§ 4.2 LINEAR ENDOMORPHISMS / OPERATORS

The most part of the definitions and results presented in the previous section (**§ 4.1**) also hold in the particular case when the two vector spaces U and V are identical. Only minor changes have to be operated in the corresponding equations / formulas, especially in the notations. But new (and specific) notions and properties also occur, as they are presented in the body of this Section.

Informally speaking, the linear endomorphisms (operators) are linear transformations (or morphisms) defined on a vector space V and taking values *in the same space* V. Hence, the definition of a linear morphism (*Def. 1.1* in **§ 4.1**) has to be slightly adapted to the case when the two spaces there involved are identical : $U \equiv V$. The formal definition is

Definition 2.1. Let V be a vector space over the field $\mathbf{K}$ (= $\mathbb{R}$ / = $\mathbb{C}$). A mapping $f : V \longrightarrow V$ is said to be a *linear endomorphism* of the vector space V (or a *linear operator* on space V) if it satisfies the following properties :

$(\mathbf{LO}_1)$ $\quad (\forall\, x_1, x_2 \in V)\ f(x_1 + x_2) = f(x_1) + f(x_2)$;

$(\mathbf{LO}_2)$ $\quad (\forall\, \lambda \in \mathbf{K})(\forall\, x \in V)\ f(\lambda\, x) = \lambda\, f(x)$.

◊

Notational convention. From this point on we adopt a specific notation for the endomorphisms / operators on a space V, which is not only somehow simpler but it also suggests the difference between general morphisms and endomorphisms. We replace the symbol f by L and we will write $L\,x\ (= y)$ instead of $f(x)$, that is the vector argument x is no more put between parentheses, except the case when the argument of L is not a simple vector but a sum of two (or more vectors), a vector multiplied by a scalar or a linear combination of vectors. This way to denote the action of an endomorphis L on a vector x is accepted and used in rather many textbooks of LINEAR ALGEBRA, for instance in [G. Strang, 1988], [I. Creangă *et al*, 1962] ; we also used this notation in the textbook [A. Carausu, 1999]. With this notational change, *Definition 2.1* becomes:

A mapping $L : V \longrightarrow V$ is said to be a *linear endomorphism* or a *linear operator* on space V if it satisfies the axioms

$$(\mathbf{LE}_1) \quad (\forall\, x_1, x_2 \in V)\ L(x_1 + x_2) = L\,x_1 + L\,x_2\ ; \tag{2.1}$$

$$(\mathbf{LE}_2) \quad (\forall\, \lambda \in \mathbf{K})(\forall\, x \in V)\ L(\lambda\, x) = \lambda\, L\,x\ . \tag{2.2}$$

As in the case of the linear transformations / morphisms, these two defining properties may be replaced by a single property – the *linearity*, leading to an equivalent definition :

Definition 2.1'. Let V be a vector space over the field $\mathbf{K}$ (= $\mathbb{R}$ / = $\mathbb{C}$). A mapping $L : V \longrightarrow V$ is a *linear endomorphism* or a *linear operator* on space V) if it satisfies the property

$(\forall\, \lambda_1, \lambda_2 \in \mathbf{K})(\forall\, x_1, x_2 \in V)$

(LIN)

$$\boxed{L(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 L x_1 + \lambda_2 L x_2 .} \tag{2.3}$$

The equivalence between *Definition 2.1* and *Def. 2.1'* was already proved in **§ 3.1**. It simply suffices to change f for L and the membership of x_1, x_2 from space U to V. Let us also remark that this property (LIN) or (2.3) can be replaced by the simpler one

$$(\forall\, \alpha \in \mathbf{K})(\forall\, x_1, x_2 \in V)\;\; L(x_1 + \alpha\, x_2) = L x_1 + \alpha L x_2 . \tag{2.4}$$

As in the case of the linear forms of linear morphisms, the property (LIN) can be extended from linear combination of two vectors to arbitrary combinations of several vectors ; thus, PROPOSITION 1.1 becomes

Proposition 2.1. *If* $L : V \longrightarrow V$ *is a linear endomorphism / operator then*

$$(\forall\, \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbf{K})(\forall\, x_1, x_2, \ldots, x_m \in V)$$

$$\boxed{L\Big(\sum_{i=1}^{m} \lambda_i x_i\Big) = \sum_{i=1}^{m} \lambda_i L x_i .} \tag{2.5}$$

Obviously, the proof in **§ 4.1** holds, with the corresponding notational changes.

This property of extended linearity, (2.5), can be rewritten by using our “matrix notations”, as we did it in **§ 4.1**, at page 124. Let us recall those notations :

$$\mathcal{X} = [x_1\; x_2 \ldots\, x_m] \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix} ; \tag{2.6}$$

With (2.6), a linear combination may be written as

$$\sum_{i=1}^{m} \lambda_i x_i = \Lambda^{\mathrm{T}} \cdot \mathcal{X}^{\mathrm{T}} = \mathcal{X} \cdot \Lambda = \sum_{i=1}^{m} x_i \lambda_i . \tag{2.7}$$

It follows from (2.5) with (2.7) that

$$L\Big(\sum_{i=1}^{m} \lambda_i x_i\Big) = L(\Lambda^{\mathrm{T}} \cdot \mathcal{X}^{\mathrm{T}}) = \Lambda^{\mathrm{T}} \cdot L\,\mathcal{X}^{\mathrm{T}} . \tag{2.5'}$$

In (2.5'), $L\,\mathcal{X}^{\mathrm{T}}$ represents the column vector of the values $L x_i,\; i = \overline{1, m}$. If the row of scalars and the column of vectors that occur in (2.5') are transposed, this property is equivalent to

$$\boxed{L(\mathcal{X} \cdot \Lambda) = L\,\mathcal{X} \cdot \Lambda .} \tag{2.8}$$

In this formula, the operator's values of (2.5) appear as the components of a row vector :

$$L\mathcal{X} = [\,L x_1 \;\; L x_2 \ldots L x_m\,]. \tag{2.9}$$

In what follows, we will prefer the notational alternative (2.8).

Any linear operator on a (finitely generated) vector space V can be characterized by a matrix, once a basis is fixed in this space.

Proposition 2.2. *If $L : V \longrightarrow V$ $(\dim V = n)$ is a linear morphism and V is spanned by basis $A = [\,a_1 \; a_2 \; \ldots \; a_n\,]$, then the morphism L uniquely determines an n-by-n matrix L_A defined by*

$$\boxed{L A^{\mathrm{T}} = L_A \cdot A^{\mathrm{T}}.} \tag{2.10}$$

Proof. The proof is similar to that of PROPOSITION 1.2 in the previous section, for general morphisms ; but it is simpler.

For any $i \in \{1, 2, \ldots, n\}$, $La_i \in V = \mathcal{L}(A)$. Therefore $L a_i$ admits a *unique* linear expression in the basis A of V :

$$L a_i = \sum_{j=1}^{n} \lambda_{ij}\, a_j = \Lambda_i \cdot A^{\mathrm{T}}. \tag{2.11}$$

More explicitly, the scalars λ_{ij} $(j = 1, 2, \ldots, n)$ are the coordinates of La_i in basis A. In the last expression of (2.11) the row of scalars that multiplies the transpose of row matrix A is

$$\Lambda_i = [\,\lambda_{i1} \; \lambda_{i2} \; \ldots \; \lambda_{in}\,], \;\; i = \overline{1, n}. \tag{2.12}$$

The n linear expressions of the form (2.11) can be written one under the other resulting a system of equations (equalities) which is equivalent to the matrix equation

$$L A^{\mathrm{T}} = L_A \cdot A^{\mathrm{T}} \;\;\text{with}\;\; L_A = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_n \end{bmatrix}, \; A^{\mathrm{T}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}. \tag{2.13}$$

Therefore the linear expression (2.10) is proved and it uniquely defines the matrix of the operator in the (given) basis. The proof is practically over, but we consider to be useful a more explicit form of the equation in (2.13).

$$L A^{\mathrm{T}} = L \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \ldots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \ldots & \lambda_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda_{n1} & \lambda_{n2} & \ldots & \lambda_{nn} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}. \tag{2.14}$$

As a technical detail, let us mention that we have used the "C-dot" in the right-hand sides of Eqs.

(2.11) and (2.13) - (2.14) just for emphasizing that La_i and LA^{T} in the left-hand sides are images of a vector in the basis, respectively of the whole basis written as a column, while the matrix-type products in the right sides represent linear combinations like that of (2.7).

■

The next result, corresponding to a more general one for the morphisms between two (possibly) different spaces, regards the analytic expression of a morphism L in a basis A of the space V (see PROPOSITION 1.3 in **§ 4.1**, at pages 125-126).

Proposition 2.3. *If the vector space V is spanned by the basis* $A = [a_1 \; a_2 \; \dots \; a_m]$, *and the matrix of the linear endomorphism* $L: V \longrightarrow V$ *in the basis* A *is* L_A *then the image* Lx *of the vector* $x = AX_A = X_A^{\mathrm{T}} A^{\mathrm{T}} \in V$ *is*

$$\boxed{Lx = X_A^{\mathrm{T}} \cdot L_A \cdot A^{\mathrm{T}}.} \tag{2.15}$$

Proof. Formula (2.15) follows from PROPOSITIONS 2.1 & 2.2 and – more precisely – from the property of extended linearity (2.7) by replacing

$$\mathcal{X} \to A \text{ and } \Lambda \to X_A, \text{ with } x = AX_A.$$

Equivalently, it can be immediately obtained from Eq. (1.13) in PROPOSITION 1.3 of **§ 4.1**, with

$$f \to L,\; f(x) \to Lx,\; F_{A,B} \to L_A,\; B^{\mathrm{T}} \to A^{\mathrm{T}}. \tag{2.16}$$

■

Remarks 2.1. The proof is over, but we can give a more explicit (expanded) version of this formula (2.15), recalling – from **§ 1.1** – that

$$X_A^{\mathrm{T}} = [\xi_1 \; \xi_2 \; \dots \; \xi_m] \Rightarrow x = \sum_{i=1}^{n} \xi_i a_i \underset{(2.5)}{\Rightarrow} Lx = \sum_{i=1}^{n} \xi_i L a_i =$$

$$= \sum_{i=1}^{n} \xi_i \sum_{j=1}^{n} \lambda_{ij} a_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \lambda_{ij} a_j = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} \xi_i \lambda_{ij} \Big) a_j\,. \tag{2.17}$$

Formulas (2.15) and (2.17) effectively give the analytical expression of the image Lx of a vector x through the endomorphism L in basis A of space V.

On another hand, PROPOSITION 2.3 may be considered as giving the converse result to PROPOSITION 2.2 : indeed, if a square matrix $[\lambda_{ij}]_{n,n}$ is given and the linear expression of a vector x of the form

$$x = AX_A = X_A^{\mathrm{T}} A^{\mathrm{T}} \in V$$

is known, then $Y = X_A^{\mathrm{T}} \cdot [\lambda_{ij}] \cdot A^{\mathrm{T}}$ is just the image of vector x through L if $LA^{\mathrm{T}} = L_A \cdot A^{\mathrm{T}}$.

Hence, if such an explicit expression of the image is given, the matrix L_A of L in basis A can be derived. We saw (in the previous Section) that a morphism $f: U \longrightarrow V$ has a uniquely determined matrix in a pair of bases (A, B) and – conversely – an m-by-n matrix $[\varphi_{ij}]_{m,n}$ plus a pair of bases uniquely determine a morphism $f: U \longrightarrow V$. Like in that more general case, the "equivalence" between an endomorphism and its matrix should not be formally understood.

Once a basis A a basis of space V is selected, any square matrix M taken as the matrix L_A in Eq. (2.10) – that is $L\,A^{\mathrm{T}} = M \cdot A^{\mathrm{T}}$ – a corresponding operator L is effectively determined by the formula (2.15).

