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IMPULSIVE CONTROL STRATEGIES FOR PEST MANAGEMENT

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In this paper, we propose two impulsive differential systems concerning biological and, respectively, integrated pest management strategies. In each case, it is observed that there exists a globally asymptotically stable susceptible pest-eradication periodic solution on condition that the amount of infective pests released periodically is larger than a certain critical value. When the amount of infective pests released is less than this critical value, the system is shown to be permanent, which implies that the trivial susceptible pest-eradication solution loses its stability. Further, the existence of a non-trivial periodic solution is also studied by means of numerical simulations. In the case in which a single control is used, one can only use the amount of infective pests which are periodically released in order to control pests at desirable low levels, while in the case in which integrated management is used, one can use the proportion of pests removed by means of spraying chemical pesticides together with the amount of infective pests which are periodically released to control pests at desirable low levels.

Keywords: Impulsive Control; Chemical Pesticides; Epidemics; Susceptible Pests; Infective Pests.

1. Introduction

The history of pest control probably began with our primitive ancestors who ever swatted a mosquito or picked off a louse. From the fossil record, we know that all major taxa of biting flies and external parasites already existed by the time Homo sapiens first appeared on Earth. Phthirus and Pediculus, the two genera of lice that feed on humans, have a host range that is limited to primates (apes and monkeys). Pest control strategies were mentioned occasionally in writings of the ancient Chinese, Sumerian, and Egyptian scholars. Many of these tactics were embedded in religion or superstition, but a few had real scientific merit. Predatory ants, for example, were used in China as early as 1200 BC to protect citrus groves from caterpillars and wood boring beetles. A passage in Homer's Iliad (eighth century BC) describes the use of fire to drive locusts into the sea, and the ancient Egyptians organized long lines of human drovers to repel swarms of invading locusts. Nowadays, we can choose from many different methods as we plan our strategy for controlling pests. Sometimes a non-chemical method of control is as effective and convenient as a chemical alternative. For many pests, total elimination is almost impossible, but it is still possible to keep pests at acceptably low levels. In this regard, knowing the options is the key to pest control. Methods available include pest prevention, non-chemical pest controls, and chemical pesticides. The most effective strategy for pest control may consist in combining the above methods in an approach known as integrated pest management (IPM) that emphasizes reducing pests to tolerable levels, with little cost to the grower and minimal possible hazard to people, property, and environment. The concept of integrated pest management (IPM) was introduced in the late 1950s and was widely practised during the 1970s and 1980s.[1]

Non-chemical pest control methods really work, and they have many advantages. Compared to chemical treatments, non-chemical methods are generally effective for longer periods of time. They are also less likely to create pest populations that develop the ability to resist pesticides and many non-chemical pest controls can be used with fewer safeguards, because they are generally thought to pose virtually no hazard to human health or the environment. An example of non-chemical pest control methods is biological treatment. Biological control is, generally, man's use of a suitably chosen living organism in order to control a particular pest. This chosen organism might be a predator, parasite, virus or bacterium that either kills the harmful pest or interferes with its biological processes.[2–8]

For example, the scientific approach to biological control began with the dramatic and successful control of the cottony-cushion scale, *Icerya purchasi* (*Mask*)., by the introduction of the vedalis beetle (ladybird), *Rodolia cardinalis* (*Muls.*), into California in 1888.[9] Another example is the control of the Asian Tiger Mosquito (*Aedes albopictus*), which can transmit viruses, especially dengue fever virus, Ross River fever virus, Barmah Forest virus and Japanese encephalitis virus. To control the spread of the Asian Tiger Mosquito, we could spray with Bti, which is a variety of the bacterium *Bacillus thuringiensis* (BT), which occurs naturally and is commonly found in soils worldwide. BT was first discovered infecting silkworms over 90 years ago in Japan, where it became known as Sotto disease. A commercial BT product was first registered in the United States in 1958; by 1960 it was cleared for use on food crops and in 1961 it was registered for use in Canada. It is now the most widely used naturally occurring pest control product in the world.

