# 1. Matrices and Determinants

## 1.1 *Determinants*

**1.1** Calculate the following determinants.

$(i)\quad \det\begin{bmatrix} -1 & 0 & 2 \\ 3 & 8 & 11 \\ -2 & 0 & 5 \end{bmatrix}$; $(ii)\quad \det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$; $(iii)\quad \det\begin{bmatrix} \sin\alpha & \cos\alpha \\ \sin\beta & \cos\beta \end{bmatrix}$;

$(iv)\quad \det\begin{bmatrix} a & a & a \\ -a & a & x \\ -a & -a & x \end{bmatrix}$; $(v)\quad \det\begin{bmatrix} 1 & i & 1+i \\ -i & 1 & 0 \\ 1-i & 0 & 1 \end{bmatrix}$ $(i^2 = -1)$.

**1.2** Develop the following determinants along the specified row / column.

$(i)\quad \begin{vmatrix} 1 & 0 & -1 & -1 \\ 0 & -1 & -1 & 1 \\ a & b & c & d \\ -1 & -1 & 1 & 0 \end{vmatrix}$, along its third row ;

$(ii)\quad \begin{vmatrix} 2 & 1 & 1 & x \\ 1 & 2 & 1 & y \\ 1 & 1 & 2 & z \\ 1 & 1 & 1 & t \end{vmatrix}$, along its fourth column.

**1.3** Calculate the following determinants using transformations on their rows / columns.

$(i)\quad \det\begin{bmatrix} 13547 & 13647 \\ 28423 & 28523 \end{bmatrix}$; $(ii)\quad \begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix}$; $(iii)\quad \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$;

$(iv)\quad \begin{vmatrix} 5 & 6 & 0 & 0 & 0 \\ 1 & 5 & 6 & 0 & 0 \\ 0 & 1 & 5 & 6 & 0 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 1 & 5 \end{vmatrix}$; $(v)\quad \begin{vmatrix} x & 0 & -1 & 1 & 0 \\ 1 & x & -1 & 1 & 0 \\ 1 & 0 & x-1 & 0 & 1 \\ 0 & 1 & -1 & x & 1 \\ 0 & 1 & -1 & 0 & x \end{vmatrix}$.

***Hint*** : In the case of higher order determinants it is recommended to bring them to a triangular form.

$\boxed{\mathbf{1.4}}$ Check three properties of the determinants on the particular determinant given below, namely : 1) the sign of a determinant changes if two rows / columns are interchanged ; 2) the value of a determinant does not change if a row is replaced by that row plus another row multiplied by a scalar (a number) ; 3) the value of a determinant does not change if a column is replaced by that column plus another column multiplied by a scalar. It can be taken the first determinant in exercise $\boxed{1.1} - (i)$ and the possible transformations could be $R_2 \leftrightarrows R_3$, $R_2 - 3R_3$, $C_1 + 2C_3$. The respective determinants may be denoted $|A_1|$, $|A_2|$, $|A_3|$ and it has to be checked that $|A| + |A_1| = 0$, $|A_2| = |A|$, $|A_3| = |A|$.

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## *Matrices – operations*

$\boxed{\mathbf{1.5}}$ Calculate the products of the following matrices and check that $|AB| = |A| \cdot |B|$.

$$(i) \quad A = \begin{bmatrix} 4 & 5 & 6 \\ 9 & 1 & 4 \\ 0 & 0 & 7 \end{bmatrix}, \qquad B = \begin{bmatrix} 7 & 0 & 8 \\ 1 & 2 & 1 \\ 3 & 2 & 0 \end{bmatrix};$$

$$(ii) \quad A = \begin{bmatrix} 2 & -3 & 0 \\ -2 & 5 & -1 \\ 3 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -6 & -3 & 3 \\ -5 & -2 & 2 \\ -13 & -7 & 4 \end{bmatrix}.$$

Check that the two matrices in $(ii)$ commute : $AB = BA$.

$\boxed{\mathbf{1.6}}$ Determine the most general square matrix of order 3 which commutes with

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

***Hint*** : The required matrix can be considered under the form $X = \begin{bmatrix} x & y & z \\ u & v & w \\ r & s & t \end{bmatrix}$.

$\boxed{\mathbf{1.7}}$ It is considered the polynomial $f(X) = X^2 - 7X + 11 I_2$. Calculate $f(A)$ for

$$A = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}.$$

$\boxed{\mathbf{1.8}}$ Determine which of the following matrices is / are nonsingular.

$$A=\begin{bmatrix}2&1&3\\1&-1&2\\4&5&-2\end{bmatrix},\quad B=\begin{bmatrix}-1&1&1\\2&1&1\\1&3&5\end{bmatrix},\quad C=\begin{bmatrix}1&0&2\\-1&1&1\\2&-1&1\end{bmatrix}.$$

**1.9** For the matrix $A=\begin{bmatrix}b&c&0\\a&0&c\\0&a&b\end{bmatrix}$ calculate its determinant and the product $AA^{\mathrm{T}}$ ($A^{\mathrm{T}}$ = the transpose of $A$). Using these results, show that

$$\det\begin{bmatrix}b^2+c^2&ab&ca\\ab&a^2+c^2&bc\\ca&bc&a^2+c^2\end{bmatrix}=4a^2b^2c^2.$$

**1.10** Check the property of the transposing operator for three matrices, that is $(ABC)^{\mathrm{T}}=C^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}}$, for the particular matrices

$$A=\begin{bmatrix}-1&2\\5&0\end{bmatrix},\quad B=\begin{bmatrix}2&4\\1&8\end{bmatrix},\quad C=\begin{bmatrix}3&-1\\1&8\end{bmatrix}.$$

**1.11** Given the matrices

$$A=\begin{bmatrix}1&2&-3\\2&-1&4\end{bmatrix}\ \&\ B=\begin{bmatrix}1&7&-13\\5&0&5\end{bmatrix},$$

check whether there exists a matrix $C$ such that $B=CA$.

***Hint*** : It is possible to look for the required matrix in a similar way as in the *Hint* to exercise $\boxed{1.5}$ : $C=\begin{bmatrix}x&y\\z&t\end{bmatrix}$. The components of the solution to the resulting linear system (of size 6-by-4) will give the entries of matrix $C$ .

**1.12** Find the $n-$ th power $A^n$ of the matrix

$$A=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}.$$

After determining $A^2$ and $A^3$, it is recommended to check the expression of $A^n$ by induction with respect to $n$. The north-eastern entry of $A^n$ will be found as $n(n-1)/2$ .

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## 1.2 *Matrices – the rank*

**`Short Introduction`**. The rank of a matrix can be defined as the maximum order of its (square) minors that are nonzero. Practically, this definition can be used by calculating the minors of increasing orders, in terms of the *bordering method*. A minor of low order (1 or 2) of a given matrix $A$ can be easily found. Next, it has to be sequentially bordered by adding each of the rows and columns outside this minor. The procedure can be continued until all the minors of order $r+1$ that are obtained by bordering a nonzero minor of order $r$ are found to be $=0$. In this case, $\operatorname{rank} A = r$.

A more convenient method consists in bringing the given matrix to a simpler form, namely a *qusi-triangular form*. More details can be found in author's textbook [A. Carausu, ***Linear Algebra - Theory and Applications***, Matrix Rom Publishers, Bucharest 1999]. In brief, any $m-by-n$ matrix $A$ can be brought to a matrix $\overline{A}$ which has a triangular submatrix or order $r$ with nonzero entries on its (main / secondary) diagonal while the other $m-r$ rows or $n-r$ columns are zero vectors in $\mathbb{R}^n$, respectively in $\mathbb{R}^m$. The rank-preserving transformations that may be applied for getting such a simpler form are of three types: $(\mathbf{t}_1)$ interchange of two rows or columns ; $(\mathbf{t}_2)$ multiplying a row / column by a nonzero scalar ; $(\mathbf{t}_2)$ adding a row / column, multiplied by a scalar, to another row / column. If the resulting matrix is $\overline{A}$, the previous definition of the rank plus the properties of the determinants ensure that $\operatorname{rank} A = \operatorname{rank} \overline{A} = r,$ but the rank of the latter matrix is quite obvious. In the case when some entries of the matrix contain parameters, the bordering method is more appropriate for finding the rank as a function of that / those parameters(s). In the case of a square matrix it is recommended to calculate its determinant and to find its roots, that is the roots of equation $|A(\lambda, \mu, \dots)| = 0$ and to continue with determining the rank(s) for the particular matrices with the parameter(s) replaced by those roots.

**1.13** Find the ranks of the following matrices.

$$(i)\quad A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}; \qquad (ii)\quad B = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 0 & 1 & 1 \\ 3 & -1 & 3 & 2 \\ 0 & 2 & -3 & 1 \end{bmatrix};$$

$$(iii)\quad C = \begin{bmatrix} -2 & 7 & 2 & -3 & 5 \\ 2 & 1 & -1 & 2 & 1 \\ 0 & 8 & 1 & -1 & 6 \end{bmatrix}; \qquad (iv)\quad D = \begin{bmatrix} 0 & 3 & -3 & 1 \\ 5 & 9 & -10 & 3 \\ -1 & 0 & 5 & -2 \\ 2 & 1 & -3 & 1 \end{bmatrix}.$$

**1.14** Discuss the possible values of the the ranks of the following matrices as depending on the parameter $\alpha \in \mathbb{R}$ / $\beta \in \mathbb{R}$.

$$(i)\quad A(\alpha) = \begin{bmatrix} 1 & 3 & 5 & 6+\alpha \\ 2 & 3 & 4-\alpha & 2 \\ 1 & 1-\alpha & -2 & -5 \\ 1 & 6 & 12 & 19 \end{bmatrix}; \qquad (ii)\quad B(\beta) = \begin{bmatrix} 1 & 1 & -1 & 2 \\ \beta & 1 & 1 & 1 \\ 1 & -1 & 3 & -3 \\ 4 & 2 & 0 & \beta \end{bmatrix}.$$

**Answers:** $(i)$ $\alpha \in \mathbb{R} \setminus \{1, -14\} \Rightarrow \operatorname{rank} A(\alpha) = 4$ ;
$\alpha = 1 \Rightarrow \operatorname{rank} A(1) = 2$ ; $\alpha = -14 \Rightarrow \operatorname{rank} A(-14) = 3$ .
$(ii)$ $\beta \in \mathbb{R} \setminus \{3\} \Rightarrow \operatorname{rank} A(\alpha) = 4$ ; $\beta = 3 \Rightarrow \operatorname{rank} B(3) = 3.$

## 2. Linear Systems

### 2.1 *Linear Systems – Solutions, Consistency*

**Short Introduction**. A *linear system* (over the real field $\mathbb{R}$) is a set consisting - in general - of $m$ *linear equations* in $n$ unknowns. Each of the $m$ equations is of the form $f_i(x_1, x_2, \ldots, x_n) = b_i$ $\left(i = \overline{1,m}\right)$. The function $f_i$ is linear in the $n$ (real) variables, that is it looks like $f_i(x_1, x_2, \ldots, x_n) = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n$. Hence, the general form of a linear system (of size $m -$ by $-n$) is

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & b_2, \\ \vdots \quad\quad \vdots \quad\quad\quad\quad \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & b_m. \end{cases} \tag{2.1}$$

The (real) numbers $a_{ij}$ $(i = \overline{1,m}$ & $j = \overline{1,n})$ are the *coefficients* of the system while $b_i$ $\left(i = \overline{1,m}\right)$ are its *free terms*.