Let us now see how the matrix and the analytic expression of an operator of the form

$$L: \mathbf{K}^n \longrightarrow \mathbf{K}^n \ / \ L: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

can be obtained from each other. Like in the case of linear forms, bilinear forms and linear morphisms, the most applications with such mappings between (finite-dimensional) Euclidean spaces are stated in terms of its analytic expression or image in terms of the components of the "input" vectors X, Y (for BLFs) or only X in the case of LFs and endomorphisms / operators. But it is equally possible that an operator L is specified by its matrix L_E in the standard basis $E_n \underset{\text{not}}{=} E$ of the space $\mathbf{K}^n$ / $\mathbb{R}^n$: equation (2.10) becomes

$$\boxed{L\,E^{\mathrm{T}} = L_E \cdot E^{\mathrm{T}}.} \tag{2.18}$$

This should not be understood as a simple change of a symbol (letter) denoting the basis. Eq. (2.18) is a vector equation giving the coordinates of the images $Le_1, Le_2, \ldots, Le_n$ in the standard basis $E_n = E$, on the rows of matrix L_E. But Eq. (2.18) may be equivalently regarded as a matrix equation whose left side. is just $[\,Le_1 \;\; Le_2 \;\; \ldots \;\; Le_n\,]^{\mathrm{T}}$ and whose right side is

$$L_E \cdot E^{\mathrm{T}} = L_E \cdot I_n = L_E \tag{2.19}$$

since the vectors $e_1, e_2, \ldots, e_n$ – written as a row of columns – give the identity matrix I_n and the same matrix is obtained if these elementary vectors are transposed and written together as a stack of rows : $E = E^{\mathrm{T}} = I_n$. It follows from (2.18) and (2.19) that

$$L\,E^{\mathrm{T}} = L\,[\,e_1 \;\; e_2 \;\; \ldots \;\; e_n\,]^{\mathrm{T}} = L_E. \tag{2.20}$$

Possibly, this equation (2.20) is not so clearly derived. But let us write, as in the case of an arbitrary basis A, the image of a vector $e_i = [0 \; \ldots \; 0 \; 1 \; 0 \; \ldots \; 0]^{\mathrm{T}}$ through L as in Eq. (2.11) at page 165, with

$$A \rightarrow E \Rightarrow [a_i \rightarrow e_i \;\&\; a_j \rightarrow e_j], \; \lambda_{ij} \rightarrow \varepsilon_{ij}:$$

$$Le_i = \sum_{j=1}^{n} \varepsilon_{ij}\, e_j = [\,\varepsilon_{i1} \;\; \varepsilon_{i2} \; \ldots \; \varepsilon_{in}\,] \cdot \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} \varepsilon_{i1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \ldots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \varepsilon_{in} \end{bmatrix} = \begin{bmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \\ \vdots \\ \varepsilon_{in} \end{bmatrix}. \tag{2.21}$$

Writing the n equations (2.21) one aside the other – for $i = 1, 2, \ldots, n$ – it follows that

$$L\,E = L\,[\,e_1\ e_2\ \ldots\ e_n] = L_E^{\mathrm{T}} \underset{\text{transp}}{\Rightarrow} (2.20)\,.$$

Let us finally see that Eq. (2.20) immediately follows from (2.18) by applying the transpose operator and taking (again) into account that $E = E^{\mathrm{T}} = I_n$.

The analytical expression of the image $L\,X$ through an operator L whose matrix in the standard basis is L_E follows rather simply from PROPOSITION 2.3, with appropriate changes in notations :

Proposition 2.4. *If the mapping* $L : \mathbf{K}^n \longrightarrow \mathbf{K}^n$ / $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is a linear operator with the matrix* L_E *in the standard basis of the space and* X *is a vector in this space then*

$$\boxed{L\,X = X^{\mathrm{T}} \cdot L_E \cdot E^{\mathrm{T}}.} \tag{2.22}$$

Proof. Formula (2.22) follows from PROPOSITION 2.3 by replacing

$$A \to E \Rightarrow X_A \to X_E = X,\ L_A \to L_E\,. \tag{2.23}$$

Obviously, (2.15) & (2.23) $\Rightarrow$ (2.22). But the same formula can be obtained from the corresponding expression $Y = f(X)$ of a vector $X \in \mathbf{K}^m$ / $\mathbb{R}^m$ through a (general) morphism $f : \mathbf{K}^m \longrightarrow \mathbf{K}^n$ / $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$: see formula (1.19) at page 151. Formula (2.22) obviously follows from that expression by taking

$$m = n \Rightarrow E_m = E_n = E\,,\ f \to L\,,\ F_{E_m, E_n} \to L_E\,. \tag{2.24}$$

This alternative way to obtain Eq. (2.22) offers - practically - a second proof of this **P. 2.4**.

■

The two sides of formula (2.22) are column vectors : according to our convention in **§ 1.1**, the vectors in the space $\mathbf{K}^n$ / $\mathbb{R}^n$ are usually written as column vectors, hence both X and its image $L\,X$ are column vectors. Let us denote

$$L\,X = Y \underset{\text{not}}{=} Y^{\mathrm{T}} \cdot E_n^{\mathrm{T}} = Y^{\mathrm{T}} \cdot I_n. \tag{2.25}$$

It follows, from expression (2.22), that

$$\boxed{Y^{\mathrm{T}} = Y^{\mathrm{T}} \cdot L_E \Rightarrow Y = L\,X = L_E^{\mathrm{T}} \cdot X\,.} \tag{2.26}$$

We have derived this expression (2.26) of the image of a vector in $\mathbf{K}^n$ (or in $\mathbb{R}^n$) from the general formula (2.15). But this is just the way an operator

$$L : \mathbf{K}^n \longrightarrow \mathbf{K}^n \ / \ L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is given in the most problem books of LINEAR ALGEBRA, or in most exercises with linear maps included in such textbooks. In fact, we earlier discussed this aspect in the previous **§ 4.1** : the image of a vector $X = [x_1 \ x_2 \ \ldots \ x_n]$ appears as a column vector whose components *are linear forms in the components of* X. Let us recall that we wrote $f(X) = M \cdot X$ where M was the transpose of morphism f's matrix in the pair of standard bases. It was Eq. (1.52) at page 136, namely $M = F^{\mathrm{T}}_{E_m, E_n}$. From this formula, replacings of (2.24) and expression (2.26) we get

$$\boxed{LX = M \cdot X} \quad \text{with} \quad \boxed{M = L_E^{\mathrm{T}} \iff L_E = M^{\mathrm{T}}.} \tag{2.27}$$

Let us conclude this discussion by mentioning that, once given an endomorphism by its matrix L_E in the standard basis of the n-dimensional space, the matrix L_E can be immediately obtained by a simple transposition, together with the expression of the image $Y = LX$ under the form (2.26). Conversely, if the operator is given as

$$Y = LX = \begin{bmatrix} \ell_1(X) \\ \ell_2(X) \\ \vdots \\ \ell_n(X) \end{bmatrix} = M \cdot X, \tag{2.28}$$

then L's matrix L_E in the standard basis follows by (2.27).

Example 2.1. The operator $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is given by the image of $X \in \mathbb{R}^3$,

$$Y = LX = \begin{bmatrix} 2x_1 - 3x_2 \\ -x_1 \quad + 3x_3 \\ 5x_1 - x_2 - x_3 \end{bmatrix}. \tag{2.29}$$

It is required to write the matrix L_E of L and to find the image of this operator, respectively the image / counter-image of the vectors

$$X = \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}, \ Y = \begin{bmatrix} 3 \\ 8 \\ -1 \end{bmatrix}. \tag{2.30}$$

The image of $X \in \mathbb{R}^3$ of (2.29) and Eq. (2.28) give the matrix

$$M = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 0 & 3 \\ 5 & -1 & -1 \end{bmatrix} \Rightarrow L_E = \begin{bmatrix} 2 & -1 & 5 \\ -3 & 0 & -1 \\ 0 & 3 & -1 \end{bmatrix}.$$

The image LX of the first vector in (2.30) is obtained with formula (2.28) :

$$L\,X = M \cdot X = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 0 & 3 \\ 5 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 0 & 3 \\ 5 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ -10 \\ 22 \end{bmatrix}.$$

The counter-image of $Y \in \mathbb{R}^3$ of (2.29) is obtained from the matrix equation $M \cdot X = Y$ that is equivalent to a non-homogeneous system of augmented matrix

$$[M \mid Y] = \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ -1 & 0 & 3 & 8 \\ 5 & -1 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 14 & -3 & 0 & 5 \\ -5 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} -12 & 0 & 0 & -2 \\ -14/3 & 1 & 0 & -5/3 \\ -5 & 1 & 1 & 1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/6 \\ -14/3 & 1 & 0 & -5/3 \\ -1/3 & 0 & 1 & 8/3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/6 \\ 0 & 1 & 0 & -8/9 \\ 0 & 0 & 1 & 49/18 \end{array}\right] \Rightarrow$$

$$\Rightarrow X = L_{-1}(Y) = \frac{1}{18} \begin{bmatrix} 3 \\ -16 \\ 49 \end{bmatrix}. \tag{2.31}$$

As a detail, we have used the subscript -1 in (2.31) since we had not checked that L of (2.29) is invertible ; in fact, it is an isomorphism but we are going to present such properties of endomorphisms (operators) in what follows. Let us close this example by checking the result in (2.31) by use of formula (2.28), with X of Eq. (2.31) :

$$M \cdot X = \frac{1}{18} \cdot \begin{bmatrix} 2 & -3 & 0 \\ -1 & 0 & 3 \\ 5 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -16 \\ 49 \end{bmatrix} = \ldots = \begin{bmatrix} 3 \\ 8 \\ -1 \end{bmatrix},$$

and the vector Y in the statement has been retrieved. □

***Definition* 2.2 (Kernel and Image) .** Let V be a vector space over the field $\mathbf{K}$ ($= \mathbb{R} / = \mathbb{C}$) and $L : V \longrightarrow V$ a linear operator. Then the *kernel* and the *image* (or *range*) of L are respectively defined by

$$\boxed{\operatorname{Ker} L \underset{\text{def}}{=} \{x \in V :\ L\,x = \mathbf{0} \in V\}\,;} \tag{2.32}$$

$$\boxed{\operatorname{Im} L \underset{\text{def}}{=} \{y \in V :\ (\exists\, x \in V)\ \ y = L\,x\}\,.} \tag{2.33}$$

◊

Like in the case of the (general) morphisms, these two subsets of space V can be equivalently written as

$$\boxed{\operatorname{Ker} L = L_{-1}(\mathbf{0}) \subseteq V\,;} \tag{2.32'}$$

$$\boxed{\operatorname{Im} L = LV = \{Lx : x \in V\} \subseteq V.} \tag{2.33'}$$

These two subsets are (as in the case of morphisms between two vector spaces) more than subsets, *they are subspaces*. Without needing an effective new proof, PROPOSITION 1.4 of **§ 4.1** can be "transferred" to the endomorphisms with appropriate replacements :

PROPOSITION 2.5. *Let* V *be a vector space over the field* $\mathbf{K}$ *and* $L : V \longrightarrow V$ *a linear opderator. Then* $\operatorname{Ker} L$ *and* $\operatorname{Im} L$ *are subspaces of* V:

$$\operatorname{Ker} L \subseteq_{\text{subsp}} V, \qquad \operatorname{Im} L \subseteq_{\text{subsp}} V. \tag{2.34}$$

Proofs. The necessary replacements in the proof of PROPOSITION 1.4 (at page 151) are :

$$U \to V \ (U \equiv V),\ f \to L,\ f(\lambda_1 x_1 + \lambda_2 x_2) \to L(\ldots),\ \text{etc.} \qquad \blacksquare$$

As a consequence of this Proposition, both the kernel and the image of the EM L contain the zero vector ; this follows from a property of any subspace (presented in **§ 1.3**). Hence we have

$$\{\mathbf{0}\} \subseteq \operatorname{Ker} L \subseteq_{\text{subsp}} V, \qquad \{\mathbf{0}\} \subseteq \operatorname{Im} L \subseteq_{\text{subsp}} V. \tag{2.35}$$

Like the morphisms, the endomorphisms / operators can also be classified from the point of view of properties of general functions (or mappings).

