the diagonal matrix

$$\boxed{P^{\mathrm{T}} [\alpha] P = \lceil \lambda_1 \ \lambda_2 \ \dots \ \lambda_n \rfloor .} \tag{3.146}$$

Remarks 3.10. This last step of the algorithm may be omitted. If all its previous steps were correctly applied, the matrix of φ in the canonical basis C is just the diagonal matrix in (3.146). However, this transformation (in the l.h.s. of (3.137), (3.146)) is recommended as a way to check the correctness of the calculations. The analytical expression of the Q-form in the basis C is

$$\boxed{\overline{\varphi}(\overline{X}) = \overline{X}^{\mathrm{T}} [\lambda_{i = \overline{1,n}}] \overline{X} = \sum_{i=1}^{n} \lambda_i \overline{x}_i^{\,2}} \ \text{ where } \ \overline{X} = X_C. \tag{3.147}$$

This algorithm could be perceived as a rather cumbersome method, but it offers the advantage to provide the canonical basis C (when it is applicable : see step **OT.3**).

We are going to illustrate the application of this **ORTHOGONAL TRANSFORMATION** (or **EVV** based) **METHOD** by a couple of examples.

Example 3.7. A quadratic form on space $\mathbb{R}^3$ is defined by its analytical expression

$$\varphi(X) = 5x_1^2 + 6x_2^2 + 7x_3^2 - 4x_1 x_2 + 4x_2 x_3 . \tag{3.148}$$

It is required to diagonalize it, by the EVV Method.

The Q-form's matrix in the standard basis E_3 of $\mathbb{R}^3$, is

$$[\varepsilon] =_{\text{not}} A_X = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}. \tag{3.149}$$

OT.1. The corresponding characteristic polynomial is

$$P_{A_X}(\lambda) \underset{(3.149)}{=} \det \begin{bmatrix} 5-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{bmatrix} = -\lambda^3 + 18\lambda^2 - 99\lambda + 162. \tag{3.150}$$

The characteristic equation $P_{A_X}(\lambda) = 0$ has the three (distinct) roots

$$\lambda_1 = 3, \ \lambda_2 = 6, \ \lambda_3 = 9. \tag{3.151}$$

OT.2. The three corresponding eigenvectors are found by solving the homogeneous linear systems of the form (3.142). We successively solve them by Gaussian elimination.

$$A_X - \lambda_1 I_3 = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

gives the general solution

$$U_1(\alpha) = \begin{bmatrix} -2\alpha \\ -2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}. \tag{3.152}$$

Hence, the first eigenvector corresponding to $\lambda_1 = 3$ is $U_1 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$.

Similarly, for $\lambda_2 = 6$,

$$A_X - \lambda_2 I_3 = \begin{bmatrix} -1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow U_2(\beta) = \begin{bmatrix} -2\beta \\ \beta \\ -2\beta \end{bmatrix} \Rightarrow U_2 = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}; \tag{3.153}$$

$$A_X - \lambda_3 I_3 = \begin{bmatrix} -4 & -2 & 0 \\ -2 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ -2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow U_3(\gamma) = \begin{bmatrix} \gamma \\ -2\gamma \\ -2\gamma \end{bmatrix} \Rightarrow U_2 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}. \tag{3.154}$$

In (3.152-154) we have re-denoted the secondary (parametric) variables of the three systems : $x_1 = \alpha,\ x_2 = \beta,\ x_3 = \gamma.$

OT.3. The strict inequality $h_j < k_j$ is not met for any eigenvalue since the three roots of (3.150) are simple.