**Classification** (by the free terms). If all free terms are $= 0$ then the system is said to be *homogeneous*. If at least one free term is non-zero, hence if $b_i \neq 0$ for some $i$ $(1 \leq i \leq m)$ then the system is *nonhomogeneous*.

**Solutions – Consistency**. A solution of a linear system of the form (2.1) is an ordered $n-$ tuple of (real) numbers $X^0 = (x_1^0, x_2^0, \ldots, x_n^0)$ whose $n$ components satisfy the $m$ equations in (2.1). Such an $n-$ tuple is – in fact – a vector in the real $n-$ dimensional space $\mathbb{R}^n$, but the *vector* (or *linear*) *spaces* are presented in the second chapter of the course of ***LinearAlgebra***.

**Classification** (by the solutions). Let $S$ denote the set of solutions to a linear system like (2.1) and let $X = (x_1, x_2, \ldots, x_n)$ be the $n-$ vector of unknowns or variables.. Then

$$S = \left\{X \in \mathbb{R}^n : f_i(X) = b_i \left(i = \overline{1,m}\right)\right\}. \tag{2.2}$$

Three possible situations exist on the nature of this set $S$ :

$$\begin{cases} \mathbf{1}^\circ \quad S = \emptyset \underset{\text{def}}{\Rightarrow} \text{ the system is } \textit{inconsistent , it has no solutons} \text{ ;} \\ \mathbf{2}^\circ \quad S \neq \emptyset \underset{\text{def}}{\Rightarrow} \text{ the system is } \textit{consistent} \text{ ;} \begin{cases} \mathbf{2.1}^\circ \;\; S = \{X^0\} \underset{\text{def}}{\Rightarrow} (2.1) \text{ is } \textit{determined} \text{ ;} \\ \mathbf{2.2}^\circ \;\; \text{card}\, S > 1 \underset{\text{def}}{\Rightarrow} (2.1) \text{ is } \textit{undetermined.} \end{cases} \end{cases}$$

**Remarks.** As a matter of terminology, the terms (qualifications) for the sub-cases $\mathbf{2.1}^\circ$ & $\mathbf{2.2}^\circ$ (inconsistent vs. consistent) are met in English textbooks of Linear Algebra and not in textbooks written in Romanian. As regards the terms for systems falling in the cases $\mathbf{1}^\circ$ / $\mathbf{2}^\circ$, the respective terms of *compatible* / *incompatible* system however occur – for instance – in [E. Sernesi : ***Linear Algebra***, Chapman & Hall Mathematics, London 1993]. It is said that a system in the case $\mathbf{2.1}^\circ$ *has a unique solution*, while a system in case $\mathbf{2.2}^\circ$ *has more than one solutions*. It will be easily seen that a system admitting more than one (i.e., at least two) solutions *has infinitely many solutions*. This assertion will be easily checked in terms of linear systems in matrix format.

**A matrix format for linear systems**. There exists a strong connection between the linear systems and the matrices. Any system of the form (2.1) may be equivalently written as a *matrix equation* of the form

$$A\,X = b \ \text{ where } \ A = [a_{ij}] \quad (i = \overline{1,m} \ \ \& \ \ j = \overline{1,n}), \tag{2.3}$$

$$X = [x_1 \quad x_2 \quad \dots \quad x_n]^{\mathrm{T}} \quad \& \quad b = [b_1 \quad b_2 \quad \dots \quad b_m]^{\mathrm{T}}. \tag{2.4}$$

It is easy to see that the components of product $A\,X$ – which is a column vector – are just the left sides of the $m$ equations in (2.1), respectively equal to the components of the colum vector $b$ of (2.4). Therefore, the linear system (2.1) can be equivalently written as the matrix equation

$$\boxed{A\,X = b\,.}$$

With this matrix notation, the set of solutions to a linear system can be easily written as

$$S = \left\{X \in \mathbb{R}^n : AX = b.\right\} \tag{2.5}$$

An $m-\text{by}-(n+1)$ matrix can be associated to a linear system like this, namely

$$\widetilde{A} = [A|b]. \tag{2.6}$$

It is called the *augmented matrix* of the system and it clearly consists of the coefficient matrix $A$ which is bordered at right by the column vector of the free terms. The two matrices $A$ & $\widetilde{A}$ are involved in an important characterization of the consistent (or compatible) linear systems:

**Kronecker - Capelli's Theorem** :

*The linear system* (2.1) *is consistent if and only if* $\operatorname{rank}\widetilde{A} = \operatorname{rank}A.$

**Remark.** If the system is *homogeneous*, that is $b = \mathbf{0} \in \mathbb{R}^m,$ it is always consistent since it admits the (triovial) zero solution $\mathbf{0} \in \mathbb{R}^n : A\mathbf{0} = \mathbf{0}.$ Clearly, the two zero vectors in the two sides of this equation are different when $m \neq n.$ Hence, for any homogeneous system $AX = \mathbf{0} \ \ \mathbf{0} \in S.$

**Solving linear systems**. There exist several ways / methods for solving linear systems. They depend on the nature of each system, on the structure and rank of matrices $A$ & $\widetilde{A}$. Some methods are known from the Linear Algebra learned in the highschools, but more efficient methods work on the augmented matrix $\widetilde{A}$, respectively on $A$ when the system is homogeneous. A couple of examples are here appropriate.

**Example 2.1.** [A. Carausu, 1999 - page 45] Check for consistency / compatibility the non-homogeneous system

$$\begin{cases} 2x_1 + x_2 - x_3 + x_4 = 1, \\ 3x_1 - 2x_2 + 2x_3 - 3x_4 = 2, \\ 5x_1 + x_2 - x_3 + 2x_4 = -1, \\ 2\,x_1 - x_2 + x_3 - 3x_4 = 4. \end{cases} \tag{2.7}$$

It is known that a linear system can be turned into another and equivalent system if the equations of the given system are combined by simple (or elementary) operations of the three types considered in the *Hint* to exercise $\boxed{1.4}$ , with reference to determinants :

1) two equations may be interchanged since the order in which they are written is not relevant ;

2) an equation may be multiplied (amplified) by a non-zero number.

3) an equation, multiplied (amplified) by a non-zero number, may be side-by-side added to another equation.

But these transformations *can be performed* ***on the rows*** *of the augemnted matrix of the system* and not on the effective equations. The resulting system is equivalent to the initial one in the sense that *they have exactly the same solutions*. The choices for the most efficient transformations are not unique but they are – in principle – aimed to create as many as possible zero entries in the transformed matrix. In the just cited textbook, a possible sequence of transformations was applied to the augmented matrix (and they were presented in detail on pages 45-46). We recall them below.

$$(2.7) \Rightarrow \widetilde{A} = \left[\begin{array}{rrrr|r} 2 & 1 & -1 & 1 & 1 \\ 3 & -2 & 2 & -3 & 2 \\ 5 & 1 & -1 & 2 & -1 \\ 2 & -1 & 1 & -3 & 4 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & -3 & 3 & -4 & 1 \\ 0 & 7 & -7 & 9 & 1 \\ 0 & 16 & -16 & 22 & -6 \\ 0 & -2 & 2 & -4 & 3 \end{array}\right] \to$$

$$\to \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -13 & 31 \\ 0 & -1 & -1 & -3 & 10 \\ 0 & 0 & 0 & -10 & 18 \\ 0 & 0 & 0 & -10 & 23 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -13 & 31 \\ 0 & -1 & -1 & -3 & 10 \\ 0 & 0 & 0 & -10 & 18 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right].$$

The fourth row in the last matrix above represents the equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 5 \Rightarrow 0 = 5$$

what is impossible. Hence the system (2.7) is inconsistent : in has no solution, $S = \varnothing$.

□

**Example 2.2.** [A. Carausu, 1999 - page 52] This was an exercise, 2-A.7, requiring to bring the following matrix to a quasi-triangular and then quasi-diagonal form. In fact, the latter form allows to derive the general solution of the homogeneous system $AX = \mathbf{0}$ with that coefficient matrix.

$$A = \begin{bmatrix} -2 & 7 & 2 & -3 & 5 \\ 2 & 1 & -1 & 2 & 1 \\ 0 & 8 & 1 & -1 & 6 \end{bmatrix}. \tag{2.8}$$

The triangular block (submatrix) could be obtained on columns $A^1$, $A^3$, $A^4$ since their components have lowe values and a zero already exists in $A^1$ ; it is also possible to operate on the last three columns since the transformations are easier when entries like 1 or $-1$ exist. Let us take the latter alternative. Our notation [A.C., 1999] for the rows of a matrix $A \in M_{m,n}$ is $A_i$ $(i = \overline{1,m})$, respectively $A^j$ $(j = \overline{1,n})$; hence the matrix in (2.8) can be "sliced" as follows :

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} A^1 & A^2 & A^3 & A^4 & A^5 \end{bmatrix}. \tag{2.9}$$

The transformations with $A_2$ on column $A^3$ (that is, $A_1 + 2A_2$ & $A_3 + A_2$) lead to

$$A^{(1)} = \begin{bmatrix} 2 & 9 & 0 & 1 & 7 \\ 2 & 1 & -1 & 2 & 1 \\ 2 & 9 & 0 & 1 & 7 \end{bmatrix}. \tag{2.10}$$

The first and third rows are identical, hence a simple subtraction $A_1 - A_3$ gives

$$A^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 2 & 1 \\ 2 & 9 & 0 & 1 & 7 \end{bmatrix}. \tag{2.11}$$

and this is already a quasi-triangular form with the $2$–by–$2$ triangular submatrix on rows $2$ and $3$, and on columns $3$ and $4$. Since each matrix in such a sequence of transfomations represents a system which is equivalent to $AX = \mathbf{0}$ with its initial matrix in (2.8), we see that the system with matrix $A^{(2)}$ is reduced to two equations only, hence the first row in this amatrix may be deleted. But let us apply the prior transformations $A_2 - 2A_3$ and $-\overline{A}_2$. (we put a bar on a transformed vecetor or matrix). The resulting matrices are

$$A^{(3)} = \begin{bmatrix} -2 & -17 & -1 & 0 & -13 \\ 2 & 9 & 0 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 2 & 17 & 1 & 0 & 13 \\ 2 & 9 & 0 & 1 & 7 \end{bmatrix}. \tag{2.12}$$

Let us denote the second (right) matrix in (2.12) as $\overline{A}$. It contains the identity matrix of order $2$, $I_2$, as a submatrix. This is the diagonal block or submatrix, and its column superscripts identify the principal variables, $x_3$ & $x_4$. The other three variables may take any values in $\mathbb{R}$. It is said that they are parameters, and the principal (or main) components of the general solution will naturally depend on them. The two equations corresponding to the matrix $\overline{A}$ are

$$\begin{cases} 2x_1 + 17x_2 + x_3 + 13x_5 = 0, \\ 2x_1 + 9x_2 + x_4 + 7x_5 = 0. \end{cases} \tag{2.13}$$

$$(2.13) \Rightarrow \begin{cases} x_3 = -2x_1 - 17x_2 - 13x_5, \\ x_4 = -2x_1 - 9x_2 - 7x_5. \end{cases} \tag{2.14}$$

The three secondary variables (parameters) may be redenoted as $x_1 = \lambda$, $x_2 = \mu$, $x_5 = \nu$. It follows that the general solution of the homogeneous system in the statement can be written as the column vector in $\mathbb{R}^5$

$$X(\lambda, \mu, \nu) = \begin{bmatrix} \lambda \\ \mu \\ -2\lambda - 17\mu - 13\nu \\ -2\lambda - 9\mu - 7\nu \\ \nu \end{bmatrix}. \qquad (\lambda, \mu, \nu \in \mathbb{R}) \tag{2.15}$$