Note that insect pathogens are used in two ways. In the first method, a small amount of pathogen is introduced into a pest population with the expectation that it will generate an epidemic which will subsequently remain endemic. In the second method, an insect pathogen is used like biopesticides. In this case, the pathogen is applied whenever the pest population reaches an economically significant level and there is no expectation that the pathogen will survive for an appreciable amount of time.

Another important method for pest control is chemical control. Since the late 1800s, entomologists and chemists have made outstanding progress in the technology of pest control. A chemical pesticide is defined as a synthetic substance, used for protecting plants, wood or other plant products from harmful organisms, for killing harmful organisms or for controlling the effects of harmful organisms. This definition includes fungicides, insecticides, herbicides and rodenticides. Chemical pesticides kill the pest directly, usually by exposing it to lethal substances or unsuitable environmental conditions, reduce the reproductive potential of a pest population, often by modifying its environment (biotic or abiotic) or by restricting its movement, and modify the behavior of the pest to make it less troublesome (attract, repel, confuse, exclude or mislead it). Farmers can use relatively simple techniques to monitor the increase in insect pest numbers. Combining with an understanding of their life cycles, farmers spray the correct amount of pesticides at the effective time in order to maintain pest population at tolerable level.

To the best of our knowledge, there is a vast amount of literature on the applications of entomopathogens or chemical pesticides to suppress pests (see Refs. 10 to 17 and bibliographies cited therein). However, there are only a few papers and books on mathematical models of the dynamics of microbial diseases and chemical synthetic substances in pest control.[18–20] How many infective pests do we release (we are interested in the situation when environmental conditions do not allow a significant epidemic to be generated if only a small amount of pathogen is introduced into a pest population)? What proportion do we need to kill the pests by spraying chemical pesticides? How do we evaluate the maximum amount (or the maximum period) of an impulsive effect according to the parameters of the system? These are the questions to be answered in order to ensure the success of our pest control strategy.

The main purpose of this paper is to construct two realistic models of systems of impulsive control strategy for pest management, and investigate their dynamics. In this regard, equations with impulsive effects describing evolution processes are characterized by the fact that at certain moments of time they abruptly experience a change of state. Processes of such character are studied in almost every domain of applied science. Numerous examples are given in certain books.[21,22] Impulsive systems have been recently introduced into population dynamics in relation to impulsive vaccination,[23,24] population ecology,[25–29] chemostat model,[30] the chemotherapeutic treatment of disease,[31] impulsive birth,[32] and boundary value problems.[33]

In Sec. 2, we introduce our above-mentioned realistic models. In Sec. 3, we give some notations and lemmas. In Sec. 4, by using Floquet's theorem, small-amplitude perturbation methods and comparison techniques, we consider the local stability and global asymptotic stability of the so-called susceptible pest-eradication periodic solution corresponding to each model. Next, we prove that each system is permanent. In the last section, numerical simulations are used to show the existence of positive periodic endemic solutions and of other rich dynamics of our models. A brief discussion of the results is also given.

2. Model Formulation

In Goh[19] the following two models were proposed:

$$\begin{cases} \dot{S} = -rSI, \\ \dot{I} = rSI - wI \end{cases} \tag{2.1}$$

and

$$\begin{cases} \dot{S} = -rSI, \\ \dot{I} = rSI - wI + u, \end{cases} \tag{2.2}$$

where $S(t)$ denote the number of susceptible pests and $I(t)$ denote the number of infective pests. The parameter w is the death rate of the infective pest population, and r is the infection rate. The control variable $u(t)$ represent the release rate of pests infected in a laboratory.
The following assumptions are made in order to formulate our mathematical models.

(A1): In the absence of the pathogen, the susceptible pest population S grows according to a logistic fashion with carrying capacity $K(> 0)$, and with an intrinsic birth rate equal to 1. Infective pests do not reproduce neither in the biological control model nor in the biological and chemical control model. Also, in the biological and chemical control model below, the infective pest population I contributes together with the susceptible pest population S to population growth towards the carrying capacity of the environment. However, population I does not contribute to population growth towards the carrying capacity of the environment for the biological control model.