OT.4 & OT.5 : It is easy to see that the three vectors are pairwise orthogonal :

$$i \neq j \ \Rightarrow\ U_i \cdot U_j = 4 - 4 = 0 \quad (1 \leq i, j \leq 3),$$

while their norms are equal : $\| U_1 \| = \| U_2 \| = \| U_3 \| = 3$. Therefore, the three unit and mutually orthogonal eigenvectors are

$$u_1 = \begin{bmatrix} -2/3 \\ -2/3 \\ 1/3 \end{bmatrix},\ u_2 = \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix},\ u_3 = \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}. \tag{3.155}$$

The orthogonal matrix of (3.141) - page 108 is

$$P = [u_1\ u_2\ u_3] \underset{(1.155)}{=} \begin{bmatrix} -2/3 & -2/3 & 1/3 \\ -2/3 & 1/3 & -2/3 \\ 1/3 & -2/3 & -2/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -2 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & -2 & -2 \end{bmatrix}. \tag{3.156}$$

OT.6. The orthogonal transformation, applied to the initial matrix A_X of (3.149) gives

$$P^{\mathrm{T}} A_X P = P A_X P \underset{(1.156)}{=} \frac{1}{9}\begin{bmatrix} -2 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} -2 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & -2 & -2 \end{bmatrix} =$$

$$= \frac{1}{9}\begin{bmatrix} -6 & -6 & 3 \\ -12 & 6 & -12 \\ 9 & -18 & 18 \end{bmatrix} \cdot \begin{bmatrix} -2 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & -2 & -2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 27 & 0 & 0 \\ 0 & 54 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \lceil 3 \quad 6 \quad 9 \rfloor.$$

Hence the diagonal matrix of (3.146), with the eigenvalues of (3.151) has been found.

The corresponding canonical expression of the Q-form is

$$\overline{\varphi}(\overline{X}) = 3\,\overline{x}_1^{\,2} + 6\,\overline{x}_2^{\,2} + 9\,\overline{x}_3^{\,2}. \tag{3.157}$$

The coordinates that occur in (3.157) are (the components of) $\overline{X} = X_C$, with C = the ortho-normal basis whose (eigen)vectors are the ones of (3.155). □

The next example shows how the orthogonal vectors $U_1, U_2, \ldots, U_n$ can be obtained when some of such vectors, obtained by solving the homogeneous systems (3.142), are not orthogonal.

Example 3.8. A quadratic form on space $\mathbb{R}^4$ is defined by its analytical expression

$$\varphi(X) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_1x_2 + 2x_1x_3 + 2x_1x_4 - 2x_2x_3 - 2x_2x_4 - 2x_3x_4. \tag{3.158}$$

It is required to diagonalize it by the EVV Method.

$$(3.158) \Rightarrow \quad A_X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \tag{3.159}$$

The characteristic polynomial of this matrix is

$$P_{A_X}(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & -1 & -1 \\ 1 & -1 & 1-\lambda & -1 \\ 1 & -1 & -1 & 1-\lambda \end{bmatrix} = \ldots = (\lambda-2)^3(\lambda+2). \tag{3.160}$$

$$(3.160) \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 2,\ \lambda_4 = -2. \tag{3.161}$$

The eigenvectors corresponding to the triple root are obtained by solving the H-system of matrix

$$\Rightarrow U_{1-3}(\alpha,\beta,\gamma) = \begin{bmatrix} \alpha+\beta+\gamma \\ \alpha \\ \beta \\ \gamma \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3.162}$$

The three column vectors that occur in then rightmost side of (3.162) are – of course – U_1, U_2, U_3. The fourth eigenvector comes from

$$A_X + 2I_4 = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & -1 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 4 & -8 & 4 \\ 0 & 4 & -4 & 0 \\ 1 & -1 & 3 & -1 \\ 0 & 0 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow U_4(\delta) = \begin{bmatrix} -\delta \\ \delta \\ \delta \\ \delta \end{bmatrix} = \delta \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \tag{3.163}$$

Checking for orthogonality the four eigenvectors in (3.162-163) :

$$U_1 \cdot U_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 1 \ ; \ U_1 \cdot U_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 1 \ ; \ U_2 \cdot U_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 1 \,. \tag{3.164}$$