*Suggestion.* The interested readers can check this general solution by taking its components into the three equations of the system. But it is easier to check the matrix eqution

$A(\lambda,\ \mu,\ \nu)X = \mathbf{0}$, that is

$$\begin{bmatrix} -2 & 7 & 2 & -3 & 5 \\ 2 & 1 & -1 & 2 & 1 \\ 0 & 8 & 1 & -1 & 6 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \\ -2\lambda - 17\mu - 13\nu \\ -2\lambda - 9\mu - 7\nu \\ \nu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

□

**Exercises (Subjects for Homework / Tests)**

### Non-homogeneous systems

$\boxed{\mathbf{2.1}}$ Check the consistency and find the solution(s) when it (they) exist, for the following linear systems $AX = b$ given either explicitly or by their augmented matrices of the form $\widetilde{A} = [A|b]$.

$$(i)\quad \begin{cases} x_1 + 2x_2 + 3x_3 = 1, \\ 2x_1 + x_2 + 4x_3 = 2, \\ 3x_1 - 3x_2 + x_3 = 1. \end{cases} ;\qquad (ii)\quad \begin{cases} x_3 + 2x_4 = 3, \\ 2x_1 + 4x_2 - 2x_3 = 4, \\ 2x_1 + 4x_2 - x_3 + 2x_4 = 7. \end{cases} ;$$

$$(iii)\ AX = b \text{ with } \widetilde{A} = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 31 \\ 5 & 1 & 2 & 29 \\ 3 & -1 & 1 & 10 \end{array}\right];\quad (iv)\ \widetilde{A} = \left[\begin{array}{ccc|c} 5 & 3 & -11 & 13 \\ 4 & -5 & 4 & 18 \\ 3 & -13 & 19 & 22 \end{array}\right];$$

$$(v)\quad \widetilde{A} = \left[\begin{array}{cccc|c} 1 & -4 & 2 & 0 & -1 \\ 2 & -3 & -1 & -5 & -7 \\ 3 & -7 & 1 & -5 & -8 \\ 0 & 1 & -1 & -1 & -1 \end{array}\right];\quad (vi)\quad \widetilde{A} = \left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 1 \\ 3 & 2 & 1 & -1 & 1 \\ 2 & 3 & 1 & 1 & 1 \\ 2 & 2 & 2 & -1 & 1 \\ 5 & 5 & 2 & 0 & 2 \end{array}\right].$$

(*vii*) Establish the nature of the solution set $S$ – depending on the real parameters $m, n \in \mathbb{R}$ – for the next nonhomogeneous system and find its solution(s) when it is consistent (or compatible):

$$\begin{cases} mx + y - 2z = 2, \\ 2x + y + 3z = 1, \\ (2m-1)x + 2y + z = n. \end{cases}$$

(*viii*) Determine the values of the parameter $m \in \mathbb{R}$ for which the following system is consistent and solve it when possible.

$$\begin{cases} 2x + \; y \; + z \; = \; 1, \\ x - \; y \; - z \; = \; m, \\ 3x + \; y \; + 2z \; = \; -1, \\ x + my \; + z \; = \; m. \end{cases}$$

***Hints***. Answers / solutions to exercises (*vii*) & (*viii*) can be found in [A. Carausu, *Linear Algebra* .... 1999] at pages 280, respectively 281-282 ; their numbers in that textbook were **2-A.15** and **2-A.18**. For *the former system*, a single value of parameter $m$ will be found which gives $\det A(m) = 0$. With that root instead of $m$, the augmented matrix will be of the form $\widetilde{A}(n)$ and the nature of the solution set (space) $S$ will have to be established as depending of $n$. For *the latter system*, $\det \widetilde{A}(m) = -(m^2 - 5m + 6)$. The two roots of this polynomial should be taken instead of parameter $m$ and the resulting systems should be analyzed.

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## Homogeneous systems

$\boxed{\mathbf{2.2}}$ (*i*)
$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0, \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 0, \\ x_1 + 3x_2 + 6x_3 + 10x_4 = 0, \\ x_1 + 4x_2 + 10x_3 + 20x_4 = 0. \end{cases} ;$$

(*ii*)
$$\begin{cases} x_1 + 2x_2 + x_3 - x_4 = 1, \\ 3x_2 - x_3 + x_4 = 2, \\ 2x_1 + 7x_2 + x_3 - x_4 = 1. \end{cases} ;$$

(*iii*) $AX = \mathbf{0}$ with
$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 2 & -1 \\ 1 & 1 & 0 & 3 & -2 \\ 1 & 1 & 3 & 0 & 1 \end{bmatrix};$$

(*iv*) Discuss the nature of solutions to the system with the matrix $A(\lambda)$ given below and find the non-trivial (general) solution.

$$A(\lambda) = \begin{bmatrix} 1-\lambda & 0 & 2 & -1 \\ 1-\lambda & 0 & 4 & 2 \\ 2 & -1 & -\lambda & 1 \\ 2 & -1 & -1 & 2-\lambda \end{bmatrix}.$$

(*v*) Solve the homogeneous system (written as the equivalent matrix equation).

$$\begin{bmatrix} 2 & 1 & 4 & 2 & 0 \\ 1 & 2 & 2 & 5 & 3 \\ 0 & -1 & 2 & -2 & 2 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad X = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5]^{\mathrm{T}}.$$

***Hint***. The interested readers could try to obtain the identity matrix $I_3$ on the columns $A^2, A^4, A^5$ (but other choices are also possible). For the recommended choice, the two parameters that will occur in the general solution will be $x_1 = m$ & $x_3 = n$. The general solution has to be verified by the matrix product $AX(m,n) = \mathbf{0} \in \mathbb{R}^3$.

**Remark**. This exercise $(v)$, proposed to be solved by transformations on the rows in the coefficient matrix, shows that the identity submatrix (in this case $I_3$) can be obtained not necessarily as a compact block but also as a *scattered* submatrix : its columns keep their position, but they are "mixed" with the other 2 columns, that correspond to the secondary variables $x_1$ & $x_3$. This example illustrates the way to define a submatrix in terms of the string of its column indices as a substring or *scattered substring* of $1\ 2\ \ldots\ n$, and the same holds for the row indices (subscripts) by a substring or scattered substring of $1\ 2\ \ldots\ m$. In the case of the just stated *Hint*, the suggested structure of the coefficient matrix correspond to the (main) column indices $2\ 4\ 5 <_{\mathrm{sw}} 1\ 2\ 3\ 4\ 5$, and the final (quasidiagonal) matrix will be of the form $\overline{A} = [\overline{A}_{11} \star I_3]$. More details, examples and exercises follow in the next subsection.

## 2.2 *Linear Systems – Gaussian elimination*

Certain (particular) linear systems can be solved by specific methods. For instance, the non-homogeneous systems admitting a unique solution can be solved by the well-known *Cramer's rule*. But this method is limited to the (nonhomogeneous) systems of the form $AX = b$ with $A$ – a nonsingular square matrix : $A \in M_n$ & $\det A \neq 0$. An apparently more general case is the one when $A \in M_{m,n}$ with $m > n$ but $\operatorname{rank} A = n$. In this case, the system naturally reduces to a square systems (of type $n-$ by $-n$) consisting of only $n$ independent equations whose corresponding rows in the coefficient matrix $A$ intersect a *nonzero determinant of order* $n$.

Cramer$'$s rule (for solving an $n-$ by $-n$ nonhomogeneous systems) requires to calculate $\det A$ and other $n$ determinants, by one for each variable (or unknown) $x_j$ $(j = \overline{1,n})$. Each such determinant is obtained by replacing the $j-$ th column of matrix $A$ by the column of the free terms. Let us denote $\det A = \Delta$ and the $n$ determinants corresponding to $x_1, \ldots, x_j, \ldots, x_n$ by $\Delta_j$. According to the just given descsription,

$$\Delta_1 = \left| b\ A^2 \ldots A^n \right|, \ \ldots\ ,\ \Delta_j = \left| A^1 \ldots A^{j-1} b\, A^{j+1} \ldots A^n \right|, \ \ldots,\ \Delta_n = \left| A^1 \ldots A^{n-1} b \right|. \tag{2.16}$$

It can be proved that the $n$ components of the unique solution of such a Cramer-type system are given by

$$x_j = \frac{\Delta_j}{\Delta}, \ (j = \overline{1,n}). \tag{2.17}$$

As a quite minor detail, if the order $n$ is low and the unknowns of the system are not denoted as $x_1, x_2, \ldots, x_n$ but $x, y, z$ (for $n = 3$), the determinants of (2.16) are usually detoted as $\Delta_x, \Delta_y, \Delta_z$ .

**Example 2.3.** Solve, by Cramer's rule, the system

$$\begin{cases} 2x_1 - x_2 - x_3 = 4, \\ 3x_1 + 4x_2 - 2x_3 = 11, \\ 3x_1 - 2x_2 + 4x_3 = 11. \end{cases} \tag{2.18}$$

$$(2.18) \Rightarrow \widetilde{A} = \left[\begin{array}{rrr|r} 2 & -1 & -1 & 4 \\ 3 & 4 & -2 & 11 \\ 3 & -2 & 4 & 11 \end{array}\right] \ \& \ \det A = \begin{vmatrix} 2 & -1 & -1 \\ 3 & 4 & -2 \\ 3 & -2 & 4 \end{vmatrix} = \ldots = 60. \tag{2.19}$$

Next, the chain of three determinants corresponding to the three unknowns is

$$\Delta_1 = \begin{vmatrix} 4 & -1 & -1 \\ 11 & 4 & -2 \\ 11 & -2 & 4 \end{vmatrix} = 180\,;\ \Delta_2 = \begin{vmatrix} 2 & 4 & -1 \\ 3 & 11 & -2 \\ 3 & 11 & 4 \end{vmatrix} = 60\,;\ \Delta_3 = \begin{vmatrix} 2 & -1 & 4 \\ 3 & 4 & 11 \\ 3 & -2 & 11 \end{vmatrix} = 60.$$

The interested readers are invited to check the values of system's determinant of (2.19) and of the other three as well. According to the formulas (2.17), the three components of the unique solution of the system (2.18) are

$$x_1 = \frac{180}{60} = 3, \quad x_2 = \frac{60}{60} = 1, \quad x_3 = \frac{60}{60} = 1 \Rightarrow X = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}. \tag{2.20}$$

In an explicit presentation, the expression (2.20) means that

$$X = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 3 \\ x_2 = 1 \\ x_3 = 1 \end{cases}.$$

The readers can check this solution by replacing the three components of the solution into system's equations (2.18). But an easier way to check the solution

The same system (2.18) can be solved by another method which *does not need to calculate any determinant*. It is based on the property of linear systems to keep the set $S$ of solutions (in particular the unique solution) if the equations are subjected to "elementary" transformations of types $(\mathbf{t}_1)$, $(\mathbf{t}_2)$ and $(\mathbf{t}_3)$ that were preseneted in the **Introduction** to the subsection *Matrices - the Rank*. But it is important to keep in mind that the allowable transformaations must be applied ***to the rows of the augmented matrix*** $\widetilde{A}$ ***only***. One of the properties of square matrices is :

*Any nonsingular square matrix of order n can be transformed, by elementary operations with its rows / columns, into the identity matrix* $I_n$.