Properties & Classification of Linear Endomorphisms

Definition 2.3. Let V be a vector space over the field $\mathbf{K}$ (= $\mathbb{R}$ / = $\mathbb{C}$) and $L : V \longrightarrow V$ a linear operator. Then L is

(i) *injective* $\underset{\text{def}}{\Longleftrightarrow}$ $(\forall\, x_1, x_2 \in V)\ x_1 \neq x_2 \Rightarrow Lx_1 \neq Lx_2\,;$, or (2.36)

$$(\forall\, x_1, x_2 \in V)\ Lx_1 = Lx_2 \Rightarrow x_1 = x_2. \tag{2.37}$$

(ii) *surjective* (or *onto*) if $\underset{\text{def}}{\Longleftrightarrow}$ $\operatorname{Im} L = LV = V \Longleftrightarrow (\forall\, y \in V)\ L_{-1}(y) \neq \varnothing.$ (2.38)

A linear endomorphism $L : V \longrightarrow V$ is *bijective* if it is both injective and surjective. ◊

Definition 2.4. **(Special endomorphisms, composite endomorphisms).** Let V be a vector space over the field $\mathbf{K}$ (= $\mathbb{R}$ / = $\mathbb{C}$). The *identical* morphism on V was defined at page 140 (*Def. 1.5*) but we recall that

$$\mathrm{id}_V : V \longrightarrow V,\ (\forall\, x \in V)\ \mathrm{id}_V\, x = x. \tag{2.39}$$

The *zero operator* is defined by

$$\mathrm{O}_V : V \longrightarrow V, \ (\forall\, x \in V) \ \ \mathrm{O}_V x = \mathbf{0} \in V. \tag{2.40}$$

Given two operators $L : V \longrightarrow V$ & $M : V \longrightarrow V$, the *composite* operator of L with M is defined by

$$M \circ L : V \longrightarrow V \ (\forall\, x \in V) \ \ (M \circ L)x \underset{\text{def}}{=} M(Lx). \tag{2.41}$$

◊

It follows from PROPOSITION 1.7 in **§ 4.1** that $M \circ L \underset{\text{not}}{=} N$ is an endomorphism, and Eq. (1.77) in the same previous section gives its matrix in a basis A of space V:

$$N_A = (M \circ L)_A = L_A \cdot M_A. \tag{2.42}$$

The properties of an operator (in *Def. 2.3*) can be characterized in terms of the kernel and image of L and also by the composite endomorphism, as they were earlier characterized for general morphisms (PROPOSITION 1.6 in **§ 4.1,** page 139).

PROPOSITION 2.6. *Let V be a vector space over the field* $\mathbf{K}$ *and* $L : V \longrightarrow V$ *a linear endomorphism. Then*

1° L *is injective* $\iff \operatorname{Ker} L = L_{-1}(\mathbf{0}) = \{\mathbf{0}\}$; (2.42)

2° L *is bijective* $\iff [\operatorname{Ker} L = L_{-1}(\mathbf{0}) = \{\mathbf{0}\}$ & $\operatorname{Im} L = LV = V]$. (2.43)

3° L *is bijective* (hence *invertible*) $\iff$ *there exists another endomorphism* $M : V \longrightarrow V$ *such that*

$$M \circ L = L \circ M = \mathrm{id}_V \underset{\text{not}}{=} \mathbf{1}_V. \tag{2.44}$$

Proofs. The proofs of these properties for morphisms, in PROPOSITION 1.6 in **§ 4.1** (pages 139-140), can be easily adapted to operators by appropriate replacements / notational changes. Let us give, however, the proof of property **1°**.

($\Rightarrow$) Let us assume that L is injective but $\operatorname{Ker} L \neq \{\mathbf{0}\}$. It follows from the latter assumption that $(\exists\, \boldsymbol{u} \in \operatorname{Ker} L)\ \boldsymbol{u} \neq \mathbf{0}$. Hence, for any vector $x \in V$, $x + u \neq x$. But

$$Lu = \mathbf{0} \Rightarrow L(x+u) = Lx + Lu = Lx$$

and this contradicts the injectivity of L as defined by (2.36). It follows that $\operatorname{Ker} L = \{\mathbf{0}\}$.

($\Leftarrow$) Conversely, let $\operatorname{Ker} L = \{\mathbf{0}\}$. and assume that L would be not injective. If we take two equal images, $Lx_1 = Lx_2$, it follows from the linearity of L that

$$Lx_1 - Lx_2 = \mathbf{0} \Rightarrow L(x_1 - x_2) = \mathbf{0} \Rightarrow x_1 - x_2 \in \operatorname{Ker} L = \{\mathbf{0}\} \Rightarrow x_1 - x_2 = \mathbf{0} \Rightarrow x_1 = x_2 \Rightarrow$$

$\Rightarrow L$ is injective according to (2.37). As a minor notational detail, we have denoted, in (2.39), the image of $x \in V$ as $\mathbf{id}_V x\,(= x)$ since this identity mapping is here considered as an endomorphism (operator) and not as a morphism, like in *Definition 1.5*. The proofs of the other properties **2°** and **3°** follow from those for PROPOSITION 1.6, as we have just mentioned. ■

As a matter of terminology, a bijective endomorphism is said to be an *automorphism* of space V. The operator M that occurs in $\mathbf{3}^\circ$ is the inverse of L : $M = L^{-1}$. It is involved in the next result - a consequence (or a particular case) of THEOREM 1.1 in **§ 4.1**. But an exact definition of the inverse of an endomorphism should be firstly stated.

***Definition** 2.5.* **(The Inverse of an Operator).** Let V be a vector space over the field $\mathbf{K}$ and let $L : V \longrightarrow V,\ M : V \longrightarrow V$ be two linear endomorphisms. The morphism M is the inverse of L iff

$$(\forall\, x \in V)\ L x = y \in V \Rightarrow\ My = x. \tag{2.45}$$

The usual notation for the inverse of the endomorphism L is $M =_{\text{not}} L^{-1}$. Hence $L^{-1} : V \longrightarrow V$ and the definition of (2.45) becomes

$$(\forall\, x \in V)\ L x = y\ \Rightarrow\ L^{-1} y = x. \tag{2.45'}$$

♢

THEOREM 2.1. *The following properties of linear endomorphisms and automorphisms hold* :

① $\mathrm{Hom}(V, V)$ *has the natural structure of a vector space.*

② *If* L *is an automorphism then* $M = L^{-1}$ *is an automorphism, too.*

③ *If* L *is an automorphism and* $M = L^{-1}$ *is its inverse then*

$$L^{-1} \circ L = \mathbf{1}_V\ \ \&\ \ L \circ L^{-1} = \mathbf{1}_V. \tag{2.46}$$

④ *If* $L : V \longrightarrow V,\ M : V \longrightarrow V$ *are inverse automorphisms, that is* $M = L^{-1}$ *and if* A *is a basis of space* V *then*

$$\boxed{M_A = (L^{-1})_A = L_A^{-1}.} \tag{2.47}$$

Proofs. ① For any $L, M \in \mathrm{Hom}(V, V)$ the two linear operations with these morphisms are naturally defined by

$$(\forall\, x \in V)\ (L + M) x \underset{\text{def}}{=} L x + M x\ \ \&\ \ (\lambda L) x \underset{\text{def}}{=} \lambda L x. \tag{2.48}$$

It is very easy to check that the two operations in (2.48) satisfy the ten axioms of a vector space (*Definition 1.1* in **§ 1.1**.), just like the morphisms in $\mathrm{Hom}(U, V)$. The *zero operator* was defined in *Def. 2.4* - Eq. (2.40) :

$$\mathrm{O}_V : V \longrightarrow V,\ (\forall\, x \in V)\ \mathrm{O}_V\, x = \mathbf{0} \in V \Rightarrow \mathrm{Ker}\ \mathrm{O}_V = V\ \&\ \mathrm{Im}\ \mathrm{O}_V = \{\mathbf{0}\}.$$

The proof of ② follows from the same property for morphisms in THEOREM 1.1 (page 145). It suffices to replace $U \to V,\ f \to L,\ g = f^{-1} \to M = L^{-1}$. However, we offer an effective proof of this important property. $L^{-1} : V \longrightarrow V$ is defined by (2.45') :

$$(\forall\, y \in V)\ L^{-1} y = x \iff L x = y. \tag{2.49}$$

If now $y_1, y_2 \in V$, the surjectivity of L ensures that

$$(\exists\, x_1, x_2 \in V)\ y_1 = L x_1 \ \ \&\ \ y_2 = L x_2 . \tag{2.50}$$

By *Definition 1.1'* in **§ 1.1** of a vector space,

$$(\forall\, \lambda_1, \lambda_2 \in \mathbf{K})(\forall y_1, y_2 \in V)\ \ \lambda_1 y_1 + \lambda_2 y_2 \in V \underset{(2.50)}{\Rightarrow} \lambda_1 L x_1 + \lambda_2 L x_2 \underset{\text{(LIN) of } L}{=}$$

$$\Rightarrow\ L(\lambda_1 x_1 + \lambda_2 x_2) \underset{\text{not}}{=} u \in V.$$

It follows from (2.46) that

$$L^{-1}(u) = \lambda_1 x_1 + \lambda_2 x_2 = \lambda_1 L^{-1}(y_1) + \lambda_2 L^{-1}(y_2).$$

Thus $M = L^{-1}$ is linear, hence it is an endomorphism (a linear operator). This allows to omit the parentheses around its arguments (from now on). The surjectivity of L^{-1} follows its definition in (2.45'). Indeed, let us suppose that there exists (at least) one vector

$$x \in V\colon (\nexists\, y \in V)\ x = L^{-1} y .$$

But any $x \in V$ has an image $y = L x$; hence, in view of (2.49), $(\forall\, x \in V)\ x = L^{-1} y$. Let us now assume that L^{-1} would not be injective, that is

$$(\exists\, y_1, y_2 \in V)\ \ y_1 \neq y_2 \ \text{but}\ \ L^{-1} y_1 = L^{-1} y_2 . \tag{2.51}$$

The surjectivity of L^{-1} implies that $y_1 = L x_1$ & $y_2 = L x_1$ with $x_1 \neq x_2$ since a single vector cannot have two different images through a mapping. But it would follow from (2.51) with the definition in (2.49) of the inverse that $x_1 = x_2$ – an obvious contradiction. As a conclusion, L^{-1} is a bijective endomorphism, hence an *automorphism* of the space V.

Property ③ similarly follows from the corresponding property in THEOREM 1.1 with its proof. The same replacements are appropriate. By the definition of the inverse of an automorphism L - Eqs. (2.45) - (2.45') on the previous page -

$$(\forall\, x \in V)\ (M \circ L) x \underset{(2.41)}{=} M(Lx) = My \underset{(2.45)}{=} x \Rightarrow M \circ L = \mathbf{1}_V \Rightarrow \text{(2.46)} .$$

Properties ④ & ⑤ in THEOREM 1.1 are no more relevant for the endomorphisms / operators.

The proof of ④ follows (together with its proof) from part ⑥ in THEOREM 1.1. The necessary replacements in Eq. (1.103) are

$$F_{A,B} \rightarrow L_A,\ \ G_{B,A} \rightarrow M_A,\ \ (F_{A,B})^{-1} \rightarrow (L_A)^{-1}.$$

Therefore, $M = L^{-1}$ & $G_{B,A} = F_{A,B}^{-1} \Rightarrow M_A = (L_A)^{-1}$.

In fact, property (2.47) also follows without making reference to a property of the general (composite) morphisms. Let us recall property (2.46) in the statement (already proved) :

$$L^{-1} \circ L = \mathbf{1}_V \ \ \&\ \ L \circ L^{-1} = \mathbf{1}_V \Rightarrow L_A \cdot (L^{-1})_A = I_n \Rightarrow \text{(2.47)} .$$

■

The notions of kernel and image, as well the properties of composite and invertible operators, are illustrated in the examples that follow.

Example 2.2. Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear operator with its matrix in the standard basis E_3 given as

$$L_{E_3} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 6 \\ 1 & 4 & 9 \end{bmatrix}. \tag{2.52}$$

It is required to find the kernel and the image of this operator. It is also considered another operator $M : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ whose matrix (in the same standard basis) is

$$M_{E_3} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix}. \tag{2.53}$$

It is required to find $\operatorname{Ker} L$, $\operatorname{Im} L$, $\operatorname{Ker}(M \circ L)$, $\operatorname{Im}(M \circ L)$, the matrix of $M \circ L$, the image of $X = [-1 \;\; 3 \;\; 2]^{\mathrm{T}}$ through L and through $M \circ L$.