It follows from (3.164) that *no pair of orthogonal vectors exists* among U_1, U_2, U_3. Instead, $U_1 \perp U_4$, $U_2 \perp U_4$, $U_3 \perp U_4$. Two vectors among U_1, U_2, U_3. should be replaced in order to get an orthogonal family of vectors. Each candidate should be in the solution (sub)space of system with the matrix in (3.159). Hence, each of them should be of the form (3.162). We successively impose the condition of orthogonality on U_1 and between U_2 & U_3:

$$U_1 \cdot U_2(\alpha, \beta, \gamma) = 0 \ \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \alpha + \beta + \gamma \\ \alpha \\ \beta \\ \gamma \end{bmatrix} = 0 \ \Rightarrow \ 2\alpha + \beta + \gamma = 0 \ ; \tag{3.165}$$

$$U_1 \cdot U_3(\alpha, \beta, \gamma) = 0 \ \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \alpha + \beta + \gamma \\ \alpha \\ \beta \\ \gamma \end{bmatrix} = 0 \ \Rightarrow \ 2\alpha + \beta + \gamma = 0 \ ; \tag{3.166}$$

The two equations of (3.165) & (3.166) coincide, but we can obtain two distinct vectors as follows.

$$\text{Eq. (3.165)} \Rightarrow \gamma = -2\alpha - \beta \ \Rightarrow \ U_2(\alpha, \beta) = \begin{bmatrix} -\alpha \\ \alpha \\ \beta \\ -2\alpha - \beta \end{bmatrix}. \tag{3.167}$$

It is easy to see that $U_2(\alpha, \beta)$ of (3.167) is orthogonal on U_4 for *any* values of the two parameters. Giving particular values to the two parameters that occur in (3.167) we can get the second and the third eigenvectors to replace U_2 & U_3:

$$U_2(1, 1) = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -3 \end{bmatrix} \underset{\text{not}}{=} U_2' \ \text{and} \ U_2(-1, 2) = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \underset{\text{not}}{=} U_3'. \tag{3.168}$$

(3.162), (3.163) & (3.168) $\Rightarrow$

$$\Rightarrow \ U_1 \perp U_2', \ U_1 \perp U_3', \ U_2' \perp U_3' \ \& \ \{U_1, U_2', U_3'\} \perp U_4 \,. \tag{3.169}$$

Therefore the four eigenvectors are mutually orthogonal and the must now be turned to unit vectors. The norms of the four vectors are

$$\| U_1 \| = \sqrt{2}\ ,\ \ \| U_2' \| = 2\sqrt{3}\ ,\ \ \| U_3' \| = \sqrt{6}\ ,\ \ \| U_4 \| = 2\,. \tag{3.170}$$

(3.162), (3.163), (3.168) & (3.170) $\Rightarrow$

$$\Rightarrow \quad u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ u_2 = \frac{1}{2\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -3 \end{bmatrix},\ u_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix},\ u_4 = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \tag{3.171}$$

The four unit eigenvectors of (3.171) form the orthonormal basis C in which the Q-form's matrix should be diagonal : $f(C^{\mathrm{T}}, C) = \lceil \lambda_1\ \ \lambda_2\ \ \lambda_2\ \ \lambda_4 \rfloor$.

We can **check** this conclusion by introducing our data in the left-hand side of Eq. (3.146), with

$$P = [u_1\ \ u_2\ \ u_3\ \ u_4] \underset{(3.171)}{=} \frac{1}{2\sqrt{3}}\begin{bmatrix} \sqrt{6} & -1 & \sqrt{2} & -\sqrt{3} \\ \sqrt{6} & 1 & -\sqrt{2} & \sqrt{3} \\ 0 & 1 & 2\sqrt{2} & \sqrt{3} \\ 0 & -3 & 0 & \sqrt{3} \end{bmatrix}. \tag{3.172}$$

$$(3.172) \quad \Rightarrow \quad P^{\mathrm{T}} = [u_1\ \ u_2\ \ u_3\ \ u_4]^{\mathrm{T}} = \frac{1}{2\sqrt{3}}\begin{bmatrix} \sqrt{6} & \sqrt{6} & 0 & 0 \\ -1 & 1 & 1 & -3 \\ \sqrt{2} & -\sqrt{2} & 2\sqrt{2} & 0 \\ -\sqrt{3} & \sqrt{3} & \sqrt{3} & \sqrt{3} \end{bmatrix}. \tag{3.173}$$