According to this property, an augmented matrix $\widetilde{A} = [A|b]$ with $|A| \neq 0$ can be turned into a matrix of the form $[I_n|\bar{b}]$. This augmented matrix corresponds to a system represented by the matrix equation

$$I_n X = \bar{b} \quad \Rightarrow \quad \boxed{X = \bar{b}.} \tag{2.21}$$

A sequence of transformations to turn $\widetilde{A} = [A|b] \to \ldots \to [I_n|\bar{b}]$ for the augmented matrix (2.19) can be

$$\widetilde{A} = \left[\begin{array}{rrr|r} 2 & -1 & -1 & 4 \\ 3 & 4 & -2 & 11 \\ 3 & -2 & 4 & 11 \end{array}\right] \backsim \left[\begin{array}{rrr|r} 3 & 4 & -2 & 11 \\ 2 & -1 & -1 & 4 \\ 3 & -2 & 4 & 11 \end{array}\right] \backsim \left[\begin{array}{rrr|r} 11 & 0 & -6 & 27 \\ -2 & 1 & 1 & -4 \\ -1 & 0 & 6 & 3 \end{array}\right] \backsim$$

$$\backsim \left[\begin{array}{rrr|r} 10 & 0 & 0 & 30 \\ -2 & 1 & 1 & -4 \\ -1 & 0 & 6 & 3 \end{array}\right] \backsim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ -2 & 1 & 1 & -4 \\ -1 & 0 & 6 & 3 \end{array}\right] \backsim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 6 & 6 \end{array}\right] \backsim$$

$$\backsim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \backsim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow X = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

Hence, the solution in (2.20), obtained by Cramer's method, *has been retrieved.*

Although the transformations in the above sequence are rather obvious, we specify them below.

$(\mathbf{t}_1) : A_1 \rightleftarrows A_2 ;\ (\mathbf{t}_3) : A_1 - 2A_2\ \&\ A_3 + 4A_2 ;\ (\mathbf{t}_2) : -A_2 ;\ (\mathbf{t}_3) : A_1 + A_3 ;$
$(\mathbf{t}_3) : A_2 + 2A_1\ \&\ A_3 + A_1 ;\ (\mathbf{t}_2) : \frac{1}{6}A_2 ;\ (\mathbf{t}_3) : A_2 - A_3 .$

As a matter of notation, we have (in this sequence) the sign $\backsim$ to mean that the two (augmented) matrices *are equivalent* ; in fact, the non–homogeuneous systems they represent are *equivalent*, that is they have exactly the same solution. But the arrow $\to$ may be also used, with the significance (the matris at left) *turns into* (the matrix at right).

□

**Discussion.** This simple example illustrates two alternative methods for solving a non-homogeneous system with a unique solution. It would be hard to estimate if one of them is more efficient than the other. However, the difference becomes obvious when the system is of a larger dimension $(n > 3)$ : for isntance, for $n = 4$ Cramer's rule needs to calculate 5 determinants of order 4, and each of them *also requires transformations* for bringing it to – for instance – a triangular form. Instead, the method of elementary transformations needs to write a sequence of matrices of size $4-$ by $-5$. It is clear that the computational effort for applying Cramer's rule rapidly increases with the order $n$. In fact, this method simplifies the coefficient matrix by turning it into the identity matrix $I_n$, and - from the point of view of variables (or unknowns) this means that each equation corresponding to the "last" matrix $[I_n|\bar{b}]$ contains a single variable which does not occur in any of the other $n-1$ equations. That is why the solution method for linear systems, schematically represented by the sequence

$$\boxed{\widetilde{A} = [A|b] \to \ldots \to [I_n|\bar{b}]\,,}$$

is said to be an ***elimination method***. More exactly, it is known as the **Gaussian elimination method** (or Gauss-Jordan elimination). This term appears in most textbooks of Linear Algebra, for instance in [E. Sernesi, 1993] or in [Gilbert Strang, *Linear Algebra and Its Applications*, Harcourt Brace Jovanovich, New York - Orlando - ... - Tokyo, 1988].

**Historical Note. Johann Carl Friedrich GAUSS** was a quite outstanding German mathematician. He was born in Brunswick - Braunschweig, on April 30, 1777. He brought extremely important contributions in mathematical fields as *Number theory*, *Algebra*, *Statistics*, *Analysis* (or *Calculus*), *Differential Geometry*, ***Matrix Theory***, but also in other domains of science like *Geodesy*, *Astronomy*, etc. Even during his life he was viewed / called *Princeps mathematicorum – The Prince of Mathematics*. He died in Göttingen (Kingdom of Hanover) on February 23, 1855. This city of Göttingen was – probably – Europe's most important matematical centre in the XVIII-th – XIX-th centuries. Academician Professor Alexandru Myller studied there. He founded the Romanian school of *Differential Geometry* and also the *Mathematical Seminar* of the "A.I. Cuza" University in Iasi.

[Informations on **J.C.F. Gauss** found by Wikipedia]

As it will be seen in the next examples, this elimination method *has many more applications* besides the solution of linear systems (with unique solution). For instance, we suggested its use in the *Short Introduction* to the section *Matrices – the rank*. The difference consisted in the possibility to apply elementary transformations of types $(\mathbf{t}_1)$, $(\mathbf{t}_2)$ and $(\mathbf{t}_3)$ not only to the rows but also to the columns of a given matrix, in order to bring it to a quasi-triangular form. It can be applied to the calculation of determinants (of higher order, $n > 3$) as well.

## Determining the nature and solution(s) of general linear systems by Gaussian elimination.

This issue was addressed in the (earlier cited) textbook [A. Carausu, Matrix Rom 1999], **Chapter 1** - § **1.2**, pages 37-54. The essentials of this method are briefly recalled in what follows, as a support for the subsequent examples and applications (subject for seminars, homework, tests).

As earlier mentioned, any linear system is equivalent to the matrix equation

$$\boxed{A\,X = b\,.} \tag{2.22}$$

All information on such a system is contained in its augemnted matrix

$$\widetilde{A} = [A|b] \text{ with } A \in M_{m,n}(\mathbb{R}) \text{ and } b \in \mathbb{R}^m, \tag{2.23}$$

The coefficient matrix $A$ has a rank, $\operatorname{rank} A \in \{1, 2, \ldots, \min\{m,n\}\} =_{\text{not}} r$. The zero rank is specific to the zero matrix $O$. The upper bound $\min\{m,n\}$ for the rank obviously follows from the definition of the rank : any $m$ – by –$n$ matrix cannot have a nonzero minor (a square submatrix) of its order greater than $m$ or $n$. Hence, if $\operatorname{rank} A = r$ then the coefficient matrix $A$ necessarily contains a square submatrix $A_p$ of order $r$ with $|A| \neq 0$. It follows that the structure of the matrix $A$ can be

$$A = \begin{bmatrix} A_p & A_s \\ A_{21} & A_{22} \end{bmatrix} \text{ with } A_p \in M_r, \quad A_s \in M_{r,\,n-r} \tag{2.24}$$

while the lower blocks are of sizes $(m-r) \times r$ and $(m-r) \times (n-r)$, respectively. We have used the subscripts $p$ and $s$ for principal / secondary. These terms are met in the school manuals dealing with linear systems, where $r$ variables are taken as principal while $n-r$ are saaid to be secondary. The lower blocks $A_{21}$ and $A_{22}$ are less important since it can be proved that they can be turned to *zero blocks* by tranformations with rows of the upper block $[A_p \ A_s]$. This follows from an alternative definition of the rank of a matrix which – however – needs the notion(s) of *linear independence* / *dependence* which will be presented in Chapter 2 - VECTOR SPACES. With this remark, the matrix in (2.24) can be transformed into a matrix with the last $m - r$ rows having *zero entries only*, by elementary transformations with the upper $r$ rows of $A$. Thus, the new structure of the transformed matrix $\overline{A}$ can be

$$\overline{A} = \begin{bmatrix} \overline{A}_p & \overline{A}_s \\ O & O \end{bmatrix}. \tag{2.25}$$

The transformations on the rows of matrix $A$ have to be applied *on the augmented matrix* $\widetilde{A}$ in order to establih whether the system (2.22) is consistent (or compatible) and – moreover – to find its general solution in the positive case. These rows have $n + 1$ entries.

**`Important remark.`** The structure in (2.25) would suggest that the four submatrices or blocks would be compact, but this is not necessary. Such a structure – with compact blocks – can be easily obtaianed if the columns are moved or interchanged, in order to obtain the principal block in the north-western position. But such changes would need to re-number the variables or redenote them – a simply useless effort. It is more convenient for the principal columns to keep their positions but, in this case, the columns of the block $\overline{A}_p$ are "mixed" with the columns of $\overline{A}_s$. In terms of the *Theory of Formal Languages*, the upper $r \times n$ block is a *shuffle product* of the columns of submatrices $\overline{A}_p$ and $\overline{A}_s$. The symbol $\star$ can be used to denote this operaation. Thus, the more general structure of the transformed matrix in (2.25) is

$$\overline{A} = \begin{bmatrix} \overline{A}_p \star \overline{A}_s \\ O \end{bmatrix}. \tag{2.26}$$

This discussion could seem to be rather abstract, but the idea to obtain a non-singular submatrix of order $r$ *without any changes with the columns* can simplify (or reduce) the calculations. By the way, such a way to perform transformations was earlier recommended in the **`Remark`** following the statement of Exercise $\boxed{\mathbf{2.2}}$ – $(v)$ with a homgeneous system.

As we have just mentioned, the transformations must be performed on the augmented matrix, except the case when the system is homogeneous and therefore the column vector of free terms, $b = \mathbf{0}$, is not relevant. The transformed augmented matrix can thus get the structure

$$\overline{\widetilde{A}} = \begin{bmatrix} \overline{A}_p \star \overline{A}_s & | & \overline{b}_p \\ O & | & \overline{b}_s \end{bmatrix}. \tag{2.27}$$

The lower $r \times (n + 1)$ block corrsponds to $m - r$ equations of the form

$$0x_1 + 0x_2 + \ldots + 0x_n = \overline{b}_\ell \quad (r + 1 \le \ell \le n). \tag{2.28}$$

Hence they reduce to $0 = \overline{b}_\ell$, where $\ell$ is a row index and has nothing in common with the subscripts $p$ and $s$ that occur in (2.26) and (2.27). Equations (2.28), equivalent to $0 = \overline{b}_\ell$ $(r + 1 \le \ell \le n)$, allow for a straightforward discussion on the nature of the set $S$ of the solutions to the linear system. The two possible cases are

$$\begin{cases} (i) \quad \overline{b}_\ell = 0 \ \text{ for } \quad \ell = \overline{r+1,n} \ \Rightarrow S \neq \varnothing \,, \\ (ii) \ (\exists \ \ell \in \overline{r+1,n}\,) \quad \overline{b}_\ell \neq 0 \quad \Rightarrow S = \varnothing \,. \end{cases} \tag{2.29}$$

Obviusly, in case $(i)$ the system is *consistent* and it reduces to the first $r$ (principal) equations. In the other case $(ii)$ the system is *inconsistent*, it has no solutions since it contains (at least) an impossible equation. The first case admits two subcases, depending on the rank of matrix $A$ as compared to the number of unknowns (or columns) $n$ :

$$\begin{cases} (i-1) \quad r < n \ \Rightarrow \text{card}\, S > 1 \,, \\ (i-2) \quad r = n \ \Rightarrow S = \{X_0\} \,. \end{cases} \tag{2.30}$$

In the first subcase of (2.30), the system is undeterminate. It admits more than one solution, in fact it admits *infinitely many solutions.* This is very easy to prove. Indeed, if $S$ contains (at least) two solutions and we consider an arbitrary scalar $\lambda \in \mathbb{R}$, then

$$X_1, X_2 \in S \Rightarrow \left[\overline{A}_p \ \star \ \overline{A}_s\right] \begin{bmatrix} X_{1\ p} \\ \star \\ X_{1\ s} \end{bmatrix} = \overline{b}_p \ \ \& \ \left[\overline{A}_p \ \star \ \overline{A}_s\right] \begin{bmatrix} X_{2\ p} \\ \star \\ X_{2\ s} \end{bmatrix} = \overline{b}_p \Rightarrow$$

$$\Rightarrow \left[\overline{A}_p \ \star \ \overline{A}_s\right] \begin{bmatrix} \lambda X_{1\ p} + (1-\lambda) X_{2\ p} \\ \star \\ \lambda X_{1\ s} + (1-\lambda) X_{2\ s} \end{bmatrix} = \lambda \overline{b}_p + (1-\lambda)\overline{b}_p = \overline{b}_p \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \lambda X_{1\ p} + (1-\lambda) X_{2\ p} \\ \star \\ \lambda X_{1\ s} + (1-\lambda) X_{2\ s} \end{bmatrix} = \lambda X_1 + (1-\lambda) X_2 = \overline{b}_p \Rightarrow$$

$$\Rightarrow \lambda X_1 + (1-\lambda) X_2 = X_\lambda \in S. \tag{2.31}$$

The solution $X_\lambda$ depends on the arbitrary real parameter $\lambda \in \mathbb{R}$ , hence $S$ is infinite ; its cardinal number is $\aleph_1 -$ the cardinal of the real numbers.