In order to find (quicker) the kernels and the images of the three operators it is more convenient to work with the transposes of the matrices in (2.52) and (2.53) :

$$L_{E_3}^{\mathrm{T}} = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 4 \\ 0 & 6 & 9 \end{bmatrix}; \quad M_{E_3}^{\mathrm{T}} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix}. \tag{2.54}$$

$\operatorname{Ker} L = S_1 =$ the set of solutions of the homogeneous system whose matrix is the first matrix in (2.54) :

$$L_{E_3}^{\mathrm{T}} = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 4 \\ 0 & 6 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 \\ 0 & 6 & 9 \\ 0 & 6 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 \\ 0 & 1 & 3/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 3/2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow X(\alpha) = \alpha \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}. \tag{2.55}$$

The image of L can be easily obtained if we write the image a vector $X \in \mathbb{R}^3$ as $Y = LX =$

$$= L_{E_3}^{\mathrm{T}} \cdot X \underset{(2.54-1)}{=} \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 4 \\ 0 & 6 & 9 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_3 \\ -x_1 + 3x_2 + 4x_3 \\ 6x_2 + 9x_3 \end{bmatrix} =$$

$$= x_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}. \tag{2.56}$$

Hence, the image of X through L is spanned by the three (column) vectors that occur in (2.56). However, they are not linearly independent as it follows from the chain of transformations resulting in the kernel of (2.55). The rank of matrix $L_{E_3}^{\mathrm{T}}$ is $= 2,$ hence any two columns among the ones in (2.56) can form a basis for $\operatorname{Im} L,$ for instance the family consisting of $b_1 = [2 \ \ -1 \ \ 0]^{\mathrm{T}}$ and $b_2 = [0 \ \ 1 \ \ 2]^{\mathrm{T}} = 1/3$ of the second column.

The subspace $\operatorname{Ker}(M \circ L)$ can be similarly determined, but the matrix of the composite operator must be found. It is $(M \circ L)_{E_3} = L_{E_3} \cdot M_{E_3},$ according to Eq. (2.42) at page 206, written for a general basis A. However, it is more appropriate to write the *transpose* of this matrix, for an easier determination of $\operatorname{Im}(M \circ L)$:

$$(M \circ L)_{E_3}^{\mathrm{T}} = M_{E_3}^{\mathrm{T}} \cdot L_{E_3}^{\mathrm{T}} \underset{(2.54)}{=} \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 4 \\ 0 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 9 & 15 \\ 4 & 12 & 20 \\ 5 & 15 & 25 \end{bmatrix}. \tag{2.57}$$

This matrix can be transformed by rank-preserving transformations (see **§ 1.2**).

$$\begin{bmatrix} 3 & 9 & 15 \\ 4 & 12 & 20 \\ 5 & 21 & 25 \end{bmatrix} \sim \begin{bmatrix} 3 & 9 & 15 \\ 1 & 3 & 5 \\ 1 & 9 & 5 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 1 & 9 & 5 \end{bmatrix} \Rightarrow \operatorname{Im}(M \circ L) = \mathscr{L}\left(\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} \right). \tag{2.58}$$

Certainly, other choices for the spanning family of $\operatorname{Im}(M \circ L)$ are equally acceptable. The third matrix in the chain of (2.58) also allows to find the kernel :

$$(M \circ L)_{E_3}^{\mathrm{T}} \underset{(2.58)}{\sim} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 1 & 9 & 5 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 5 \end{bmatrix} \Rightarrow \operatorname{Ker} L = \mathscr{L}\left(\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right). \tag{2.59}$$

The matrix of $M \circ L$ is the transpose of the matrix in (2.57) :

$$(M \circ L)_{E_3} = L_{E_3} \cdot M_{E_3} = \ldots = \begin{bmatrix} 3 & 4 & 5 \\ 9 & 12 & 15 \\ 15 & 20 & 25 \end{bmatrix}. \tag{2.60}$$

The images of $X = [-1 \ \ 3 \ \ 2]^{\mathrm{T}}$ through L and through $M \circ L$ can be obtained from the matrices in (2.54-2) and – respectively – (2.57), applying the formula (2.26) at page 201, equivalent to (2.27) at page 202 :

$$L X = L_{E_3}^{\mathrm{T}} \cdot X = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 4 \\ 0 & 6 & 9 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \\ 36 \end{bmatrix}. \tag{2.61}$$

$$(M \circ L) X = (M \circ L)_{E_3}^{\mathrm{T}} \cdot X = \begin{bmatrix} 3 & 9 & 15 \\ 4 & 12 & 20 \\ 5 & 15 & 25 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 54 \\ 72 \\ 90 \end{bmatrix}. \tag{2.62}$$

This image $(M \circ L)X$ can be checked by applying the definition of a composite morphism, that is by taking the image of $Y = LX$ of (2.61) through M:

$$(M \circ L)X \underset{\text{def}}{=} M(LX) \underset{(2.26)}{=} M_{E_3}^{\mathrm{T}} \cdot LX \underset{(2.54-2)+(2.62)}{=} \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 18 \\ 36 \end{bmatrix} = \begin{bmatrix} 54 \\ 72 \\ 90 \end{bmatrix}.$$

Hence, the image of (2.62) has been retrieved. □

Other notions and properties, presented for general morphisms, have to be transferred to the endomorphisms (operators).

***Definition* 2.6. (Rank & Nullity of an Endomorphism).** Let V be a vector space over the field $\mathbf{K} (= \mathbb{R} / = \mathbb{C})$ and $L : V \longrightarrow V$ a linear endomorphism. Then the *rank* of L is defined as the rank of its matrix L_A in any basis A of space V. The *nullity* of L is the dimension of its kernel.

Hence, if the matrix L_A is defined as in PROPOSITION 2.3 - Eq. (1.9), that is $LA^{\mathrm{T}} = L_A \cdot A^{\mathrm{T}}$, then

$$\boxed{\operatorname{rank} L \underset{\text{def}}{=} \operatorname{rank} L_A, \quad \operatorname{null} L \underset{\text{def}}{\equiv} \dim \operatorname{Ker} L.} \tag{2.63}$$

It would follow, from (2.63), that this notion of rank would be dependent on basis A. However, we shall see – a little later – how the change of bases affects the matrix of an operator, but not its rank. As regards the nullity, it does not explicitly depend on any basis, it equals the dimension of a subspace and we recall from that $\dim W \; (W \subseteq_{\text{subsp}} V)$, as well as the dimension of the entire space, do not depend on any basis.

Like for the morphisms, there exists a connection between the rank of an operator, its kernel and its image. But let us firstly notice that

$$\dim V = n \;\Rightarrow\; L_A \in \mathcal{M}_n(\mathbb{R}) \;\Rightarrow\; \operatorname{rank} L \underset{\text{not}}{=} r \le n. \tag{2.64}$$

Certainly, this inequality also holds when the two (finite-dimensional) vector spaces and the endomorphism are considered on a more general field, $\mathbf{K}$ instead of $\mathbb{R}$. The result corresponding to PROPOSITION 1.5 in the previous Section is

PROPOSITION 2.7. *Let V be a vector space over the field $\mathbf{K} (= \mathbb{R} / = \mathbb{C})$ and $L : V \longrightarrow V$ a linear operator. If* $\dim V = n$ and $\operatorname{rank} L = r$ *then*

$$\boxed{\dim \operatorname{Ker} L = \operatorname{null} L = n - r} \quad \& \quad \boxed{\dim \operatorname{Im} L = r.} \tag{2.65}$$

Proof. The proof is similar to (in fact, it just follows from) the proof of **P. 1.5** at page 134. For the dimension of the *kernel*, let us see that the coordinates of a vector in the kernel should satisfy the homogeneous system resulting from the analytic expression of Lx – Eq. (2.15) at page 200 :

$$L\,x = X_A^{\mathrm{T}} \cdot L_A \cdot A^{\mathrm{T}} \iff L\,x = A \cdot L_A^{\mathrm{T}} \cdot X_A \,. \tag{2.66}$$

(2.32) & (2.66) $\Rightarrow$

$$\Rightarrow X_A^{\mathrm{T}} \cdot L_A \cdot A^{\mathrm{T}} = [0 \;\ldots\; 0]\,A^{\mathrm{T}} \;\Rightarrow\; L_A^{\mathrm{T}} \cdot X_A = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n \tag{2.67}$$

since, as mentioned in **§ 1.1** , the coordinates of the zero vector in any basis are just the components of $\mathbf{0} \in \mathbb{R}^n / \in \mathbf{K}^n$. Therefore, the kernel of L is the set of vectors whose coordinates X_A should satisfy the homogeneous system (2.67). More precisely,

$$\operatorname{Ker} L = \{x \in V : \; x = A\,X_A \;\;\&\;\; L_A^{\mathrm{T}} \cdot X_A = \mathbf{0} \in \mathbb{R}^n / \in \mathbf{K}^n\}\,. \tag{2.68}$$

Thus, the kernel equals the solution subspace S of the homogeneous system (2.67) whose matrix is L_A^{T} with $\operatorname{rank} L_A^{\mathrm{T}} = \operatorname{rank} L_A = r$. It is known from the theory of the homogeneous systems (P. 2.9 in **§ 1.2**) that S is a subspace whose dimension is $n - r$. This implies the first equation in (2.65).

As regards the *image*, the same analytic expression (2.15), recalled in (2.66), implies that the coordinates of $y = L\,x$ in basis A result from

$$y = A\,Y_A = Y_A^{\mathrm{T}} A^{\mathrm{T}} \in \operatorname{Im} L \iff Y_A^{\mathrm{T}} = X_A^{\mathrm{T}} \cdot L_A \iff Y_A = L_A^{\mathrm{T}} \cdot X_A \,. \tag{2.69}$$

From the theory of non-homogeneous systems in **§ 1.2** , Eq. (2.69) means that the vectors of the coordinates of $y = L\,x$ in basis A is a linear combination of the columns of matrix L_A^{T}. But $\operatorname{rank} L_A^{\mathrm{T}} = \operatorname{rank} L_A = r$ implies that *only* r *columns of* L_A^{T} *are linearly independent* and can form a basis for the subspace $\mathrm{COLSP}_{L_A^{\mathrm{T}}}$. This implies the second equation in (2.65).

The proof is over, but let us still see that the kernel and image of an operator of the form $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ or $L : \mathbf{K}^n \longrightarrow \mathbf{K}^n$, with its analytic expression (Eq. (2.27) at page 169), that is

$$L\,X = M \cdot X \quad \text{with} \quad M = L_E^{\mathrm{T}} \iff L_E = M^{\mathrm{T}}, \tag{2.70}$$

can be easily determined by using just this matrix, together with their dimensions $\operatorname{null} L$ & $\operatorname{rank} L$. ■

Example 2.2'. With reference to the latest **Example 2.2**, we can see that

$$\operatorname{null} L = 1 \;\;\&\;\; \operatorname{rank} L = \dim \operatorname{Im} L = 2. \qquad \square$$

Remarks 2.2. If V is a finitely generated vector space, it is clear that both the rank and the nullity of an operator L of space V are natural (non-negative integer) numbers.

If $\dim V = n \in \mathbb{N} \cup \{0\}$ then $\operatorname{rank} L \;\&\; \operatorname{null} L \in \mathbb{N} \cup \{0\}$; more precisely,

$$\operatorname{rank} L \underset{\text{def}}{=} \operatorname{rank} L_A \underset{\text{not}}{=} r : \; L\,A^{\mathrm{T}} = L_A \cdot A^{\mathrm{T}} \Rightarrow 0 \le r \le n\,; \tag{2.71}$$

$$\operatorname{null} L \underset{\text{def}}{=} \dim \operatorname{Ker} L = \dim \{x \in V : Lx = \mathbf{0}\} = n - r \;\Rightarrow\; 0 \le n - r \le n. \tag{2.72}$$

Both inequalities in (2.71) & (2.72) follow from the definition of the rank of a square matrix of order n. The trivial case $\operatorname{Ker} L = V$ obviously holds for $L = \mathrm{O}_V =$ the zero operator. Its matrix, in any basis A of V, is the zero matrix : $(\forall\, A : V = \mathcal{L}(A))\; L_A = \mathbf{0} \in \mathcal{M}_{n,n}(\mathbb{R})$. Indeed, the zero operator O_V maps any vector $x \in V$ onto $\mathbf{0}$, hence

$$\mathrm{O}_V A^{\mathrm{T}} = \mathrm{O}_V [a_1\; a_2\; \ldots\; a_n]^{\mathrm{T}} = [\mathbf{0}\;\; \mathbf{0}\; \ldots\; \mathbf{0}]^{\mathrm{T}} \;\Rightarrow\; (\mathrm{O}_V)_A = \mathbf{0} = [0]_{n,n}.$$

The inequality in (2.72) also follows from

$$\operatorname{Ker} L \subseteq_{\text{subsp}} V \;\Rightarrow\; \dim \operatorname{Ker} L \le \dim V = n.$$

The trivial case $\operatorname{null} L = 0$ obviously holds $\Longleftrightarrow$ $\operatorname{Ker} L = \{\mathbf{0}\}$ $\Longleftrightarrow$ L is injective. The term of *nullity* comes from the *nullspace* (of a matrix / of a morphism) as an equivalent or synonim to *kernel*.