For calculating the product of three matrices $P^{\mathrm{T}} A_X P$ we take outside the two scalar factors whose product gives $1/12$. From (3.172), (3.159) and (3.173) we get

$$P^{\mathrm{T}} A_X P = \frac{1}{12}\begin{bmatrix} \sqrt{6} & \sqrt{6} & 0 & 0 \\ -1 & 1 & 1 & -3 \\ \sqrt{2} & -\sqrt{2} & 2\sqrt{2} & 0 \\ -\sqrt{3} & \sqrt{3} & \sqrt{3} & \sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{6} & -1 & \sqrt{2} & -\sqrt{3} \\ \sqrt{6} & 1 & -\sqrt{2} & \sqrt{3} \\ 0 & 1 & 2\sqrt{2} & \sqrt{3} \\ 0 & -3 & 0 & \sqrt{3} \end{bmatrix} =$$

$$= \frac{1}{12}\begin{bmatrix} 2\sqrt{6} & 2\sqrt{6} & 0 & 0 \\ -2 & 2 & 2 & -6 \\ 2\sqrt{2} & -2\sqrt{2} & 4\sqrt{2} & 0 \\ 2\sqrt{3} & -2\sqrt{3} & -2\sqrt{3} & -2\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{6} & -1 & \sqrt{2} & -\sqrt{3} \\ \sqrt{6} & 1 & -\sqrt{2} & \sqrt{3} \\ 0 & 1 & 2\sqrt{2} & \sqrt{3} \\ 0 & -3 & 0 & \sqrt{3} \end{bmatrix} =$$

$$= \frac{1}{12}\begin{bmatrix} 24 & 0 & 0 & 0 \\ 0 & 24 & 0 & 0 \\ 0 & 0 & 24 & 0 \\ 0 & 0 & 0 & -24 \end{bmatrix} = \lceil 2 \ \ 2 \ \ 2 \ \ -2 \rfloor. \tag{3.174}$$

The canonical basis in which Q-form's matrix has taken a diagonal form is just C whose vectors are the columns of the matrix in (3.174) : $C = [u_1 \ u_2 \ u_3 \ u_4]$. The four unit vectors are mutually orthogonal. Although this equation is almost identical with that of (3.173), we keep the notation C for the *canonical* basis, while P denotes the *orthogonal transformation matrix* of (3.142).

The canonical analytic expression of the Q-form (3.158) in this basis C is

$$\bar{\varphi}(\bar{X}) = \bar{X}^{\mathrm{T}} f(C^T, C)\,\bar{X} = 2\,\bar{x}_1^2 + 2\,\bar{x}_2^2 + 2\,\bar{x}_3^2 - 2\,\bar{x}_4^2, \text{ with } \bar{X} = X_C. \tag{3.175}$$

□

An interesting and rather important feature of a diagonalized Q-form is the number of the positive, negative and zero terms in a canonical expression of the form (3.49) - at page 88, or (3.147) in terms of the eigenvalues. This feature is called the signature of the Q-form. Let us denote it by

$$\operatorname{sgn} \varphi = (\pi, \nu, \zeta) \tag{3.176}$$

with $\pi =$ the number of positive terms, $\nu =$ the number of negative terms, $\zeta =$ the number of zero terms. For instance, the signature of the Q-form in the latest example is - according to (3.175) - $\operatorname{sgn} \varphi = (3, 1, 0)$. The signatures of the Q-forms in **Examples 3.6** - Eq. (3.102) and **3.7** - Eq. (3.157) are – both of them – $\operatorname{sgn} \varphi = (3, 0, 0)$. Obviously, the number of non-zero terms in a canonical expression is $\pi + \nu = r = \operatorname{rank} \varphi$. This follows from the rank of the matrix of a Q-form in an "initial" basis is kept by the transformations leading to a diagonal form, and the rank of a matrix like $D = \lceil d_1 \ d_2 \ \ldots \ d_n \rfloor$ is clearly equal to the number of non-zero entries on its main diagonal.