The nonhomogeneous system whose augmented matrix has been brought to the form in (2.27) , in the case of consistency with $\overline{b}_s = \mathbf{0}$, reduces to its first $r$ rows. Hence, it is

$$\left[ \ \overline{A}_p \ \star \ \overline{A}_s \ \mid \ \overline{b}_p \ \right] \tag{2.32}$$

It corresponds to the $r$ principal equations, represented by the matrix equation

$$\overline{A}_p X_p + \overline{A}_s X_s = \overline{b}_p. \tag{2.33}$$

The general solution will depend on the $n-r$ parameters – the secondary variables whose (transformed) coefficients are the entries of the submatrix $\overline{A}_s$. This situation occures when $r < n$ , that is in the relevant case $(i-1)$. The other case $(i-2)$ corresponds to a Cramer type system with a single solution. But the equation (2.32) does not yet provide the solution to the system (2.22). The augmented matrix (2.31) has to be further transformed until the principal submatrix $\overline{A}_p$ turns to the identity matrix $I_r$. The corresponding augmented matrix is

$$\left[\ I_r \star \overline{\overline{A}}_s \ \middle| \ \overline{\overline{b}}_p \ \right] \tag{2.34}$$

The corresponding matrix equation is

$$I_r X_p + \overline{\overline{A}}_s X_s = \overline{\overline{b}}_p \ \Rightarrow \ \boxed{X_p = \overline{\overline{b}}_p - \overline{\overline{A}}_s X_s.} \tag{2.35}$$

The secondary (column) vector is $X_s = [x_{j_1}\ x_{j_2}\ \dots\ x_{j_{n-r}}]^{\mathrm{T}}$. The $n-r$ components of this vector can be redenoted as $x_{j_\ell} = \lambda_\ell$ $(\ell = 1, n-r)$. The column vector of the $n-r$ parameters can be denoted as $\Lambda = [\lambda_1\ \lambda_2\ \dots\ \lambda_{n-r}]^{\mathrm{T}}$ and the expression of the principal vector in (2.35) becomes

$$X_p = \overline{\overline{b}}_p - \overline{\overline{A}}\,\Lambda \ \Rightarrow \ \boxed{X(\Lambda) = \begin{bmatrix} \boldsymbol{x}_{j_1}(\Lambda) \\ \boldsymbol{x}_{j_2}(\Lambda) \\ \vdots \\ \boldsymbol{x}_{j_r}(\Lambda) \\ \star \\ \Lambda \end{bmatrix} = \begin{bmatrix} \overline{\overline{b}}_1 - \overline{\overline{A}}_{s,1}\Lambda \\ \overline{\overline{b}}_2 - \overline{\overline{A}}_{s,2}\Lambda \\ \vdots \\ \overline{\overline{b}}_r - \overline{\overline{A}}_{s,2}\Lambda \\ \star \\ \Lambda \end{bmatrix}.} \tag{2.36}$$

**Note.** This method, based upon the ***Gaussian elimination***, for establishing the nature of a general (non-homogeneous) system and for getting its general solution in (2.36) can be also found in [A. Carausu, 1999] - pages 42-46. Certain minor differences in notations exist, but they are not essential. Let us also mention that the "mixing" operation $\star$ which occurs in Eqs. (2.26), (2.27), (2.30), (2.32), (2.36) denotes either the *shuffle product* of (the columns of) two submatrices, or the "vertical" shuffle product with the components of a column vector $X$ / $X(\Lambda)$. The above discussion and formulas could seem to be too technical but they are going to become clear and understandable by means of the examples that follow. We have deboted the $r$ principal components on the upper positions of the solution vector $X(\Lambda)$ in (2.36) by ***bold symbols*** $\boldsymbol{x}_{j_1}, \dots \boldsymbol{x}_{j_r}$ in order to avoid possible confusions with the secondary components of $X_s = [x_{j_1}\ x_{j_2}\ \dots\ x_{j_{n-r}}]$.

**Example 2.4.** [Gh. Procopiuc *et al*, 2001] (*Exercise* **1.24** - 2) at page 20) Establish the nature (depending of $\lambda \in \mathbb{R}$) and find the general solution of the following system, when it is consistent (compatible):

$$\begin{cases} 5x_1 - 3x_2 + 2x_3 + 4x_4 = 3 \\ 4x_1 - 2x_2 + 3x_3 + 7x_4 = 1 \\ 8x_1 - 6x_2 - x_3 - 5x_4 = 9 \\ 7x_1 - 3x_2 + \mathbf{7}x_3 + 17x_4 = \lambda \end{cases}. \tag{2.37}$$

System's augemented matrix is

$$\widetilde{A} = \left[\begin{array}{cccc|c} 5 & -3 & 2 & 4 & 3 \\ 4 & -2 & 3 & 7 & 1 \\ 8 & -6 & -1 & -5 & 9 \\ 7 & -3 & 7 & 17 & \lambda \end{array}\right]. \tag{2.38}$$

The nature of the solution set $S$ can be established after getting a quasi-triangular or even quasi-diagonal form of the matrix $A$, by Gaussian elimination (of course). A possible sequence of transformations can start on the 3-th column, with the entry $-1$.

$$\widetilde{A} \backsim \left[\begin{array}{cccc|c} 21 & -15 & 0 & -6 & 21 \\ 28 & -20 & 0 & -8 & 28 \\ -8 & 6 & 1 & 5 & -9 \\ 63 & -45 & 0 & -18 & \lambda+63 \end{array}\right] \backsim \left[\begin{array}{cccc|c} 7 & -5 & 0 & -2 & 7 \\ 7 & -5 & 0 & -2 & 7 \\ -8 & 6 & 1 & 5 & -9 \\ 63 & -45 & 0 & -18 & \lambda+63 \end{array}\right] \backsim$$

$$\backsim \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 7 & -5 & 0 & -2 & 7 \\ -8 & 6 & 1 & 5 & -9 \\ 63 & -45 & 0 & -18 & \lambda+63 \end{array}\right] \backsim \left[\begin{array}{cccc|c} 7 & -5 & 0 & -2 & 7 \\ -1 & 1 & 1 & 3 & -2 \\ 63 & -45 & 0 & -18 & \lambda+63 \end{array}\right] \backsim$$

$$\backsim \left[\begin{array}{cccc|c} 7 & -5 & 0 & -2 & 7 \\ -1 & 1 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & \lambda \end{array}\right]. \tag{2.39}$$

It follows from the matrix in (2.39) is consistent if and only if (abbreviated *iff*) $\lambda = 0$. For this value, the system reduces to only two equations and its augmented matrix is

$$\left[\begin{array}{cccc|c} 7 & -5 & 0 & -2 & 7 \\ -1 & 1 & 1 & 3 & -2 \end{array}\right] \backsim \left[\begin{array}{cccc|c} -7/2 & 5/2 & 0 & 1 & -7/2 \\ -1 & 1 & 1 & 3 & -2 \end{array}\right] \backsim$$

$$\backsim \left[\begin{array}{cccc|c} -7/2 & 5/2 & 0 & 1 & -7/2 \\ 19/2 & -13/2 & 1 & 0 & 17/2 \end{array}\right] \backsim \left[\begin{array}{cccc|c} 19/2 & -13/2 & 1 & 0 & 17/2 \\ -7/2 & 5/2 & 0 & 1 & -7/2 \end{array}\right]. \tag{2.40}$$

The rank of the matrix in (2.40) is clearly $= 2$, hence the system has infinitely many $(\infty^2)$ solutions. The general solution will depend on the parameters $x_1 = 2a$ and $x_2 = 2b$ :

$$X(a,b) = \left[\begin{array}{c} 2a \\ 2b \\ 17/2 - 19a + 13b \\ 7/2 + 7a - 5b \end{array}\right]. \tag{2.41}$$

The solution in (2.41) can be easily checked as follows :

$$AX(a,b) = \begin{bmatrix} 5 & -3 & 2 & 4 \\ 4 & -2 & 3 & 7 \\ 8 & -6 & -1 & -5 \\ 7 & -3 & 7 & 17 \end{bmatrix} \begin{bmatrix} 2a \\ 2b \\ 17/2 \ - 19a + 13b \\ -7/2 + 7a - 5b \end{bmatrix} =$$

$$= \begin{bmatrix} 10a - 6b + 17 - 38a + 26b - 14 + 28a - 20b \\ 8a - 4b + 51/2 - 57a + 39b - 49/2 + 49a - 35b \\ 16a - 12b - 17/2 + 19a - 13b + 35/2 - 35a + 25b \\ 14a - 6b + 119/2 - 133a + 91b - 119/2 + 119a - 85b \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 9 \\ 0 \end{bmatrix}.$$

It follows that the general (2 - parametric) solution of (2.41) satisfies the system (2.37) with $\lambda = 0$. If $\lambda \neq 0$, the system has no solution: $S$

**Note.** The authors of the textbook (cited in the statement) give an apparently different answer since they chose, as parameters, $x_3 = \lambda_1$ & $x_4 = \lambda_2$. Their solution was

$$X(\lambda_1,\lambda_2) = \begin{bmatrix}-5/2\,\lambda_1 - 13/2\,\lambda_2 - 3/2 & -\,7/2\,\lambda_1 - 19/2\,\lambda_2 - 7/2 & \lambda_1 & \lambda_2\end{bmatrix}^{\mathrm{T}} \qquad (2.42)$$

The interested readers are invited to check this solution in the same way the solution in (2.41) has been just checked. It is also proposed to identify our transformations on the rows of $\widetilde{A}$ that have led to the next (transformed) matrices, and also to apply the transformations which led to the solution in (2.42), with the principal block obtained on the first two columns.