Our previous remark that the rank of an endomorphism / operator seems to be basis-dependent although it does not actually depend on the basis in which its matrix is L_A will be turned clear in what follows : how the matrix of an operator changes when the basis is changed.

* * * *

The Matrix of a Linear Operator after a Change of Basis

An important problem we have to discuss in this section regards the just mentioned problem: the effect of a change of basis on the matrix of an operator. Before giving a technical presentation, let us mention that changes of bases can help the matrix L_A *to get a simpler form*, for example a diagonal form. The corresponding problem for morphisms between two (possibly different) linear spaces was clarified by PROPOSITION 1.5 in **§ 4.1**. The corresponding result for $U = V$ is

PROPOSITION 2.8. (**Changing the Basis and Matrix of an Operator**). *Let V be a vector space over the field $\mathbf{K}$ and $L : V \longrightarrow V$ an operator whose matrix in a basis A of V is L_A. If the basis A is changed to B with the transformation matrix T, that is $B^{\mathrm{T}} = T \cdot A^{\mathrm{T}}$, then*

$$\boxed{L_B = T \cdot L_A \cdot T^{-1}.} \tag{2.73}$$

Proof. This formula can be obtained by particularizing the similar (and more general) formula for general morphisms, that is Eq. (1.97) in PROPOSITION 1.12 of **§ 4.1** (page 146-147), with adequate replacements. However, a direct proof is easy to present ; it can illustrate the advantage of using our matrix notations. The essential property of the operators here involved is the *extended linearity*, that is PROPOSITION 2.1 in this section. Let us replace, in Eq. (2.5'), the row of scalars Λ^{T} by the i-th row T_i of the transformation matrix T and the column of vectors $\mathcal{X}^{\mathrm{T}}$ by A^{T}. We thus get

$$L(T_i \cdot A^{\mathrm{T}}) \underset{(2.5')}{=} T_i \cdot L A^{\mathrm{T}} \underset{(2.13)}{=} T_i \cdot L_A \cdot A^{\mathrm{T}}. \tag{2.74}$$

Equations (2.74) can be written one under the other for $i = 1, 2, \ldots, n$ resulting in the matrix equation $L(T \cdot A^{\mathrm{T}}) = T \cdot L A^{\mathrm{T}} = T \cdot L_A \cdot A^{\mathrm{T}}$ which gives – taking into account the connection between the two bases –

$$L B^{\mathrm{T}} = L(T \cdot A^{\mathrm{T}}) = T \cdot L_A \cdot A^{\mathrm{T}}. \tag{2.75}$$

But the same image of basis B^{T} through L is also obtained according to the formula (2.10) in PROPOSITION 2.2 (at page 165) :

$$L B^{\mathrm{T}} = L_B \cdot B^{\mathrm{T}} = L_B \cdot T \cdot A^{\mathrm{T}}. \tag{2.76}$$

Equations (2.74) & (2.74) give two expression of basis B (written as the column B^{T}) in the "initial" basis A^{T}. More exactly, the leftmost sides in Eqs. (2.75) & (2.76) consist of the images $L b_i \; (i = \overline{1,n})$ written one under the other, as the components of the vector $L B^{\mathrm{T}}$. The uniqueness of the coordinates of a vector in any basis implies that

$$L B^{\mathrm{T}} = T \cdot L_A \cdot A^{\mathrm{T}} = L_B \cdot T \cdot A^{\mathrm{T}} \;\Rightarrow\; T \cdot L_A = L_B \cdot T. \tag{2.77}$$

We proved, in **§ 1.1** , that the transformation matrix from a basis to another *is nonsingular*, hence *it is invertible*. Hence the last equation in (2.77) can be turned into an equivalent by post-multiplying its sides with T^{-1} :

$$(T \cdot L_A)\, T^{-1} = (L_B \cdot T)\, T^{-1} \;\Rightarrow\; T \cdot L_A \cdot T^{-1} = L_B \cdot (T \cdot T^{-1}) = L_B \cdot I_n = L_B. \tag{2.78}$$

The associativity of the matrix product has been involved in this (multiple) equation, after $\Rightarrow$. By reading Eq. (2.78) – rightmost side = leftmost side, Eq. (2.73) in the statement is thus proved.

■

Remarks 2.3. (i) The formula (2.73) that gives the matrix of an endomorphism after a change of basis is rather similar to the relation of similarity between two matrices. We considered it in connection with the diagonalization of quadratic form ; we recall *Definition 3.9.* at page 129 - Eq. (3.128) :

If $A, B \in \mathscr{M}_n(\mathbb{R})$ are two matrices, then they are said to be *similar* if

$$(\exists\, S \in \mathscr{M}_n(\mathbb{R})) : \det S \neq 0 \;\;\&\;\; B = S^{-1} A S. \tag{2.79}$$

If we compare equations (2.73) with (2.79), it is clear that the matrices L_A & L_B are similar, with the similarity matrix $S = T^{-1}$ because – for any non-singular (hence invertible) square matrix A – $(A^{-1})^{-1} = A$.

(ii) If the operator L is considered on the Euclidean space $\mathbb{R}^n$ (or $\mathbf{K}^n$), it is usually defined in the standard basis $E_n \underset{\text{not}}{=} E$ or (in many applications) by its image, the vector $Y = L X$ whose components are linear forms of the components of X. If the operator is given by its matrix in the standard basis, Eq. (2.18) at page 167 gives the connection $E \to L\,E$; if L is given by its image $Y = L\,X \in \mathbb{R}^n$, formulas (2.22) at page 168 and (2.26) - (2.27) at page 168-169 are applicable. We recall them below :

$$L\,X = X^{\mathrm{T}} \cdot L_E \cdot E^{\mathrm{T}}, \;\; L\,X = M \cdot X \;\text{ with }\; M = L_E^{\mathrm{T}} \iff L_E = M^{\mathrm{T}}. \tag{2.80}$$

If the matrix of the operator in another basis B has to be determined, the formula (2.73) is applicable, with the transformation matrix T replaced by $T = B^{\mathrm{T}}$. This issue was discussed in **§ 2.1** , in connection with the change of coordinates $X_E = X$ in another basis B. Thus, formula in (2.73) becomes

$$\boxed{L_B = B^{\mathrm{T}} \cdot L_E \cdot B^{-\mathrm{T}}.} \tag{2.81}$$

But we firstly offer on example illustrating the formula (2.73), for a basis change $A \to B$.

Example 2.3. Let $L : V \longrightarrow V$ $(\dim V = 3)$ be a linear operator with its matrix in the basis A

$$L_A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 0 \end{bmatrix}. \tag{2.82}$$

It is required to determine the matrix L_B in basis B which is obtained from A with the transformation matrix

$$T = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}. \tag{2.83}$$

It is also required to express $L(3\,a_2)$ in both bases A & B.

The inverse of matrix T of (2.82) is

$$T^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 0 & -3 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}. \tag{2.84}$$

Eq. (2.73) with the matrices in (2.82) & (2.84) gives

$$L_B = \frac{1}{2} \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & -3 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} =$$

$$= \frac{1}{2} \begin{bmatrix} -2 & 2 & -4 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & -3 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -2 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & -3 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & -4 & 0 \\ -2 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.85}$$

The image Lx of the vector $x = 3a_2 \Rightarrow X_A = [0\ \ 3\ \ 0]^{\mathrm{T}}$ can be obtained by formula (2.15) at page 166, but is easier to see, from operator's matrix in (2.82), that

$$La_2 = -a_1 + a_2 - 2a_3 \Rightarrow Lx = L(3a_2) = -3a_1 + 3a_2 - 6a_3\,. \tag{2.86}$$

The linear expression of the same vector Lx in basis B can be found in two ways. If we denote (as usually) $Lx = y$, the coordinates $Y_A = [-3\ \ 3\ \ -6]^T$ follow from (2.86). They are turned by the basis transformation according to formula (1.78) of **§ 2.1** (page 22) :

$$Y_B = (T^{\mathrm{T}})^{-1}\,Y_A \underset{\text{not}}{=} T^{-\mathrm{T}}\,Y_A = \tfrac{1}{2}\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ -3 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 3 \\ -6 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 6 \\ -12 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 0 \end{bmatrix}. \tag{2.87}$$

The second way consists in using the matrix L_B of (2.85), but this requires the coordinates of x in basis B to be previously found. They can be determined using (again) formula (1.78) in PROPOSITION 1.6 and the matrix $T^{-\mathrm{T}}$, just involved in obtaining the coordinates in (2.87) :

$$X_B = T^{-\mathrm{T}}X_A = \tfrac{1}{2}\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ -3 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}. \tag{2.88}$$

It can now be applied the formula giving the image $Lx = y$ with formula (2.15) in PROPOSITION 2.3 at page 166, written for basis B :

$$Lx = X_B^{\mathrm{T}} \cdot L_B \cdot B^{\mathrm{T}} = \tfrac{1}{2}[3\ \ 0\ \ 3] \cdot \begin{bmatrix} 2 & -4 & 0 \\ -2 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} \cdot B^{\mathrm{T}} = \ldots = [3\ \ -6\ \ 0] \cdot B^{\mathrm{T}} \Rightarrow$$

$$\Rightarrow Lx = 3b_1 - 6b_2 \Rightarrow Y_B = \begin{bmatrix} 3 \\ -6 \\ 0 \end{bmatrix} \tag{2.89}$$

and the coordinates in (2.87) have been retrieved.

This example can be extended since it matrix L_B of (2.87) is obviously singular, with

$$\operatorname{rank} L_B = 2 \Rightarrow \operatorname{Im} L \neq \mathbb{R}^3 \ \&\ \operatorname{Ker} L \neq \{\mathbf{0}\}.$$

The two subspaces can be easier obtained using just this matrix L_B (instead of L_A) : the coordinates X_B / X_B^{T} of a vector $x \in \operatorname{Ker} L$ should satisfy a homogeneous system of matrix

$$L_B^{\mathrm{T}} \underset{(2.85)}{=} \begin{bmatrix} 2 & -2 & 0 \\ -4 & 0 & 0 \\ 0 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow X_B^{\mathrm{T}} = \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix}. \tag{2.90}$$

As regards the image, the coordinates $Y_B \,/\, Y_B^{\mathrm{T}}$ of a vector $y = Lx \in \operatorname{Im} L$ can be found from

$$L_B^{\mathrm{T}} = M_B = \begin{bmatrix} 2 & -2 & 0 \\ -4 & 0 & 0 \\ 0 & -4 & 0 \end{bmatrix} \underset{(2.89)}{\Rightarrow} y = Lx = \begin{bmatrix} x_1 & & \\ & x_2 & \\ 0 & & \end{bmatrix} \cdot B^{\mathrm{T}} \Rightarrow y = \beta b_1 + \gamma b_2 . \tag{2.91}$$

Let us close this example with the remark that the expression of $y = Lx$ in (2.91) is consistent with the one in (2.89) for $\beta = 3$ & $\gamma = -6$. The property of the dimensions of the two subspaces, that is Eq. (2.65) in PROPOSITION 2.7 (page 177), is also verified :

$$\operatorname{rank} L = \dim \operatorname{Im} L = 2 \ \ \& \ \ \operatorname{null} L = \dim \operatorname{Ker} L = 1 . \qquad \square$$

Example 2.4. Let the operator $L : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be defined by the image of the standard basis $E_4 =_{\text{not}} E$.

$$LE = \begin{cases} L e_1 = e_2 , \\ L e_2 = 2 e_1 + e_3 , \\ L e_3 = e_4 , \\ L e_4 = -e_1 . \end{cases} \tag{2.92}$$

It is required to write the matrix L_E of L in the standard basis, to write its explicit expression $LX = M \cdot X$, to find the matrix L_A in the basis A defined by

$$A = [a_1 \ \ a_2 \ \ a_3 \ \ a_4] = \begin{bmatrix} 3 & 1 & 3 & -1 \\ 2 & 1 & -2 & 1 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \tag{2.93}$$

It is also required to find the image of $X_0 = [2 \ \ 0 \ \ 0 \ \ -3]^{\mathrm{T}}$ as the vector $Y = LX \in \mathbb{R}^4$ and its linear expression in the basis A, as well as the counter-image of $V_0 = [4 \ \ 1 \ \ -4 \ \ -1]^{\mathrm{T}}$.