The signature of a quadratic form is *intrinsic* to a given Q-form, in the sense that $\operatorname{sgn} \varphi$ *is the same* for any of its canonical expressions. This remarkable property is known as SYLVESTER's INERTIA THEOREM.

THEOREM 3.1. (Sylvester) *If* $\varphi : V \longrightarrow \mathbf{K}$ *is a quadratic form, then the signature of* φ *is the same for any of its diagonal (canonical) expressions* (3.49) - *page 88.*

Proof. Let $C = [u_1 \ u_2 \ \ldots \ u_n]$ be a basis in which the Q-form φ has a canonical expression,

$$\varphi(X) = \sum_{i=1}^{\pi} c_i x_i^2 - \sum_{i=\pi+1}^{\pi+\nu} c_i x_i^2 \text{ with } \pi + \nu = r = \operatorname{rank} \varphi \ \ \& \ \ (\forall\, i)\ c_i > 0. \tag{3.177}$$

If the vectors $u_i \ (i = \overline{1, r})$ are replaced by the vectors

$$v_i = \frac{1}{\sqrt{c_i}} u_i, \ (i = \overline{1,r}) \ \text{while} \ v_j = u_j \ (j = \overline{r+1,n}). \tag{3.178}$$

a new basis is obtained, namely $D = [v_1 \ v_2 \ \ldots \ v_r \ u_{r+1} \ \ldots \ u_n]$, in which the expression of (3.177) becomes

$$\bar{\varphi}(\bar{X}) = \sum_{i=1}^{\pi} \bar{x}_i^2 - \sum_{i=\pi+1}^{\pi+\nu} \bar{x}_i^2 \ \text{ where } \ \bar{X} = \sum_{i=1}^{r} \bar{x}_i \frac{1}{\sqrt{c_i}} u_i + \sum_{i=r+1}^{n} x_i u_i. \tag{3.179}$$

In this way we may assume, from the beginning, that all non-zero coefficients in the canonical expression of φ (we start from) are $= \pm 1$. The fact assumption the positive and respectively negative terms in expressions (3.177) and (3.179) appear in "compact trains" does not reduce the generality. If, in a certain canonical expression, the terms with coefficients $c_i \,/\, -c_j \ (c_i, c_j > 0)$ appear in "mixed" sequences, a simple renumbering of the variables will bring such an expression to the form in (3.177). It is just the way we forced a Q-form with $\Delta_2 = 0$ to accept a canonical expression by Jacobi's method in **Example 3.4'** - pages 117-118 (Eqs. (3.75) & (3.76)).

Let then B and B_1 be two bases of space V in which the Q-form φ has the canonical expressions

$$\varphi(X) = \sum_{i=1}^{\pi} x_i^2 - \sum_{i=\pi+1}^{r} x_i^2 \ \text{ in basis } B = [v_1 \ v_2 \ \ldots \ v_n], \ (X = X_B) \tag{3.180}$$

$$\varphi_1(Y) = \sum_{i=1}^{\rho} y_i^2 - \sum_{i=\rho+1}^{r} y_i^2 \ \text{ in basis } B_1 = [w_1 \ w_2 \ \ldots \ w_n] \ (Y = X_{B_1}). \tag{3.181}$$

Let us notice that the numbers of positive, respectively negative terms in the two expressions (3.180) & (3.181) are assumed to be possibly different. In (3.180) $\mathbf{sgn}\,\varphi = (\pi, \nu, r - \pi - \nu)$ while, in (3.181), $\mathbf{sgn}\,\varphi_1 = (\rho, r - \rho, n - r)$. Let us assume that $\pi \neq \rho$ and admit the case when $\rho > \pi$. Consider now the two subspaces spanned by two smaller sub-bases of B and B_1 :

$$U = \mathcal{L}([v_1 \ v_2 \ \ldots \ v_\pi]) \ \ \& \ \ W = \mathcal{L}([w_{\rho+1} \ w_{\rho+2} \ \ldots \ w_n]). \tag{3.182}$$