(A2): The transmission term has the form

$$\frac{\beta I(t)}{1 + aS^l(t)},$$

where a and $l(\leq 1)$ are positive constants.

(A3): In the biological control model, the action of releasing pests which are infected by a pathogen in laboratories is impulsive and periodic. In the biological and chemical control model, pesticides are also sprayed in an impulsive and periodic fashion, with the same period T but at different moments than those at which infective pests are released.

(A4): The infective pest population does not recover and cannot attack crops.

Note that for $a = 0$ one obtains Goh's infection rate in (A2) and that our model accounts for the effects of crowding, unlike Goh's. Now following the above assumptions we develop the model (2.2) and write the following impulsive epidemic models describing the evolution of susceptible and infective pests.

$$\begin{cases} \left.\begin{array}{l} \dot{S}(t) = S(t)\left(1 - \frac{S(t)}{K}\right) - \frac{\beta I(t)S(t)}{1+aS^l(t)}, \\ \dot{I}(t) = \frac{\beta I(t)S(t)}{1+aS^l(t)} - wI(t), \end{array}\right\} & t \neq nT, \\ \Delta I(t) = \mu, & t = nT \end{cases} \tag{2.3}$$

and

$$\begin{cases} \left.\begin{array}{l} \dot{S}(t) = S(t)\left(1 - \frac{S(t)+I(t)}{K}\right) - \frac{\beta I(t)S(t)}{1+aS^l(t)}, \\ \dot{I}(t) = \frac{\beta I(t)S(t)}{1+aS^l(t)} - wI(t), \end{array}\right\} & t \neq (n+\widetilde{l}-1)T, \quad t \neq nT, \\ \left.\begin{array}{l} \triangle S(t) = -p_1 S(t), \\ \triangle I(t) = -p_2 I(t), \end{array}\right\} & t = (n+\widetilde{l}-1)T, \\ \left.\begin{array}{l} \triangle S(t) = 0, \\ \triangle I(t) = \mu, \end{array}\right\} & t = nT, \end{cases} \tag{2.4}$$

where $0 < \widetilde{l} < 1$, $\triangle S(t) = S(t^+) - S(t)$, and $\triangle I(t) = I(t^+) - I(t)$. Also, $0 < p_1 < 1$ ($0 < p_2 < 1$) represents the fraction of susceptible pests (respectively of infective pests) which die due to pesticide spraying at $t = (n + \widetilde{l} - 1)T$. The quantity $\mu > 0$ represents the release amount, at $t = nT$ of infective pests which are bred in laboratories in order to drive target susceptible pests to contracting the disease. Also, $n \in Z_+$ and $Z_+ = \{1, 2, \ldots\}$. T is the period of the impulsive effect, and the biological meanings of other coefficients are the same as for the model (2.1). That is, we can use a single biological control strategy or a combination of biological and chemical methods to eradicate the susceptible pests or keep the susceptible pest population below the damage level.

3. Preliminary

In this section, we will give some definitions, notations and some lemmas which will be useful for our main results.

Let $R_+ = [0, \infty)$, $R_+^2 = \{x \in R^2 : x > 0\}$. Denote $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2)^T$ the map defined by the right hand side of the first two equations in system (2.3) and $f = (f_1, f_2)^T$ the map defined by the right hand side of the first two equations in system (2.4).

Let $V : R_+ \times R_+^2 \to R_+$. Then V is said to belong to class V_0 if

(i) V is continuous in $((n-1)T, (n+\mathcal{L}-1)T] \times R_+^2$ and $((n+\mathcal{L}-1)T, nT] \times R_+^2$ and for each $x \in R_+^2$, $n \in Z_+$, $\lim_{(t,y)\to((n+\mathcal{L}-1)T^+,x)} V(t,y) = V((n+\mathcal{L}-1)T^+, x)$ and $\lim_{(t,y)\to(nT^+,x)} V(t,y) = V(nT^+, x)$ exist and are finite, where $0 < \mathcal{L} \leq 1$.
(ii) V is locally Lipschitzian in x.