□

**Example 2.5.** The Gaussian elimination method can be also applied for solving homogeneous systems, when they admit nonzero (non-trivial) solution. The matrix $A$ of the system $AX = \mathbf{0}$ can be brought to a quasi-triangular form, for cheching whether rank $A = r < n$ or $r = n$. In the former case the system admits infinitely many solutions depending on $n - r$ parameters, while the latter case corresponds to the trivial solution $X = \mathbf{0} \in \mathbb{R}^n$ only. If $r < n$, the triangular submatrix of order $n$ has to be further transformed by Gaussian elimination until the identity matrix $I_r$ replaces it. Like in (2.34), this last matrix is of the form

$$I_r \star \overline{\overline{A}}_s \Rightarrow X_p = -\overline{\overline{A}}_s X_s \qquad (2.43)$$

since the vector of free terms remains zero : $b = \mathbf{0}$ and $\overline{\overline{b}}_p = \mathbf{0}$, too. With the notation $\Lambda = [\lambda_1 \ \lambda_2 \ \ldots \ \lambda_{n-r}]$ for the parameters, system's general solution has the same / similar form as the column vectors in (2.36) :

$$X(\Lambda) = \begin{bmatrix} \boldsymbol{x}_{j_1}(\Lambda) \\ \boldsymbol{x}_{j_2}(\Lambda) \\ \vdots \\ \boldsymbol{x}_{j_r}(\Lambda) \\ \star \\ \Lambda \end{bmatrix} = \begin{bmatrix} -\overline{\overline{A}}_{s,1}\Lambda \\ -\overline{\overline{A}}_{s,2}\Lambda \\ \vdots \\ -\overline{\overline{A}}_{s,2}\Lambda \\ \star \\ \Lambda \end{bmatrix}. \tag{2.44}$$

Next, this discussion is illustrated by the solution of a homogeneous system from the same textbook [Gh. Procopiuc *et al*, 2001] (*Exercise* **1.25** - 1) at page 20) :

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ x_1 + 3x_2 + 6x_3 + 10x_4 = 0 \\ x_1 + 4x_2 + 10x_3 + 20x_4 = 0 \end{cases}. \tag{2.45}$$

$$(2.45) \Rightarrow A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} \backsim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{bmatrix} \backsim$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{bmatrix} \backsim\backsim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \det A = 1 \neq 0 \Rightarrow S = \{\mathbf{0}\}$$

Another H-system, whose matrix is of size $3-$ by $-4$, will obviously admit non-trivial solutions (besides $\mathbf{0}$), since $r \leq 3 < 4 = n$.

$$A\,X = \mathbf{0} \text{ with } A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 3 & -1 & 1 \\ 2 & 7 & 1 & -1 \end{bmatrix} \backsim \begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 3 & -1 & 1 \\ 2 & 10 & 0 & 0 \end{bmatrix} \backsim$$

$$\begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \backsim \begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 3 & -1 & 1 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -5x_2, \\ x_4 = -3x_2 + x_3. \end{cases}$$

If we take $x_2 = \lambda$ and $x_2 = \mu$, the general solution of the given system is

$$X(\lambda,\mu) = \begin{bmatrix} -5\lambda \\ \lambda \\ \mu \\ -3\lambda + \mu \end{bmatrix}, \quad \lambda, \mu \in \mathbb{R}. \tag{2.46}$$

The solution in (2.46) can be easily verified.

□

**Exercises - Linear Systems by Gaussian elimination**

$\boxed{\mathbf{2.3}}$ Establish the nature and find the (general) solutions for the next linear systems, using the Gaussian elimination method on the augemnted matrices.

$$(i) \quad \begin{cases} ax + y + z = 4, \\ x + by + z = 3, \\ x + 2by + z = 4. \end{cases}, \quad \text{discussion in terms of parameters } a, b \in \mathbb{R};$$

$$(ii) \quad \begin{cases} 2x_1 + x_2 - x_3 + x_4 = 1, \\ 3x_1 - 2x_2 + 2x_3 - 3x_4 = 2, \\ 5x_1 + x_2 - x_3 + 2x_4 = -1, \\ 2x_1 - x_2 + x_3 - 3x_4 = 4. \end{cases} \qquad (iii) \quad \begin{bmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 4 & -2 \\ 4 & -3 & 1 & 1 \end{bmatrix} X = \mathbf{0};$$

$$(iv) \quad \begin{bmatrix} 1-\lambda & 0 & 2 & -1 \\ 1-\lambda & 0 & 4 & -2 \\ 2 & -1 & -\lambda & 1 \\ 2 & -1 & -1 & 2-\lambda \end{bmatrix} X = \mathbf{0}; \qquad (v) \quad \begin{cases} -\lambda x_1 + x_2 + x_3 = 0, \\ x_1 - \lambda x_2 + x_3 = 0, \\ x_1 + x_2 - \lambda x_3 = 0. \end{cases} .$$

**Note - Hint.** For the homogeneous systems of $(iv)$ & $(v)$, the values of the parameter $\lambda$ for which these systems admit non-zero solutions must be firstly deremined. The respective roots $\lambda_i$ of $|A(\lambda)| = 0$ should be then taken to the matrices and the general solutions of the corresponding homogeneous systems have to be found. They will depend on $n - \operatorname{rank} A(\lambda_i)$ parameters. The solutions thus obtained can be checked as in the earlier examples and exercises. If $P$ denotes the set of these parameters then the general solution will be of the form $X(P)$ and it follows to verify that $A(\lambda_i)X(P) = \mathbf{0}$ for each root of the equation $|A(\lambda)| = 0$.

$(vi)$ Determine the real values of the parameter $m$ so that the homogeneous system

$$\begin{cases} x_1 + x_2 + mx_3 - x_4 = 0, \\ 2x_1 + x_2 - x_3 + x_4 = 0, \\ 3x_1 - x_2 - x_3 - x_4 = 0, \\ mx_1 - 2x_2 \phantom{- x_3} - 2x_4 = 0 \end{cases};$$

admit non-zero solutions, too. Find the general solution obtained for the determined value of the parameter and check it.

$(vii)$ Determine (by Gaussian elimination) the general solution of the non-homogeneous system

$$\left[\begin{array}{ccccc|c} 15 & 2 & 1 & 11 & 4 & 7 \\ 8 & 4 & 0 & 2 & 2 & -4 \end{array}\right] X = \left[\begin{array}{c} 6 \\ 4 \end{array}\right].$$

**Hint.** It follows to apply elementary transformations on (with) the two rows of the augemented matrix $\widetilde{A} = [A|b]$. It is clear that its rank is $= 2$ and the first column of $I_2$ already exists. It follows to obtain its second column, for instance instead of o $A^5$. The general solution follows to be checked as suggested for exercises $(iv)$ and $(v)$.

---

## 2.3 *Multiple non-homogeneous linear systems*

The just presented **method of Gaussian elimination** can be successfully applied to find the solutions (if they exist) of *several non-homogeneous systems with specific free terms but with the same* (matrix of) *coefficients*. Formally, let us assume that it is required to solve $p$ non-H systems with the same coefficient matrix $A$, namely

$$A\,X = \boldsymbol{b}^{(1)}, \quad A\,X = \boldsymbol{b}^{(2)}, \dots, \; A\,X = \boldsymbol{b}^{(p)}. \tag{2.47}$$

Some of the systems in (2.47) could be consistent (or compatible) while other ones could have no solution. Each of these $p$ systems would have to be analyzed from the point of view of its solutions $S^{(k)}$ $(1 \leq k \leq p)$ and even solved when $S^{(k)} \neq \emptyset$. But it is very easy to avoid such a multiple $(p-$ fold) effort if the *Gaussian elimination method* is applied.

If $A$ is of size $m-\text{by}-n$, a larger augmented matrix of size $m-\text{by}-(n+p)$ can (or must) be written, namely

$$\widetilde{A} = \left[\, A|b^{(1)} \;\; b^{(2)} \;\; \dots \;\; b^{(p)} \,\right]. \tag{2.48}$$

Two essentially diferent cases can be encountered:

$$\begin{cases} \mathbf{1}^\circ \quad r = m \qquad\qquad \Rightarrow S^{(k)} \neq \emptyset\,, \\ \mathbf{2}^\circ \quad r < m \;\; \Rightarrow S^{(k)} \neq \emptyset \;\; \text{or} \;\; S^{(k)} = \emptyset\,. \end{cases} \tag{2.49}$$

In the case $\mathbf{1}^\circ$, all the $p$ systems admit (nonzero) solutions since all the vectors of free terms $b^{(k)}$ $(k = \overline{1,p})$ have exactly $m = r$ components. This case is equivalent to the case $(i)$ in the criterion (2.29) since none of the $m$ equations of any $k-$th system does not fall in the case $(ii)$ of that criterion.

The second case $\mathbf{2}^\circ$ admits two alternatives. If, for a certain index $k$ $(1 \leq k \leq p)$, the case $(ii)$ in (2.29) is met then the $k-$th system does not admit any solution: $S^{(k)} = \emptyset$. The corresponding column(s) $\boldsymbol{b}^{(k)}$ in the right block of (2.48) have to simply deleted. The other *consistent* systems follow to be further "processed" by Gaussian elimination. Let us denote by $K$ the subset of indices corresponding to inconsistent systems and let $L = \{1, 2, \dots, p\} \backslash K$. Let us also denote by $X^{(\ell)}$ the (general) solution of the $\ell-$th system. Its nature depends on the cardinal number of $S^{(\ell)}$, according to the two subcases $(i-1)$ and (respectively) $(i-2)$ at page $15-$ Eqs. (2.30). In the former case, $r < n$, all the solutions $X^{(\ell)}$ $(\ell \in L)$ will depend on $n-r$ parameters while – if $r = n$ $-$ each solution will be unique. If card $L = q \le p$, the column vectors of the free terms can be $\boldsymbol{b}^{(1)}, \ \boldsymbol{b}^{(2)}, \ \dots, \ \boldsymbol{b}^{(q)}$, after the removal of the $p - q$ free vectors giving incompatible systems. Then the sequence of transformations , for the case $r = n$, will give the $q$ unique solutions of the $q$ Cramer-type systems:

$$\widetilde{A} = \left[ A | \boldsymbol{b}^{(1)} \ \boldsymbol{b}^{(2)} \ \dots \ \boldsymbol{b}^{(q)} \right] \to \ \dots \to \left[ I_r | X^{(1)} \ X^{(2)} \ \dots \ X^{(q)} \right]. \tag{2.50}$$

For the case when $r < n$ , each general solution will be found by its expression in (2.36). The Gaussian elimination method can be represented by the sequence of tranformations

$$\widetilde{A} = \left[ A_p \star A_s | \boldsymbol{b}^{(1)} \ \boldsymbol{b}^{(2)} \ \dots \ \boldsymbol{b}^{(q)} \right] \to \ \dots \to \left[ I_r \star \overline{\overline{A}}_s | X^{(1)} \ X^{(2)} \ \dots \ X^{(q)} \right]. \tag{2.51}$$

where

$$X^{(\ell)} = X^{(\ell)}(\Lambda) = \begin{bmatrix} \overline{\overline{\boldsymbol{b}}}_p^{(\ell)} - \overline{\overline{A}}_s \Lambda \\ \star \\ \Lambda \end{bmatrix}. \tag{2.52}$$

The (column) vector of parameters is $\Lambda = [\, \lambda_1 \ \lambda_2 \ \dots \ \lambda_{n-r} ]^{\mathrm{T}}$ and its components are "mixed" with the $r$ principal components, according to the structure of $A = A_p \star A_s$.

**Remark.** This approach to the simultaneous discussion and solution of *several linear systems* also admits the case when some of systems in (2.47) when some of the systems are *homogeneous*. This means that some of the vectors of free terms in the right sides are $= \mathbf{0}$. If, for instance, $\boldsymbol{b}^{(\ell)} = \mathbf{0}$ , then the general solution of that system will be also given by (2.52) with

$$\overline{\overline{\boldsymbol{b}}}_p^{(\ell)} = \mathbf{0} \ \Rightarrow \ X^{(\ell)}(\Lambda) = \begin{bmatrix} -\overline{\overline{A}}_s \Lambda \\ \star \\ \Lambda \end{bmatrix}. \tag{2.53}$$

The examples that follow will be able to illustrate this way *to establish the nature and to solve (simultaneously) several linear systems*.