Since $\dim U + \dim W > n$, it follows from Grassmann's formula (THEOREM 3.1 in **§ 1.3**, page 65, Eq. (3.17) in ⟦𝒜. 𝒞., 1999⟧) that

$$U \cap W \neq \{\mathbf{0}\}. \tag{3.183}$$

Hence there exists a vector $v \in U \cap W, \ v \neq \mathbf{0}$, such that

$$v = x_1 v_1 + x_2 v_2 + \ldots + x_\pi v_\pi = y_{\rho+1} w_{\rho+1} + y_{\rho+2} w_{\rho+2} + \ldots + y_n w_n. \tag{3.184}$$

Since $v \neq \mathbf{0}$ it follows from (3.180) that

$$\varphi(v) = \varphi(X) = \sum_{i=1}^{\pi} x_i^2 = x_1^2 + \ldots + x_\pi^2 > 0, \ \text{with} \ X = v_B, \tag{3.185}$$

and from (3.181) that

$$\varphi(v) = \varphi(Y) = -\sum_{i=\rho+1}^{r} y_i^2 = -y_{\rho+1}^2 - \ldots - y_r^2 < 0, \text{ with } Y = v_{B_1}. \tag{3.186}$$

The contradiction between the inequalities (3.185) & (3.186) is obvious, so $\rho > \pi$ is impossible and the same holds for the symmetric inequality, so $\rho = \pi$. Consequently, the signature of a Q-form under canonical expressions is the same in any (canonical) basis. ■

Comments. A proof of SYLVESTER'S INERTIA THEOREM can be found in the textbook [C. Radu, 1996]. We have followed the way of proof in this reference until Eqs. (3.180) & (3.181) but we continued along the simpler and more elegant setup in another monograph of LINEAR ALGEBRA, namely [E. Sernesi, 1993, pages 234-235]. Certainly, we have adapted the notations used by the two authors to our notations like φ for the Q-forms, the bases written as ordered *n*-tuples (rows) of vectors like in (3.180) to (3.182), and notation (3.176) for the signature of a quadratic form. Here we must mention that the definition of this numerical characteristic is slightly different as given by E. Sernesi. It is given in terms of positive and negative terms only :

$$\pi \to p, \ \nu \to r-p, \ \operatorname{sgn}\varphi = (\pi, \nu, \zeta) \to (p, r-p).$$

Clearly, the two definitions are wholly equivalent. From a signature of the form $(p, r-p)$ it follows that $\pi = p, \ \nu = r-p, \ \zeta = n-r$. Let us close this comment with the remark that the Q-form is degenerate $\Longleftrightarrow \zeta > 0$.

In [E. Sernesi, 1993], a canonical expression of the form (3.180) is called a *normal form*. In fact, it is a canonical expression like (3.177) with the coefficients turned to ± 1 by the basis transformation (3.178).

The structure of the matrix of a Q-form with $\operatorname{sgn}\varphi = (\pi, \nu, \zeta)$ and a canonical expression like (3.180) is

$$[\delta] = f(D^{\mathrm{T}}, D) = \begin{bmatrix} I_\pi & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -I_\nu & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{3.187}$$

The zero blocks in (3.187) have appropriate sizes ; for instance, the north-eastern block is of size (π, ζ).

Definition 3.9. A quadratic form φ (defined) on a real vector space V is said to be

(i) *positive definite* if $(\forall\, x \in V)\ x \neq \mathbf{0} \Rightarrow \varphi(x) > 0$,

(ii) *positive semi-definite* if $(\forall\, x \in V)\ \varphi(x) \geq 0$,

(iii) *negative definite* if $(\forall\, x \in V)\ x \neq \mathbf{0} \Rightarrow \varphi(x) < 0$,

(iv) *negative semi-definite* if $(\forall\, x \in V)$ $\varphi(x) \le 0$,

(v) *indefinite* if φ does not satisfy any of (ii), (iv).