Definition 3.1. $V \in V_0$. Then for $(t, x) \in ((n-1)T, (n+\mathcal{L}-1)T] \times R_+^2$ and $((n+\mathcal{L}-1)T, nT] \times R_+^2 (0 < \mathcal{L} \leq 1)$, the upper right derivative of $V(t, x)$ with respect to the impulsive differential system (2.3) (or (2.4)) is defined as

$$D^+V(t,x) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h, x+hf(t,x)) - V(t,x)].$$

The solution of system (2.3) (or (2.4)), denoted by $x(t) = (S(t), I(t)) : R_+ \rightarrow R_+^2$, is continuously differentiable on $((nT, (n+1)T] \times R^2)$ (or on $((n-1)T, (n+\widetilde{l}-1)T] \times R_+^2$ and $((n+\widetilde{l}-1)T, nT] \times R_+^2$, $0 < \widetilde{l} < 1$). Obviously, the global existence and uniqueness of solutions of system (2.3) (or (2.4)) is guaranteed by the smoothness properties of $\mathfrak{f}$ (or f) (see Lakshmikanthan *et al.*[21] and Bainov and Simeonov[22] for details on fundamental properties of impulsive systems). The following Lemmas are obvious.

Lemma 3.1. *Suppose $x(t)$ is a solution of* (2.3) (*or* (2.4)) *with $x(0^+) \geq 0$. Then $x(t) \geq 0$ for $t \geq 0$. Further, if $x(0^+) > 0$, then $x(t) > 0$ for $t \geq 0$.*

Lemma 3.2. *There exists a constant $M > 0$ such that $S(t) \leq M$ and $I(t) \leq M$ for each solution $x(t)$ of system* (2.3) (*or* (2.4)) *and t large enough.*

Lemma 3.3. *Let $V : R_+ \times R^2 \rightarrow R_+$ and $V \in V_0$. Assume that*

$$\begin{cases} D^+V(t, X) \leq g(t, V(t, X)), & t \neq (n+\mathcal{L}-1)T,\ t \neq nT, \\ V(t, X(t^+)) \leq \Psi_n^{(1)}(V(t, X(t))), & t = (n+\mathcal{L}-1)T, \\ yV(t, X(t^+)) \leq \Psi_n^{(2)}(V(t, X(t))), & t = nT, \end{cases} \tag{3.1}$$

where $g : R_+ \times R_+^2 \rightarrow R_+^2$ is continuous on $((n-1)T, (n+\mathcal{L}-1)T]$ and $((n+\mathcal{L}-1)T, nT]$, $0 < \mathcal{L} \leq 1$. Assume also that for each $v \in R_+^2$ and $n \in N$,

$$\lim_{(t,v)\rightarrow((n+\mathcal{L}-1)T^+, v)} g(t, v) = g((n+\mathcal{L}-1)T^+, v)$$

and

$$\lim_{(t,y)\rightarrow(nT^+, v)} g(t, y) = g(nT^+, v)$$

exist and are finite, where $\Psi_n^{(i)}(i = 1, 2) : R_+ \rightarrow R_+^2$ are quasi-monotone non-decreasing.[22] *Let $R(t, 0, U_0)$ be the maximal solution of the scalar impulsive differential equation*

$$\begin{cases} U'(t) = g(t, U), & t \neq (n+\mathcal{L}-1)T, \quad t \neq nT, \\ U(t^+) = \Psi_n^{(1)}(U(t)), & t = (n+\mathcal{L}-1)T, \\ U(t^+) = \Psi_n^{(2)}(U(t)), & t = nT, \\ U(0^+) = U_0. \end{cases} \tag{3.2}$$

defined on $[0, \infty)$. Then $V(0^+, X_0) \leq U_0$ implies that $V(t, X(t)) \leq R(t)$, $t \geq 0$, where $X(t) = X(t, 0, X_0)$ is any solution of (2.3) *or* (2.4) *defined on* $[0, \infty)$.