The matrix L_E immediately follows from the definition (2.92) of L :

$$L_E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{2.94}$$

$$(2.94) \Rightarrow \quad M = L_E^{\mathrm{T}} = \begin{bmatrix} 0 & 2 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow Y = LX = M \cdot X = \begin{bmatrix} 2x_2 - x_4 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{2.95}$$

The matrix L_A can be found by replacing the images Le_j $(j = \overline{1,4})$ of (2.92) into $La_i = L(\sum \lambda_j e_j)$ $(i = \overline{1,4})$ of (2.93) and using the extended linearity of L. However, the matrix approach is more convenient. The transformation matrix from E to A follows from (2.93) :

$$T = A^{\mathrm{T}} = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}. \tag{2.96}$$

The matrix T of (2.96) must now be inverted, and the Gaussian transformation method (see § **1.1**) is most efficient :

$$[T|I_4] = \left[\begin{array}{cccc|cccc} 3 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 3 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 & 0 & -1 \\ 3 & -2 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cccc|cccc} 0 & 4 & 0 & 0 & 1 & 0 & -1 & 0 \\ -4 & 4 & 0 & 0 & 0 & 1 & -2 & -1 \\ 3 & -2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} -4 & 0 & 0 & 0 & -1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1/4 & 0 & -1/4 & 0 \\ 3 & -2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/4 & -1/4 & 1/4 & 1/4 \\ 0 & 1 & 0 & 0 & 1/4 & 0 & -1/4 & 0 \\ 3 & 0 & 1 & 0 & 2/4 & 0 & 2/4 & 0 \\ 2 & 0 & 0 & 1 & 1/4 & 0 & 3/4 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/4 & -1/4 & 1/4 & 1/4 \\ 0 & 1 & 0 & 0 & 1/4 & 0 & -1/4 & 0 \\ 0 & 0 & 1 & 0 & -1/4 & 3/4 & -1/4 & -3/4 \\ 0 & 0 & 0 & 1 & -1/4 & 2/4 & 1/4 & 2/4 \end{array}\right] \Rightarrow$$

$$\Rightarrow T^{-1} = A^{-\mathrm{T}} = \begin{bmatrix} 1/4 & -1/4 & 1/4 & 1/4 \\ 1/4 & 0 & -1/4 & 0 \\ -1/4 & 3/4 & -1/4 & -3/4 \\ -1/4 & 2/4 & 1/4 & 2/4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -3 \\ -1 & 2 & 1 & 2 \end{bmatrix}. \tag{2.97}$$

The formula giving the matrix L_A can now be obtained with formula (2.81) at page 181, taking $B \rightarrow A$ and using the (just obtained) matrices of (2.96), (2.94), (2.97) :

$$L_A = T \cdot L_E \cdot T^{-1} = A^{\mathrm{T}} \cdot L_E \cdot A^{-\mathrm{T}} =$$

$$= \begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \cdot \frac{1}{4} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -3 \\ -1 & 2 & 1 & 2 \end{bmatrix} = \ldots =$$

$$= \frac{1}{4} \begin{bmatrix} 4 & 4 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{2.98}$$

Note 1. The reader is invited to check, in detail, the calculations that have led to the matrix in (2.98). As regards the transformations that produced the inverse T^{-1} in (2.97), they can also be identified. For instance, the first two transformations that have yielded the column vectors e_4 and then e_3 in the left block were (respectively)

$$(T_1): R_2 \to R_2 - R_4 \;\&\; (T_2): R_1 \to R_1 - R_3,\ R_2 \to R_2 - 2R_3,\ R_4 \to R_4 + R_3.$$

The image of the vector $X_0 = [2 \;\; 0 \;\; 0 \;\; -3]^{\mathrm{T}}$, it can be obtained from (2.95) by replacing its components :

$$Y_0 = L X_0 = M \cdot X_0 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}. \tag{2.99}$$

The counterimage U_0 of $V_0 = [4 \;\; 1 \;\; -4 \;\; -1]^{\mathrm{T}}$ will be the solution to the vector equation

$$L X = V_0 \iff M \cdot X = V_0. \tag{2.100}$$

This matrix equation is, in its turn, equivalent to a non-homogeneous system whose augmented matrix is

$$[M \,|\, V_0] = \left[\begin{array}{cccc|c} 0 & 2 & 0 & -1 & 4 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 2 & 0 & -1 & 4 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -12 \end{array}\right] \Rightarrow U_0 = \begin{bmatrix} 1 \\ -4 \\ -1 \\ -12 \end{bmatrix}. \tag{2.101}$$

Note 2. It is clear that the matrix L_E in (2.94) and – of course – its transpose M in (2.95) are nonsingular ; this implies that the endomorphism is bijective, hence it is an *automorphism*. Hence, regarding the result in (2.101), we may write it as $U_0 = L^{-1} V_0$ and not $U_0 = L_{-1}(V_0)$.

Note 3. Another suggestion to the reader consists in the verification of the image in (2.99) by use of basis A. This makes necessary to find the coordinates of $X_0 = [2 \;\; 0 \;\; 0 \;\; -3]^{\mathrm{T}}$ in this basis, and then to use the matrix L_A for finding the coordinates of $Y_0 = LX_0$ in basis A, with formula (2.15) at page 200. Finally, the coordinates of $Y_0 = LX_0$ in the standard basis, hence the vector of (2.99), would have to be retrieved from $Y_0 = A^{\mathrm{T}} \cdot (Y_0)_A$.

Note 4. This exercise was found in the textbook [Gh. Procopiuc *et al*, 2001 - page 44], where only the matrix L_A was required. Let us mention a couple of notational differences : L is there denoted T, Lx is $T(x)$, the initial and the transformed bases are (respectively) $\mathcal{B} = (e_1, e_2, e_3, e_4)$ and $\mathcal{B}' = (e'_1, e'_2, e'_3, e'_4)$, the transformation matrix is there $C =$ our T^{T}, and the vectors in $\mathbb{R}^4$ are written as ordered 4-tuples (x_1, x_2, x_3, x_4) and not as column vectors. Finally, the transpose of our matrix L_A is there A'. These technical details of notation could help the possible reader of that textbook, very rich in exercises, to approach other problems and to relate them to our notations. □

Example 2.5. Let V be a real vector space of dimension 3, and let $A = [a_1 \;\; a_2 \;\; a_3]$ be one of its bases. Consider the following two bases whose vectors are expressed in those of A :

$$B:\; b_1 = a_1 + a_2,\; b_2 = 2a_1 + a_2 + a_3,\; b_3 = -2a_2 + a_3\,; \tag{2.102}$$

$$C:\; c_1 = -a_1 + a_3,\; c_2 = a_1 - 2a_2 - 3a_3,\; c_3 = a_1 + a_2 + a_3\,. \tag{2.103}$$

The transformations of bases $A \to B$ & $A \to C$ induce two linear endomorphisms. Let us denote the matrices corresponding to expression in (2.102) & (2.103) by $M_{A \to B}$ & $M_{A \to C}$, relating the pairs of bases by

$$B^{\mathrm{T}} = M_{A \to B} \cdot A^{\mathrm{T}},\quad C^{\mathrm{T}} = M_{A \to C} \cdot A^{\mathrm{T}}. \tag{2.104}$$

Since we use to denote the "initial" basis by A, we invert the notations for the vectors of bases in (1.102) and (1.103).

$$A:\; a_1 = b_1 + b_2,\; a_2 = 2b_1 + b_2 + b_3,\; a_3 = -2b_2 + b_3\,; \tag{2.105}$$

$$A:\; a_1 = -c_1 + c_3,\; a_2 = c_1 - 2c_2 - 3c_3,\; a_3 = c_1 + c_2 + c_3\,. \tag{2.105'}$$

This means that basis B, respectively C, is (are) transformed into basis A by changes

$$B \to A \;\;\&\;\; C \to A$$

of the form

$$A^{\mathrm{T}} = M_{B \to A} \cdot B^{\mathrm{T}},\quad A^{\mathrm{T}} = M_{C \to A} \cdot C^{\mathrm{T}}. \tag{2.106}$$

(i) It is required to write these matrices that occur in (2.106).

(ii) It is also required to find the coordinates of $x = 2a_1 - 4a_2 + a_3$ in basis B.

In what follows we deal with the first transformation, that is (2.105) and the first equation in (1.106). The transformation $C \to A$ remains to be discussed by the reader, as an exercise.

If a vector $x \in V$ is expressed in basis A, $x = A \cdot X_A = X_A^{\mathrm{T}} \cdot A^{\mathrm{T}}$, then

$$x = X_A^{\mathrm{T}} \cdot A^{\mathrm{T}} \underset{(2.106)}{=} X_A^{\mathrm{T}} \cdot M_{B \rightarrow A} \cdot B^{\mathrm{T}}. \tag{2.107}$$

It follows from Eq. (2.107), compared with Eqs. (1.78) - (1.79) at page 22 (PROPOSITION 1.6 in Chapter 1),

$$X_B = T^{-\mathrm{T}} \cdot X_A \iff X_B^{\mathrm{T}} = X_A^{\mathrm{T}} \cdot T^{-1},$$

resulting from $x = X_A^{\mathrm{T}} \cdot A^{\mathrm{T}} = X_B^{\mathrm{T}} \cdot B^{\mathrm{T}}$ & $B^{\mathrm{T}} = T A^{\mathrm{T}}$, that

$$X_A^{\mathrm{T}} \cdot M_{B \rightarrow A} = X_A^{\mathrm{T}} \cdot T^{-1} \Rightarrow M_{B \rightarrow A} = T^{-1} \Rightarrow T^{-\mathrm{T}} = M_{B \rightarrow A}^{\mathrm{T}}. \tag{2.108}$$

Hence the matrix that turns the coordinates $X_A \rightarrow X_B$ is just the transpose of the matrix of the basis change in (2.105-1). It is - therefore - justified *to consider a change of coordinates, induced by a change of basis, as an endomorphism* : it does not map a vector $x \in V$ onto another vector $y \in V$; instead, it changes the coordinates of the same vector $X_A \rightarrow X_B$. Let us denote it as

$$L_{A \rightarrow B} : \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ L_{A \rightarrow B}\, X_A = X_B. \tag{2.109}$$

(i) The required matrices are

$$M_{B \rightarrow A} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix},\ M_{C \rightarrow A} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & -3 \\ 1 & 1 & 1 \end{bmatrix}. \tag{2.110}$$

(ii) $X_A = [2 \ \ -4 \ \ 1]^{\mathrm{T}} \underset{(2.109)}{\Rightarrow} X_B = L_{A \rightarrow B}\, X_A = M_{B \rightarrow A}^{\mathrm{T}} \cdot X_A \underset{(2.110)}{=}$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ -5 \end{bmatrix} \Rightarrow x = -6 b_1 - 5 b_3. \tag{2.111}$$

This expression in (2.111) can be checked, with (2.108), by calculating $T^{-\mathrm{T}} \cdot X_A$ as the solution to the matrix equation $T^{\mathrm{T}} \cdot X_B = X_A$: $T^{\mathrm{T}} = M_{B \rightarrow A}^{-\mathrm{T}}$ and we have, with the matrix that occurs in (2.111),

$$[M_{B \rightarrow A}^{\mathrm{T}} \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & -1 & 1 & 0 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \Rightarrow$$

$$T^{\mathrm{T}} = M_{B \to A}^{-\mathrm{T}} = \begin{bmatrix} 3 & -2 & -4 \\ -1 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix} \Rightarrow T^{\mathrm{T}} \cdot X_B = \begin{bmatrix} 3 & -2 & -4 \\ -1 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}.$$

Thus, the coordinates of x in (ii) have been retrieved.

Comments. The result in (2.108) could be anticipated since a change of basis like (2.105) is just converse to our usual changes of basis, $A \to B$, from the initial basis to a new one. We found the changes in (2.102) & (2.103) as Example 3 at page 170 in the monograph [E. Sernesi, 1993]. But they were there considered for illustrating how two changes of bases can be composed, and how such a change can be inverted. □

Powers of Operators, Special Operators

As presented in the previous section, two morphisms $f: U \longrightarrow V$ & $g: V \longrightarrow W$ can be composed and the composite mapping $h = g \circ f$, defined in **§ 4.1** - *Def. 1.5* at page 140, Eq. (1.82), is itself a linear morphism.