**Examples 2.6.** It is required to establish the nature of solutions and to solve the following multiple systems, given by their extended matrices of the form (2.48). The first example is taken from [A. Carausu, 1999] – ***Example 2.4*** at page 47.

$$(i) \quad \widetilde{A} = \begin{bmatrix} 2 & -1 & 0 & | & 3 & 4 & 5 & 2 \\ -1 & 1 & -1 & | & 2 & 1 & 1 & 0 \\ 4 & -2 & 1 & | & -5 & 4 & -2 & 2 \end{bmatrix}. \tag{2.54}$$

$$(ii) \quad \widetilde{A} = \begin{bmatrix} 2 & 1 & 0 & | & -1 & 2 & 0 & 2 & -1 \\ 3 & -1 & 4 & | & 5 & 1 & 0 & 0 & -1 \\ 7 & 1 & 4 & | & 3 & 6 & 0 & 4 & -3 \end{bmatrix}. \tag{2.55}$$

***Solutions***. $(i)$ It is clear and the most convenient transformation to start with, for getting a simpler form of matrix $A$ in (2.54), is $\widetilde{A}_2 + \widetilde{A}_3$ (or $\widetilde{R}_2 + \widetilde{R}_3$). We recall that the selection of transformations is taken with the target to transform matrix $A$ into a quasi-diagonal or even diagonal matrix with $I_r$ as a submatrix, respectively into $I_n$ when $\operatorname{rank} A = m = n$. Hence

$$\widetilde{A} \backsim \left[\begin{array}{rrr|rrrr} 2 & -1 & 0 & 3 & 4 & 5 & 2 \\ 3 & -1 & 0 & -3 & 5 & -1 & 2 \\ 4 & -2 & 1 & -5 & 4 & -2 & -2 \end{array}\right] \backsim \left[\begin{array}{rrr|rrrr} -1 & 0 & 0 & 6 & -1 & 6 & 0 \\ -3 & 1 & 0 & 3 & -5 & 1 & -2 \\ 4 & -2 & 1 & -5 & 4 & -2 & 2 \end{array}\right] \backsim$$

$$\backsim \left[\begin{array}{rrr|rrrr} -1 & 0 & 0 & 6 & -1 & 6 & 0 \\ -3 & 1 & 0 & 3 & -5 & 1 & -2 \\ -2 & 0 & 1 & 1 & -6 & 0 & -2 \end{array}\right] \backsim \left[\begin{array}{rrr|rrrr} 1 & 0 & 0 & -6 & 1 & -6 & 0 \\ 0 & 1 & 0 & -15 & -2 & -17 & -2 \\ 0 & 0 & 1 & -11 & -4 & -12 & -2 \end{array}\right]. \quad (2.56)$$

It follows that the matrix $A$ in this example is nonsingular since $\operatorname{rank} A = m = n = 3$. The four (unique) solutions of the four linear systems corresponding to the augemented matrix in (2.54) are:

$$X^{(1)} = \begin{bmatrix} -6 \\ -15 \\ -11 \end{bmatrix}, \; X^{(2)} = \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}, \; X^{(3)} = \begin{bmatrix} -6 \\ -17 \\ -12 \end{bmatrix}, \; X^{(4)} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}. \quad (2.57)$$

**Checking**. These solutions can be easily checked (as in some earlier examples) by calculating the matrix product

$$A\left[X^{(1)}\;\; X^{(2)}\;\; X^{(3)}\;\; X^{(4)}\right] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} -6 & 1 & -6 & 0 \\ -15 & -2 & -17 & -2 \\ -11 & -4 & -12 & -2 \end{bmatrix} =$$

$$= \begin{bmatrix} 3 & 4 & 5 & 2 \\ 2 & 1 & 1 & 0 \\ -5 & 4 & -2 & 2 \end{bmatrix} = \left[\boldsymbol{b}^{(1)}\;\; \boldsymbol{b}^{(2)}\;\; \boldsymbol{b}^{(3)}\;\; \boldsymbol{b}^{(4)}\right]$$

and the solutions in (2.57) are thus correct. □

**`Errata.`** The solutions to the *Example 2.4* of [A. Carausu, 1999] were affected by some errors, just from the first transformation on the augemnted matrix. The correct solutions are just the ones in (2.57) above.

(*ii*) The augmented matrix in (2.55) can be similarly transformed.

$$\widetilde{A} = \left[\begin{array}{rrr|rrrrr} 2 & 1 & 0 & -1 & 2 & 0 & 2 & -1 \\ 3 & -1 & 4 & 5 & 1 & 0 & 0 & -1 \\ 7 & 1 & 4 & 3 & 6 & 0 & 4 & -3 \end{array}\right] \backsim \left[\begin{array}{rrr|rrrrr} 5 & 0 & 4 & 4 & 3 & 0 & 2 & -2 \\ -3 & 1 & -4 & -5 & -1 & 0 & 0 & 1 \\ 10 & 0 & 8 & 8 & 7 & 0 & 4 & -4 \end{array}\right]$$

$$\backsim \begin{bmatrix} 5 & 0 & 4 & | & 4 & 3 & 0 & 2 & -2 \\ -3 & 1 & -4 & | & -5 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & | & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2.58}$$

At this point it can be remarked that the third row of the transformed augemnted matrix is "almost" the zero vector in $\mathbb{R}^8$, except its 5-th component $= 1.$ This means that the system $AX = \boldsymbol{b}^{(2)}$ *is inconsistent,* and its (transformed) free vector $\bar{\boldsymbol{b}}^{(2)}$ (the sixth column in (2.58) *can be removed*. But the remaining third row can be also removed since the reduced multiple system consists of *two equations only*. It can be also seen that the third system of (2.55) is a homogeneous system while the other four are non-homogeneous. Hence the transformations can continue starting from the matrix

$$\begin{bmatrix} 5 & 0 & 4 & | & 4 & 0 & 2 & -2 \\ -3 & 1 & -4 & | & -5 & 0 & 0 & 1 \end{bmatrix} \backsim \begin{bmatrix} -3 & 1 & -4 & | & -5 & 0 & 0 & 1 \\ 5 & 0 & 4 & | & 4 & 0 & 2 & -2 \end{bmatrix} \backsim$$

$$\backsim \begin{bmatrix} 2 & 1 & 0 & | & -1 & 0 & 2 & -1 \\ 5 & 0 & 4 & | & 4 & 0 & 2 & -2 \end{bmatrix} \backsim \begin{bmatrix} 2 & 1 & 0 & | & -1 & 0 & 2 & -1 \\ 5/4 & 0 & 1 & | & 1 & 0 & 1/2 & -1/2 \end{bmatrix}. \tag{2.59}$$

The last matrix in (2.59) allows to write the four solutions of the remaining four systems (after neglecting the incompatible system). The first variable $x_1$ is secondary and it can be redenoted as a parameter, $x_1 = 4\lambda$ (in order to avoid some fractions in the components of the four solutions $X^{(\ell)}$ $(\ell = 1,3,4,5).$ The follow from the transformed matrix in (2.59, with $A_p$ replaced by the identity matrix $I_2$ :

$$X^{(1)} = \begin{bmatrix} 4\lambda \\ -1-8\lambda \\ 1-5\lambda \end{bmatrix}, \quad X^{(3)} = \begin{bmatrix} 4\lambda \\ -8\lambda \\ -5\lambda \end{bmatrix}, \tag{2.60}$$

$$X^{(4)} = \begin{bmatrix} 4\lambda \\ 2-8\lambda \\ 1/2-5\lambda \end{bmatrix}, \quad X^{(5)} = \begin{bmatrix} 4\lambda \\ -1-8\lambda \\ -1/2-5\lambda \end{bmatrix}. \tag{2.61}$$

**Checking**. These four solutions can be easily checked (as we did in the previous example and with other examples, as well). But the $4-$ by $-4$ matrix
$X(\lambda) = \left[ X^{(1)} \;\; X^{(3)} \;\; X^{(4)} \;\; X^{(5)} \right]$ has to be multiplied, at left, by the initial $3-$ by $-3$ coefficient matrix $A$ of (2.55), although it was a singular matrix. In this way we are going to obtain four column vectors in $\mathbb{R}^3$ that will be compared with the four free vectors in the statement.

$$A\,X(\lambda) = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \\ 7 & 1 & 4 \end{bmatrix} \begin{bmatrix} 4\lambda & 4\lambda & 4\lambda & 4\lambda \\ -1-8\lambda & -8\lambda & 2-8\lambda & -1-8\lambda \\ 1-5\lambda & -5\lambda & 1/2-5\lambda & -1/2-5\lambda \end{bmatrix} =$$

$$= \begin{bmatrix} 8\lambda - 1 - 8\lambda & 8\lambda - 8\lambda & \cdots & \cdots \\ 12\lambda + 1 + 8\lambda + 4 - 20\lambda & 12\lambda + 8\lambda - 20\lambda & \cdots & \cdots \\ 28\lambda - 1 - 8\lambda + 4 - 20\lambda & 28\lambda - 8\lambda - 20\lambda & \cdots & \cdots \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & 0 & 2 & -1 \\ 5 & 0 & 0 & -1 \\ 3 & 0 & 4 & -3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}^{(1)} & \boldsymbol{b}^{(3)} & \boldsymbol{b}^{(4)} & \boldsymbol{b}^{(5)} \end{bmatrix}.$$

Hence the four solutions in (2.60) & (2.61) satisfy the four linear systems in the statement (except the second one which has been eliminated). We leave the solutions $X^{(4)}$ & $X^{(5)}$ to to be echecked by the the interested readers.

□

## Exercises - Multiple Linear Systems

**2.4** Determine the nature and find the solutions (that exist) to the mutiple linear systems that follow. They are stated in terms of their augmented matrices, like in (2.50),

$$(i)\quad \widetilde{A} = \left[\begin{array}{rrr|rrrr} 2 & 2 & 3 & 4 & 4 & 5 & 0 \\ 1 & -1 & 0 & 2 & 2 & 0 & 2 \\ -1 & 2 & 1 & -1 & 0 & -2 & 2 \end{array}\right];$$

$$(ii)\quad \widetilde{A} = \left[\begin{array}{rrr|rrr} 1 & 1 & -3 & -1 & 0 & 0 \\ 2 & 1 & -2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 & -2 & 0 \\ 1 & 2 & -3 & 1 & 5 & 0 \end{array}\right];$$

$$(iii)\quad \widetilde{A} = \left[\begin{array}{rrrr|rr} 1 & -2 & 3 & -4 & 4 & 1 \\ 0 & 1 & -1 & 1 & -3 & 0 \\ 1 & 3 & 0 & -3 & 1 & -1 \\ -7 & 3 & 0 & 1 & -3 & 2 \end{array}\right];$$

$(iv)$ Solve the following linear systems (given by their explicit expression and free term vectors).

$$\begin{cases} 2x_1 - x_2 + x_3 \quad + 3x_5 = \\ x_1 - x_2 \quad + 2x_4 - x_5 = \end{cases} \begin{bmatrix} -1 \\ 4 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ 6 \end{bmatrix}.$$

## 2.4 *Matrix equations, inverting matrices*

A set of linear systems with the same coefficients, like those in (2.47), can be equivalently written as follows if the next notations are employed.

$$\left[\boldsymbol{b}^{(1)} \quad \boldsymbol{b}^{(2)} \quad \ldots \quad \boldsymbol{b}^{(p)}\right] \underset{\text{not}}{=} B, \quad \left[X^{(1)} \ X^{(2)} \ \ldots \ X^{(p)}\right] \underset{\text{not}}{=} X. \tag{2.62}$$