◊

Obviously, if φ is positive or negative definite it is also positive or negative semi-definite, respectively. If it is neither positive nor negative semi-definite it is *indefinite*. The normal forms corresponding to the five types of quadratic forms are more relevant to the ways their canonical expressions look. A table-like characterization is given in [E. Sernesi, 1993], page 236. We offer a slightly more complete characterization, with the signs / (in)equalities for the three components of a Q-form's signature. For the semi-definite cases it is assumed that $0 \le r \le n$ while $\pi\nu > 0 \Rightarrow r > 0$ for the indefinite case. Let us recall that $r = \mathrm{rank}\,\varphi$ & $n = \dim V$.

Table 3.1 **Types of quadratic forms**

Type	*Normal form*	*Signature*
PD - (i)	$x_1^2 + \ldots + x_n^2$	$(n, 0, 0)$, $\pi = r = n$
PsD - (ii)	$x_1^2 + \ldots + x_r^2$	$(r, 0, n - r)$, $\pi = r \le n$
ND - (iii)	$-x_1^2 - \ldots - x_n^2$	$(0, n, 0)$, $\nu = r = n$
NsD - (iv)	$-x_1^2 - \ldots - x_r^2$	$(0, \nu, n - r)$, $\nu = r \le n$
ID - (v)	$x_1^2 + \ldots + x_\pi^2 - x_{\pi+1}^2 - \ldots - x_r^2$, $\nu = r - \pi$	(π, ν, ζ), $0 < \pi, \nu < r$

With reference to a couple of earlier examples, let us write down the signatures of the Q-forms that were diagonalized. We also recall the pages where those examples appear.

Ex. 3.4	pp. 90-92,	Gauss	Eqs. (3.58) & (3.70)	$\mathrm{sgn}\,\varphi = (2, 2, 0)$;
Ex. 3.4'	pp. 93-95,	Gauss	Eqs. (3.58) & (3.79)	$\mathrm{sgn}\,\varphi = (2, 2, 0)$;
Ex. 3.5	page 98	Jacobi	Eqs. (3.98) & (3.100)	$\mathrm{sgn}\,\varphi = (3, 1, 0)$;
Ex. 3.6	page 98	Jacobi	Eqs. (3.101) & (3.102)	$\mathrm{sgn}\,\varphi = (3, 0, 0)$;
Ex. 3.7	pp. 109-111,	OT-EVV	Eqs. (3.148) & (3.157)	$\mathrm{sgn}\,\varphi = (3, 0, 0)$;
Ex. 3.8	pp. 111-115,	OT-EVV	Eqs. (3.158) & (3.175)	$\mathrm{sgn}\,\varphi = (3, 1, 0)$.

The equation numbers where the initial expressions of the Q-forms appear are written in smaller font (11 pt). The reader is asked to apply other methods among **G, J, OT** for checking **TH. 3.1.**

§ 2.3-A APPLICATIONS TO SYMMETRIC BLFS & QUADRATIC FORMS

3 – A.1 Given the symmetric BLF $f : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ with its matrix in basis E_3

$$F_{E_3} = [\varepsilon] = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 0 & -2 \\ 3 & -2 & 4 \end{bmatrix},$$

it is required to find a basis spanning each of the subspaces $\operatorname{Ker} f$ and $U^{\perp}$:
U is spanned by $a_1 = [1 \;\; 0 \;\; 2\,]^{\mathrm{T}}$, $a_2 = [\,0 \;\; -1 \;\; 1]^{\mathrm{T}}$.