Its matrix in the pair of bases (A, C), for $U = \mathcal{L}(A)$, $V = \mathcal{L}(B)$, $W = \mathcal{L}(A)$, was derived and presented in PROPOSITION 1.7 at pages 142 - Eq. (1.77).

$$H_{A,C} = F_{A,B}\, G_{B,C} \underset{\text{not}}{\Longleftrightarrow} H_{A,C} = F_{A,B} \cdot G_{B,C}. \tag{2.112}$$

Such a composition of morphisms was possible provided the space on which g was defined was the same with the one where f took its values. This condition is naturally satisfied by any two operators of space V,

$$L_1 : V \longrightarrow V \ \& \ L_2 : V \longrightarrow V. \tag{2.113}$$

The composition of two operators L, M was presented in *Def. 2.3* at page 206 but we restate it, with slight changes in notations ; the notation M for the "second" operator could be ambiguous with the notation $LX = M \cdot X$ for $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. We also consider the composite of an operator by (with) itself, and powers of an operator. The formula (2.41) takes the form in

***Definition* 2.7 (Composite Operators).** Let V be a vector space over the field $\mathbf{K}$ and L_1, L_2 the operators of (2.113). Then

$$L_2 \circ L_1 : V \longrightarrow V \ (\forall\, x \in V) \ (L_2 \circ L_1) x \underset{\text{def}}{=} L_2 (L_1 x). \tag{2.114}$$

Certainly, the two endomorphisms can be composed in the reverse order, too :

$$L_1 \circ L_2 : V \longrightarrow V \ (\forall\, x \in V) \ (L_1 \circ L_2) x \underset{\text{def}}{=} L_1 (L_2 x). \tag{2.114'}$$

If $L : V \longrightarrow V$ is an endomorphism then its second power is defined by

$$L \circ L \underset{\text{not}}{=} L^2 : (\forall\, x \in V) \ L^2 x = (L \circ L) x \underset{(2.114)}{=} L(Lx). \tag{2.115}$$

The m-th power of L is (inductively) defined by

$$L^m : V \longrightarrow V, \ (\forall\, x \in V) \ L^m x = (L \circ L^{m-1}) x = L(L^{m-1} x). \tag{2.116}$$

♢

Remarks 2.4. **(1)** The formula (2.112), adapted to the composition of two operators - Eq. (2.114) - gives the matrix of a composite operator in a basis of the space. If the space V is spanned by a basis A and the matrices of the two operators in this basis are $(L_1)_A$, $(L_2)_A$ then the matrix of the composite operator in (2.114) follows from the just recalled formula (2.112) :

$$(L_1 \circ L_2)_A = (L_1)_A \cdot (L_2)_A . \tag{2.117}$$

(2) The definition of the square of an operator - *Def. 2.7* - Eq. (2.115) and the definition of its m-th power - Eq. (2.116), combined with the definition of the matrix L_A in a basis A (formula (2.10) in PROPOSITION 2.2 at page 165) and the formula for the matrix of composite operators - Eq. (2.112) at page 188, allow to see that the matrix of the square of $L : V \longrightarrow V$ is just the square of L_A, and similarly for higher powers of L. Indeed, if the basis A spans space V then

$$L\,A \underset{(2.10)}{=} L_A \cdot A^{\mathrm{T}} \Rightarrow L^2 A \underset{(2.115)}{=} L(L\,A) = L(L_A \cdot A^{\mathrm{T}}) = (L_A \cdot L_A) \cdot A^{\mathrm{T}} = L_A^2 \cdot A^{\mathrm{T}}. \tag{2.118}$$

In deriving this formula, the extended linearity of any linear morphism was (simultaneosly) applied with the rows of L_A in $L(L_A \cdot A^{\mathrm{T}}) = \ldots$ This formula (2.118) can be inductively extended for higher powers :

$$\Rightarrow L^m A^{\mathrm{T}} \underset{(2.116)}{=} L(L^{m-1} A^{\mathrm{T}}) = \ldots = L_A^m \cdot A^{\mathrm{T}}. \tag{2.119}$$

As we earlier mentioned, if operators of the form $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are considered (mainly in exercises), the most usual way to introduce such an operator is the one presented by Eq. (2.28) at page 203 :

$$Y = L\,X = M \cdot X, \quad \text{with} \quad M = L_E^{\mathrm{T}}. \tag{2.119}$$

In this case, the formula (2.119) becomes

$$L^k X = M^k \cdot X. \tag{2.120}$$

(3) It can be expected that the operation of raising to powers an endomorphism has an effect on the two subspaces associated to it, $\mathrm{Ker}\, L$ & $\mathrm{Im}\, L$. This follows from the repeated inclusions below.

$$\mathrm{Im}\, L = L\,V \subseteq V \Rightarrow \mathrm{Im}\, L^2 = L(L\,V) \subseteq L\,V = \mathrm{Im}\, L \tag{2.121}$$

since $L(L\,V)$ gives only a part of $\mathrm{Im}\, L$ since the set-argument of L is, this time, only a subset of the whole space V. A more rigorous proof could be given but the property is quite obvious. Hence, the operator $\mathrm{Im}\, L$ – applied to the successive powers of L – generates a *decreasing chain of subsets* (in fact, subspaces) :

$$V \supseteq \mathrm{Im}\, L \supseteq \mathrm{Im}\, L^2 \supseteq \ldots \supseteq \mathrm{Im}\, L^m \supseteq \ldots \tag{2.122}$$

As regards the kernels of the successive powers of an operator, they increase with the powers:

$$\operatorname{Ker} L \subseteq \operatorname{Ker} L^2 \subseteq \ldots \subseteq \operatorname{Ker} L^m \subseteq \ldots \tag{2.123}$$

Indeed,

$$x \in \operatorname{Ker} L \underset{\text{def}}{\Rightarrow} L\,x = \mathbf{0} \Rightarrow L^2 x = L(L\,x) = L\,\mathbf{0} = \mathbf{0} \Rightarrow x \in \operatorname{Ker} L^2 \Rightarrow \ldots (2.123).$$

It remains to see when the inclusions in the two chains are strict or they are equalities. We do not go into details here, but let us recall that both the kernels and the images are related to the rank of the matrix L_A, respectively of the successive powers of this matrix. For instance, if

$$\operatorname{rank} L_A = n = \dim V \iff \det L_A \neq 0 \iff \dim \operatorname{Im} L = n$$

then $\operatorname{rank} L_A^2 = \dim \operatorname{Im} L^2 = n$ because $\det L_A^2 = (\det L_A)^2 \neq 0 \Rightarrow \operatorname{rank} L_A^2 = n$. This implies $\operatorname{Im} L = \operatorname{Im} L^2$ since *two subspaces of the same dimension are practically equal.*

***Definition** 2.8* (**Particular Types of Operators**) **.** Let $L : V \longrightarrow V$ an endomorphism (or operator) of space V. It is

1° an *automorphism* if it is bijective ;

2° a *projection* if $L \circ L \underset{\text{not}}{=} L^2 = L$;

3° an *involution* (or a *product structure*) if $L \circ L \underset{\text{not}}{=} L^2 = \mathbf{1}_V$;

4° a *complex structure* if $L \circ L \underset{\text{not}}{=} L^2 = -\mathbf{1}_V$;

5° a *nilpotent operator of index* $\mathbf{p} \geq 1$ if $L^p = \mathrm{O}_\mathrm{V}$. ♢

Some properties of these special types of operators are presented and proved in [@.C., 2014] .

***Definition** 2.9* (**Invariant Subspaces**)**.** *If* $L \in \operatorname{Hom}(V, V)$ *and* $U \subseteq_{\text{subsp}} V$ *then the subspace* U *is* L *– invariant if*

$$L\,U \subseteq U \underset{\text{not}}{\iff} (\forall\, x \in U)\ L\,x \in U. \tag{2.124}$$

♢

In practical applications, when the operator is of the form $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ or $L : \mathbf{K}^n \longrightarrow \mathbf{K}^n$, it is convenient to use the matrix M and its successive powers of (2.120).

Example 2.7. Let us consider the operator $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$$Y = L\,X = M \cdot X = \begin{bmatrix} -x_1 + 2x_2 \\ 3x_1 + x_2 + 7x_3 \\ 2x_2 + 2x_3 \end{bmatrix}. \tag{2.125}$$

$$(2.125) \Rightarrow \quad M = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 7 \\ 0 & 2 & 2 \end{bmatrix} \Rightarrow \ldots \Rightarrow M^2 = \begin{bmatrix} 7 & 0 & 14 \\ 0 & 21 & 21 \\ 6 & 6 & 18 \end{bmatrix}. \tag{2.126}$$

The first matrix in (2.126) leads to the kernel and image of L:

$$M = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 7 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -2 \\ 3 & 1 & 7 \\ -6 & 0 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow X(\alpha) = \begin{bmatrix} -2\alpha \\ -\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \in \operatorname{Ker} L.$$

$$(2.125) \Rightarrow \quad Y = \begin{bmatrix} -x_1 + 2x_2 \\ 3x_1 + x_2 + 7x_3 \\ 2x_2 + 2x_3 \end{bmatrix} = x_1 \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}. \tag{2.127}$$

The three generators of $\operatorname{Im} L$ of (2.126) are not linearly independent since it is clear (from the derivation of $\operatorname{Ker} L$) that $\operatorname{rank} M = 2$. Hence, any two of the three column vectors in (2.126-1) can be chosen to form a basis for the image. For instance,

$$\operatorname{Im} L = \mathcal{L}([b_1 \ \ b_2]) \ \text{ with } \ b_1 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \ b_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}. \tag{2.128}$$

For the squared operator L^2, the second matrix in (2.126) leads to

$$M^2 = \begin{bmatrix} 7 & 0 & 14 \\ 0 & 21 & 21 \\ 6 & 6 & 18 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \operatorname{Ker} L^2 = \operatorname{Ker} L. \tag{2.129}$$

The image $\operatorname{Im} L^2$ can be found from the expression of the form (2.28), with the matrix in (2.126-2):

$$M^2 = \begin{bmatrix} 7 & 0 & 14 \\ 0 & 21 & 21 \\ 6 & 6 & 18 \end{bmatrix} \Rightarrow L^2 X = \begin{bmatrix} 7x_1 + 14x_3 \\ 21x_2 + 21x_3 \\ 6x_1 + 6x_2 + 18x_3 \end{bmatrix} = x_1 \begin{bmatrix} 7 \\ 0 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 21 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 14 \\ 21 \\ 18 \end{bmatrix}. \tag{2.130}$$

Obviously, the second vector in (2.158) may be replaced by a "shorter" one - this vector times 1/3. But the three vectors are again linearly dependent since the chain of rank-equivalent matrices in (2.129) show that $\operatorname{rank} M^2 = 2$. For instance, the first two vector can be taken in the basis of $\operatorname{Im} L^2$:

$$\operatorname{Im} L^2 = \mathcal{L}([c_1 \ \ c_2]) \ \text{ with } \ c_1 = \begin{bmatrix} 7 \\ 0 \\ 6 \end{bmatrix}, \ c_2 = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}. \tag{2.131}$$

The two image subspaces in (2.157) and (2.160) are certainly isomorphic since they have the same dimension $(=2)$. But *they are even equal*. This follows from a vector equation resulting from the two bases of $\operatorname{Im} L$ & $\operatorname{Im} L^2$ in (2.128) and (2.131) :

$$\beta_1 \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + \beta_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \gamma_1 \begin{bmatrix} 7 \\ 0 \\ 6 \end{bmatrix} + \gamma_2 \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}. \tag{2.132}$$

Eq. (2.132) is equivalent to a homogeneous system whose matrix is

$$\begin{bmatrix} -1 & 2 & -7 & 0 \\ 3 & 1 & 0 & -7 \\ 0 & 2 & -6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 7 & 0 \\ 0 & 7 & -21 & -7 \\ 0 & 2 & -6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 7 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & -1 \end{bmatrix}. \tag{2.133}$$

Obviously, the H-system whose (equivalent) matrix is the last matrix in (2.133) has infinitely many solutions. In fact, the transformation leading to this matrix were not effectively needed. An equation of the form (2.132) clearly leads to a homogeneous system and such a system is always consistent ; but (2.133) implies that Eq. (2.132) has a double infinity of solutions, since

$$(2.132) \Rightarrow \beta_1 = -\gamma_1 + 2\gamma_2 \;\;\&\;\; \beta_2 = 3\gamma_1 + \gamma_2. \tag{2.134}$$

It is now possible to check the equality $\mathrm{Im}\, L = \mathrm{Im}\, L^2$: (2.135)

(2.131) with (2.133) & (2.134) $\Rightarrow$

$$\Rightarrow \beta_1 b_1 + \beta_2 b_2 = (-\gamma_1 + 2\gamma_2) \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + (3\gamma_1 + \gamma_2) \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \gamma_1 \begin{bmatrix} 7 \\ 0 \\ 6 \end{bmatrix} + \gamma_2 \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix} \Rightarrow (2.135).$$

Hence, it has been effectively proved that $\mathrm{Im}\, L = \mathrm{Im}\, L^2$: *the two subspaces coincide*, but (although) *they are generated by two different bases*.