Let us recall the set of $p$ linear systems of (2.47), that is $A\,X = \boldsymbol{b}^{(1)}$, $A\,X = \boldsymbol{b}^{(2)}, \ldots,$ $A\,X = \boldsymbol{b}^{(p)}$. Then it is clear that such a set of linear systems, with the same coefficients in the matrix $A$, is equivalent – with the notations in (2.62) – to a matrix equation of the form

$$\boxed{A\,X = B\,.} \tag{2.63}$$

If the matrix $A$ is *square of order* $= n$ and $\operatorname{rank} A = n$, then we have seen that each of the $p$ systems admits a unique solution ; it is a system *of Cramer type*. The augemnted matrix of (2.48) can now be written, with the first notation in (2.62), as

$$\widetilde{A} = [A|B]\,. \tag{2.64}$$

The method of Gaussian elimination allows to transform this matrix into a quasi-diagonal form, with the identity matrix $I_n$ instead of the left block $A$, that is

$$\widetilde{A} = [A|B] \rightarrow \ \ldots \ \rightarrow \left[I_n | \overline{\overline{B}}\right]. \tag{2.65}$$

The first block matrix in (2.65) corresponds to the matrix equation (2.63) which is equivalent to the multiple (nonhomogeneous) system $A X = \left[\boldsymbol{b}^{(1)} \ \boldsymbol{b}^{(2)} \ \ldots \ \boldsymbol{b}^{(p)}\right]$. The last matrix in the sequence of transformations (2.65) correponds to the matrix equation

$$I_n X = \overline{\overline{B}} \iff X = \overline{\overline{B}}\,. \tag{2.66}$$

Hence the right block in the last matrix of (2.65) gives just the matrix solution to the matrix equation (2.63). The size of the unknown matrix $X$ of (2.63) and of the solution (with the same notation) of (2.66) depends on the size of the right side matrix $B$. We have earlier assumed that $A$ is square of order $= n$ (and that $|A| \neq 0$). It follows from the first notation in (2.62) that $B$ is a matrix of size $n - \text{by} - p$. Hence the unknown / solution matrix $X$ is of the same size as $B$, $n \times p$.

This discussion shows that a matrix equation like (2.63) can be solved by the Gaussian elimination method, applied to the block matrix $[A|B]$. In the particular case when the right-side matrix is also a square matrix of order $n$, the matrix $X$ is also a square matrix of order $n$. Taking into account the *uniqueness* of the solution of any Cramer-type system, it follows that each column vector $X^{(k)}$ in matrix $X$ is unique, hence the *matrix solution* $X$ is itself *unique.*

**Example 2.7.** It is required to solve the matrix equation $A\,X = B$ where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 5 & -2 & 1 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 0 & 2 & -1 & 5 \\ 3 & 0 & 0 & 2 \\ -4 & 0 & 1 & -1 \end{bmatrix} \tag{2.67}$$

***Solution.*** (2.67) $\Rightarrow$

$$\Rightarrow [A|B] = \left[\begin{array}{rrr|rrrr} 1 & -1 & 1 & 0 & 2 & -1 & 5 \\ 2 & 1 & 1 & 3 & 0 & 0 & 2 \\ 5 & -2 & 1 & -4 & 0 & 1 & -1 \end{array}\right] \backsim \left[\begin{array}{rrr|rrrr} -4 & 1 & 0 & 4 & 2 & -2 & 6 \\ -3 & 3 & 0 & 7 & 0 & -1 & 3 \\ 5 & -2 & 1 & -4 & 0 & 1 & -1 \end{array}\right] \backsim$$

$$\backsim \left[\begin{array}{rrr|rrrr} -1 & -2 & 0 & -3 & 2 & -1 & 3 \\ -3 & 3 & 0 & 7 & 0 & -1 & 3 \\ 5 & -2 & 1 & -4 & 0 & 1 & -1 \end{array}\right] \backsim \left[\begin{array}{rrr|rrrr} 1 & 2 & 0 & 3 & -2 & 1 & -3 \\ 0 & 9 & 0 & 16 & -6 & 2 & -6 \\ 0 & -12 & 1 & -19 & 10 & -4 & 14 \end{array}\right]$$

$$\backsim \left[\begin{array}{rrr|rrrr} 1 & 2 & 0 & 3 & -2 & 1 & -3 \\ 0 & 1 & 0 & 16/9 & -2/3 & 2/9 & -2/3 \\ 0 & -12 & 1 & -19 & 10 & -4 & 14 \end{array}\right] \backsim$$

$$\backsim \left[\begin{array}{rrr|rrrr} 1 & 0 & 0 & -5/9 & -2/3 & 5/9 & -5/9 \\ 0 & 1 & 0 & 16/9 & -2/3 & 2/9 & -2/3 \\ 0 & 0 & 1 & 21/9 & 18 & -4/3 & 18/3 \end{array}\right]. \tag{2.68}$$

**Remarks**. The solution of equation $AX = B$, with the matrices in (2.67), is the right block of the $3 \times 7$ matrix in (2.68):

$$X = \begin{bmatrix} -5/9 & -2/3 & 5/9 & -5/9 \\ 16/9 & -2/3 & 2/9 & -2/3 \\ 21/9 & 18 & -4/3 & 18/3 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -5 & -6 & 5 & -5 \\ 16 & -6 & 2 & -6 \\ 21 & 162 & -12 & 54 \end{bmatrix}. \tag{2.69}$$

The interested reader can check the solution in (2.68), with $A$ & $B$ of (2.67). An alternative way consists in obtaining the inverse $A^{-1}$ and calculating the product $A^{-1}B$ which sould be $= X$ of (2.69). A method for getting the inverse bay elementary transformations is presented in what follows, but it is possible to finde the inverse by the formula involving the determinant of the matrix and the cofactor (or adjugate) matrix $A^\star$. We give this inverse below:

$$A^{-1} = \frac{1}{9} \begin{bmatrix} 3 & -1 & -2 \\ 3 & -4 & 1 \\ -9 & -3 & 3 \end{bmatrix}.$$

□

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## The inverse of a matrix.

**Definition 2.1**. Given a square matrix (of order $n$) $A$, another matrix $B$ of the same size is said to be an *inverse* of $A$ if

$$\boxed{AB = BA = I_n.} \tag{2.70}$$

**Proposition 2.1**. *An inverse of a matrix, if it exists, is unique.*

***Proof***. The assertion in the statement can be easily proved (by "*reductio ad absurdum*"). If we assume that another inverse would exist, let it be $C$, then it ought also satisfy the defining equation (2.70) :

$$AC = CA = I_n. \tag{2.71}$$

It was seen, in the subsection on operations with square matrices, that the identity matrix $I_n$ is a *unit element* with respect to to matrix product. Hence

$$(\forall\ Q \in M_n) \quad Q I_n = I_n Q = Q. \tag{2.72}$$

Starting from the matrix $C$, the multiple equations that follow can be derived.

$$C \underset{(2.72)}{=} C I_n \underset{(2.71)}{=} C(AB) = (CA)B \underset{(2.71)}{=} I_n B \underset{(2.72)}{=} B.$$

Thus the uniqueness of the inverse of a matrix is proved and – in what follows – one must speak of *the inverse* and not longer of *an inverse* of a given matrix.

■

The usual notation for the inverse of a matrix is $A^{-1}$ and it follows from (2.70) that its defining equation is

$$\boxed{AA^{-1} = A^{-1}A = I_n.} \tag{2.73}$$

The notion of the inverse and the inverting operator have several remarcable properties.

**Proposition 2.2**. (***Properties of the inverses***)

$(i)$ $A \in M_n$ is invertible $\Leftrightarrow$ $\det A \neq 0$ ;

$(ii)$ $A \in M_n$ is invertible $\Leftrightarrow$ $\det A^{-1} = 1/\det A$ ;

$(iii)$ If $A_1, A_2, \ldots, A_m$ are $m$ square and invertible matrices then

$$\left(A_1\ A_2\ \ldots\ A_m\right)^{-1} = A_m^{-1}\ A_{m-1}^{-1}\ \ldots\ A_1^{-1}. \tag{2.74}$$

***Proofs***. $(i)$ This property follows from the definition in (2.70) and from a property of the determinants involving the matrix product:

$$\det(AB) = \det A\ \det B. \tag{2.75}$$

$$(2.73)\ \&\ (2.75) \Rightarrow \det(AB) = \det A\ \det A^{-1} = \det I_n = 1 \quad \Rightarrow (ii).$$

Thus the second property in the statement is proved, and it obvioulsy implies $(i)$ as well. The equation $\det I_n = 1$ readily follows from the definition of a determinant.

$(iii)$ This property can be obtained by taking the product of $A_1, A_2, \ldots, A_m$ by the right-side expression of its inverse $\left(A_1\ A_2\ \ldots\ A_m\right)^{-1}$ in (2.71) , using the associativity of the matrix product.

$$\left(A_1\ A_2\ \ldots\ A_m\right)\left(A_m^{-1}\ A_{m-1}^{-1}\ \ldots\ A_1^{-1}\right) = A_1\ A_2\ \ldots\ A_{m-1}\left(A_m\, A_m^{-1}\right)A_{m-1}^{-1}\ \ldots\ A_1^{-1} \underset{(2.73)}{=}$$
$$= A_1\ A_2\ \ldots\ A_{m-1}\ I_n\ A_{m-1}^{-1}\ \ldots\ A_1^{-1} = A_1\ A_2\ \ldots\ \left(A_{m-1}\ A_{m-1}^{-1}\right)\ \ldots\ A_1^{-1} = \ldots$$
$$\ldots = A_1\ I_n\ A_1^{-1} = A_1\ A_1^{-1} \underset{(2.70)}{=} I_n.$$

■

### Inverting a matrix by Gaussian elimination

In view of the definition (2.70) of $A^{-1}$ and the matrix equation (2.63), it follows that this inverse is the (unique) solution of the equation

$$\boxed{A\,X = I_n\ .} \tag{2.76}$$

Since $A$ is a non-singular square matrix, a sequence of transformations like the one in (2.65) can be applied to the block matrix $\left[\,A\,|\,I_n\,\right]$ :

$$\boxed{\left[A\,|\,I_n\right] \to\ \ldots\ \ \to [I_n\,|\,A^{-1}]\,.} \tag{2.77}$$

The theoretical ground of this method to obtain the inverse of an invertible matrix is offered by the (earlier presented) technique for solving multiple non-homogeneous systems, by the *Gaussian elimination.* Indeed, we have to take – instead of the columns of $B$ in (2.62) – the columns of the identity matrix $I_n$ :

$$\boldsymbol{b}^{(j)} \curvearrowright e^j,\quad e^j = \left[0\ \ldots\ \ 0\ 1\ 0\ \ldots\ \ 0\ \right]^{\mathrm{T}}\quad (j = \overline{1,n}) \tag{2.78}$$

with the component $1$ on the $j-$ th position. Each column vector of the matrix solution to Eq. (2.73) will be the corresponding column of the inverse matrix $A^{-1}$.

**Example 2.8.** It is required to find (by Gaussian elimination) the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}.$$

$$\left[A\,|\,I_3\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 1 & 0 \\ 1 & 5 & 12 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 9 & -1 & 0 & 1 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 3 & 2 & -3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2/3 & -1 & 1/3 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11/3 & -3 & 1/3 \\ 0 & 1 & 0 & -7/3 & 3 & -2/3 \\ 0 & 0 & 1 & 2/3 & -1 & 1/3 \end{array}\right] \Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}.$$

The reader is invited to check this inverse by the definition (2.73).

□