3 – A.2 Diagonalize the following quadratic forms $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$, using either of the available methods :

a) $\varphi(X) = x_1^2 + 4x_1x_2 + 2x_1x_3 + x_2^2 + 2x_2x_3 + 3x_3^2$;

b) $\varphi(X) = 2x_1x_2 + 2x_2x_3 + 4x_1x_3 - 2x_2^2 + x_3^2 + x_1^2$;

c) $\varphi(X) = x_1^2 + x_2^2 + 3x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$;

d) $\varphi(X) = x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3$;

e) $\varphi(X) = 3x_1^2 + 4x_2^2 + 3x_3^2 + 4x_1x_2 - 4x_2x_3$;

f) $\varphi(X) = 2x_1^2 + 5x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_1x_3 + 8x_2x_3$;

g) $\varphi(X) = -5x_1^2 + x_2^2 + 4x_1x_2 + 6x_1x_3$;

h) $\varphi(X) = 2x_1x_2 + 6x_2x_3$.

Specify the final (canonical) bases when eigenvalues and eigenvectors are used, and try to check the results with Eq. (3.146) - page 109, as in **Examples**

3.7 - page 98 and **3.8** - page 111-115, see transformations leading to the diagonal matrix in (3.174) .

3-A.3 Diagonalize the Q-form

$$\varphi(X) = x_1^2 + 7x_2^2 + x_3^2 - 8x_1x_2 - 6x_1x_3 - 8x_2x_3$$

using eigenvalues and eigenvectors ; specify a basis in which φ is diagonal.

3-A.4 Write the analytical expression of the symmetric BLF whose associated Q-form is the preceding one (in **3-A.3**).

3-A.5 Given the symmetric bilinear form $f: \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ by its matrix in the standard basis E,

$$f(E^{\mathrm{T}}, E) = [\varepsilon] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

it is required to write the analytical expression of the associated Q-form φ, to determine the subspaces $\operatorname{Ker} f$ and $U^{\perp f}$ where U is spanned by the vectors $u_1 = [1 \;\; 0 \;\; 1]^{\mathrm{T}}$, $u_2 = [-1 \;\; 0 \;\; 0]^{\mathrm{T}}$. Find a basis (or a vector) spanning $U^{\perp f}$ and check whether the intersection $U \cap U^{\perp f}$ is trivial or not. Then write the general expression of a vector in $U + U^{\perp f}$.

3-A.6 Write the following Q-forms under the matrix form $f(X) = X^{\mathrm{T}}[\varepsilon]X$ and bring each of them to a diagonal (canonical) expression :

(a) $\quad \varphi(X) = 9x_1^2 - x_2^2 + 4x_3^2 + 6x_1x_2 - 8x_1x_3 + x_2x_3$;

(b) $\quad \varphi(X) = x_1^2 + x_2^2 - 3x_3^2 - 5x_1x_2 + 9x_1x_3$;

(c) $\quad \varphi(X) = x_1x_2 + x_1x_3 + x_2x_3$;

(d) $\quad \varphi(X) = \sqrt{2}\,x_1^2 - \sqrt{3}\,x_3^2 + 2\sqrt{2}\,x_1x_2 - 8\sqrt{3}\,x_1x_3$;

(e) $\quad \varphi(X) = x_1^2 + x_2^2 - x_3^2 - x_4^2 + 2x_1x_2 - 10x_1x_4 + 4x_3x_4$.

3-A.7 Diagonalize the quadratic form $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$\varphi(X) = 2x_1^2 + 5x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_1x_3 - 8x_2x_3$$

using the method of orthogonal transformations - **EVV**. Check the result with formula (3.146) at page 109 and write the corresponding canonical and normal expressions.

Note : The subspace corresponding to the eigenvalue $\lambda = 1$ is two-dimensional, and an orthogonal basis spanning it has to be found.

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3 – A.8

Show that the connection from a Q-form φ to the symmetric BLF f which determines it , by $\varphi(x) = f(x, x)$, that is formula (3.41) in PROPOSITION 3.6 at page 110, can be replaced by the somehow simpler formula

$$f(x, y) = \frac{1}{2}[\varphi(x + y) - \varphi(x) - \varphi(y)]$$

since it is equivalent to the connection (3.41).