This example is practically completed. Let us (however) remark that the way we have checked the just mentioned equality, via Eq. (2.132) and the corresponding H-system, was used in **§ 2.2** for finding (a basis spanning) *the intersection of two subspaces*. It would be interesting to continue with calculating the matrix of L^3. Eqs. (2.126) $\Rightarrow$

$$\Rightarrow M^3 = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 7 \\ 0 & 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 7 & 0 & 14 \\ 0 & 21 & 21 \\ 6 & 6 & 18 \end{bmatrix} = \begin{bmatrix} -7 & 42 & 28 \\ 63 & 63 & 189 \\ 12 & 54 & 78 \end{bmatrix}. \tag{2.136}$$

$$(2.164) \Rightarrow M^3 \sim \begin{bmatrix} -1 & 6 & 4 \\ 1 & 1 & 3 \\ 2 & 9 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 7 & 7 \\ 0 & 7 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \underset{(2.129)}{\Rightarrow} \mathrm{Ker}\, L^3 = \mathrm{Ker}\, L^2 = \mathrm{Ker}\, L.$$

Since $\dim \mathrm{Ker}\, L^3 = 1$, $\dim \mathrm{Im}\, L^3 = 2$, it can be expected that the effective image of L^3 , spanned by a certain basis, will again coincide with $\mathrm{Im}\, L = \mathrm{Im}\, L^2$. We leave this verification as an exercise to the reader.

As a conclusion, the chains of inclusions of (2.122) & (2.123) seem to become – for the operator of (2.125) – stable, with *equalities instead of proper inclusions*, except the first one :

$\mathbb{R}^3 \supset \mathrm{Im}\, L = \mathrm{Im}\, L^2 = \ldots = \mathbb{R}^2$. □

Other specific properties of the endomorphisms follow in **§ 4.3** that deals with definitions and methods for *bringing the matrix of an operator to simpler forms* like the diagonal matrices. A similar problem was approached and presented in **§ 3.3** : the diagonalization of the quadratic forms. Notions like eigenvalues and eigenvector / eigen-subspaces will be met again, in another context.

We are going to close this section with a figure intended to illustrate the "action" of an endomorphism which is neither surjective nor injective.

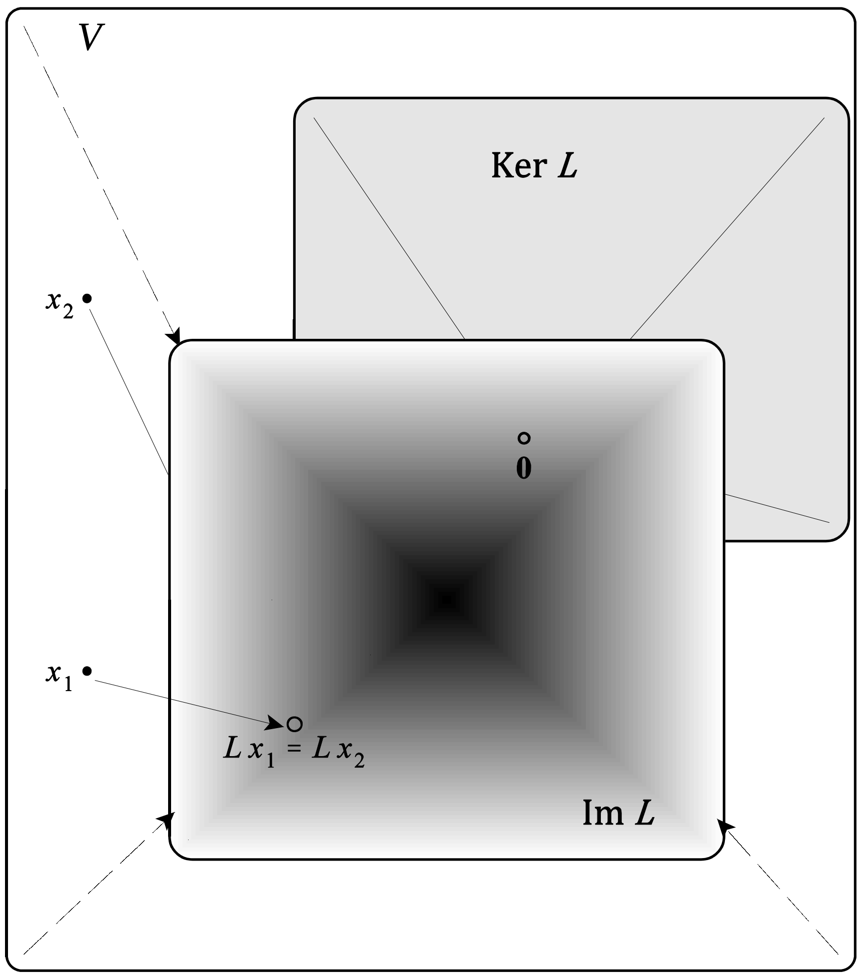

Fig. 2.1 A non-surjective and non-injective endomorphism

§ 4.2-A APPLICATIONS TO LINEAR OPERATORS

2 – A.1 Check whether the following mappings are linear endomorphisms or not and establish which of them are isomorphic.

a) $L : \mathscr{M}_{2,2}(\mathbb{R}) \longrightarrow \mathscr{M}_{2,2}(\mathbb{R}),\ L\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & a-b \\ b-c & a+d \end{bmatrix}.$

b) $L : \mathrm{POL}_2(\mathbb{R}) \longrightarrow \mathrm{POL}_2(\mathbb{R}),$

$$L(a_0 + a_1 t + a_2 t^2) = a_0 + a_1 + (a_1 + a_2)\,t + (a_0 + a_2)\,t^2.$$

c) $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ LX = [x_1 + x_2 \quad x_2 + x_3 \quad x_3 + x_1]^{\mathrm{T}}.$

d) $L : \mathscr{M}_2(\mathbb{R}) \longrightarrow \mathscr{M}_2(\mathbb{R}),\ LA = A \cdot B,$ where B is a fixed matrix.

e) $L : \mathrm{POL}_n(\mathbb{R}) \longrightarrow \mathrm{POL}_n(\mathbb{R}),\ Lp(t) = p(-t).$

Find $\mathrm{Ker}\,L$ and $Im\,L$ for those mappings that are endomorphisms / operators.

2 – A.2 Find the matrix of the endomorphism $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ which maps the vectors

$$U_1 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix},\ U_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},\ U_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{onto} \quad W_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ W_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},\ W_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

respectively.

2 – A.3 The operator $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ maps the vectors of the standard basis E on three given vectors as follows :

$$Le_1 = [1 \quad 1 \quad 1]^{\mathrm{T}},\ Le_2 = [0 \quad 1 \quad 1]^{\mathrm{T}},\ Le_3 = [0 \quad 1 \quad 0]^{\mathrm{T}}.$$

It is required to find $L\,[1 \quad 2 \quad 3]^{\mathrm{T}}$, then the matrix of L in the basis

$$B : b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ b_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and – finally – to determine $\mathrm{Ker}\,L$ and $Im\,L$.

2 – A.4 The following operators $L_i : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ are given :

a) $L_1 X = (x_{n+1}, x_{n+2}, \ldots, x_{2n}, -x_1, -x_2, \ldots, -x_n)$;

b) $L_2 X = (x_{n+1}, x_{n+2}, \ldots, x_{2n}, 0, 0, \ldots, 0)$;

c) $L_3 X = (0, 0, \ldots, 0, -x_1, -x_2, \ldots, -x_n)$;

d) $L_4 X = (x_{n+1}, x_{n+2}, \ldots, x_{2n}, x_1, x_2, \ldots, x_n)$.

It is required to write their matrices in the standard basis of the space $\mathbb{R}^{2n}$, to determine their dimensions and – finally – to check that

$$L_1^2 = -\mathrm{id}\,,\ L_2^2 = L_3^2 = \mathrm{O}\,,\ L_4^2 = \mathrm{id}\ \ \&\ \ L_2 \circ L_3 + L_3 \circ L_2 = -\mathrm{id}\,,$$

where id is the identity mapping and O is the zero morphism of space $\mathbb{R}^{2n}$.

Hint. The matrices of the four operators $L_i : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ can be easily written *under block forms*. In fact, we suggest to write (firstly) their transposes M_i. For example, the matrix M of the operator defined by

$$L X = (x_1, x_2, \ldots, x_n, -x_{n+1}, -x_{n+2}, \ldots, -x_{2n}) \text{ is } \begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & -I_n \end{bmatrix}.$$

2 – A.5 Let the operator $L : \mathrm{POL}_2(\mathbb{R}) \longrightarrow \mathrm{POL}_2(\mathbb{R})$ be defined by

$$L p(t) = t p'(t).$$

Find its matrix in the basis $B = \{1,\ 1+t,\ (1+t)^2\}$.

2 – A.6 Show hat the endomorphism $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfying the equation

$$L^2 - L + \mathrm{id}_{\mathbb{R}^n} = \mathrm{O}_{\mathbb{R}^n}$$

is an automorphism (that is, it is both linear and bijective).

2 – A.7 The matrix of the linear endomorphism $L : V \longrightarrow V$ in a basis A is

$$L_A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Find its matrix in a basis B that is obtained from A with the transformation matrix

$$T = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix} \quad (B^{\mathrm{T}} = T \cdot A^{\mathrm{T}})$$

and express the image $L(2a_1 - a_2 - 3a_3)$ in both bases A & B.

2 – A.8 Show that the space $\mathscr{M}_2(\mathbb{R})$ of the square matrices is isomorphic to the space $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ and identify a standard basis in each of them.

2 – A.9 Check that the mappings $p_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $p_i(X) = x_i \; (1 \le i \le n)$ is a projection (the "canonical" projection).
Hint : It must be checked that $p_i \; (1 \le i \le n)$ are linear.

2 – A.10 Let the linear operator $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by

$$L X = \begin{bmatrix} -x_1 + x_2 + 2x_3 \\ 3x_1 + 3x_2 + 4x_3 \\ 2x_1 + x_2 + 2x_3 \end{bmatrix}. \tag{10-1}$$

Show that L is an automorphism. Determine

$$L^{-1}[0 \;\; 1 \;\; 2]^{\mathrm{T}} \;\;\&\;\; L^{-1}[1 \;\; 5 \;\; 2]^{\mathrm{T}}. \tag{10-2}$$

Write the explicit expression, similar to (10-1), of the inverse endomorphism $X = L^{-1} Y$. Calculate the matrices $(L^2)_E$ & $(L^3)_E$, or their transposes. Check that L^2 is also an automorphism. Find the matrix L_A of the operator in the basis

$$A = [a_1 \;\; a_2 \;\; a_3] = \begin{bmatrix} 1 & -4 & -1 \\ -1 & 6 & 3 \\ 0 & -10 & -9 \end{bmatrix}.$$

Find the image of the vector $x = 3a_1 + 4a_2 - a_3$ through L & L^2 using the matrix L_A, and also the matrix L_E (or the definition (10-1) of the operator).

2 – A.11 Let $L : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a linear operator defined by

$$L X = \begin{bmatrix} x_2 + x_3 \\ -x_1 - x_2 + x_4 \\ x_1 + x_2 - x_4 \\ -x_1 + x_3 + x_4 \end{bmatrix}.$$

Show that $\operatorname{Ker} L = \operatorname{Im} L$ and find a basis spanning this subspace.

2 – A.12 Find the 4-by-4 matrix CP_E (or $CP_E = M$) of the (endo)morphism that represents a cyclic permutation : each vector $X = [x_1 \;\; x_2 \;\; x_3 \;\; x_4]^{\mathrm{T}}$ is transformed to $CP\,X = [x_2 \;\; x_3 \;\; x_4 \;\; x_1]^{\mathrm{T}}$. What is the effect of CP^2 ? Show that $CP^3 = CP^{-1}$.

2 – A.13 The space of all 2-by-2 matrices has the four basis (canonical) "vectors" :

$$E = [e_{11} \;\; e_{12} \;\; e_{21} \;\; e_{22}] : \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Consider the linear transformation T of *transposing* every 2-by-2 matrix, and find its matrix T_E with respect to this basis. Why is $T^2 = \mathrm{id}$?