Proof. For $t \in [0, \mathcal{L}T]$, we have by the classical comparison theorem $V(t, X(t)) \leq R(t)$. Hence, according to the facts that $\Psi_1^{(1)}$ is quasi-monotone non-decreasing and $V(\mathcal{L}T, X(\mathcal{L}T)) \leq R(\mathcal{L}T)$, we obtain

$$V(\mathcal{L}T^+, X(\mathcal{L}T^+)) \leq \Psi_1^{(1)}(V(\mathcal{L}T, X(\mathcal{L}T))) \leq \Psi_1^{(1)}(R(\mathcal{L}T)) = R(\mathcal{L}T^+).$$

For $t \in (\mathcal{L}T, T]$, it follows, using again the classical comparison theorem, that $V(t, X(t)) \leq R(t)$. Since $\Psi_1^{(2)}$ is quasi-monotone non-decreasing and $V(T, X(T)) \leq R(T)$, we get

$$V(T^+, X(T^+)) \leq \Psi_1^{(2)}(V(T, X(T))) \leq \Psi_1^{(2)}(R(T)) = R(T^+).$$

Thus, for $t \in [0, T]$, it follows $V(t, X(t)) \leq R(t)$. Repeating this argument, we finally arrive at the desired result. This completes the proof. □

When all the directions of the inequalities in (3.1) are reversed, by using a method similar to the one employed in the above similar method, it easily follows from $V(0^+, X_0) \geq U_0$ that $V(t, X(t)) \geq R(t)$. Note that if we have some smoothness conditions of g to guarantee the existence and uniqueness of solutions for (3.2), then $R(t)$ is exactly the unique solution of (3.2).

Next, we consider the following sub-systems of systems (2.3) and (2.4), respectively:

$$\begin{cases} I'(t) = -wI(t), & t \neq nT, \\ \Delta I(t) = \mu, & t = nT, \\ I(0^+) = I_0. \end{cases} \tag{3.3}$$

and

$$\begin{cases} I'(t) = -wI, & t \neq (n + \widetilde{l} - 1)T, t \neq nT, \\ \triangle I(t) = -p_2 I(t), & t = (n + \widetilde{l} - 1)T, \\ \triangle I(t) = \mu, & t = nT, \\ I(0^+) = I_0. \end{cases} \tag{3.4}$$

Lemma 3.4. *The system* (3.3) *has a positive periodic solution* $I_1^*(t)$ *and for every solution* $I(t)$ *of* (3.3), $|I(t) - I_1^*(t)| \to 0$ *as* $t \to \infty$, *where*

$$I_1^*(t) = \frac{\mu e^{-w(t-nT)}}{1 - e^{-wT}}, \quad nT < t \leq (n+1)T$$

and

$$I_1^*(0^+) = \frac{\mu}{1 - e^{-wT}}.$$

Lemma 3.5. *The system* (3.4) *has a positive periodic solution* $I_2^*(t)$ *and for every solution* $I(t)$ *of* (3.4), $|I(t) - I_2^*(t)| \to 0$ *as* $t \to \infty$, *where*

$$I_2^*(t) = \begin{cases} \dfrac{\mu \exp(-w(t - (n-1)T))}{1 - (1 - p_2)\exp(-wT)}, & (n-1)T < t \leq (n + \widetilde{l} - 1)T, \\ \dfrac{\mu(1 - p_2)\exp(-w(t - (n-1)T))}{1 - (1 - p_2)\exp(-wT)}, & (n + \widetilde{l} - 1)T < t \leq nT, \end{cases}$$

$$I_2^*(0^+) = I_2^*(nT^+) = \frac{\mu}{1 - (1 - p_2)\exp(-wT)}, I_2^*(\widetilde{l}T^+) = \frac{\mu(1 - p_2)\exp(-w\widetilde{l}T)}{1 - (1 - p_2)\exp(-wT)}.$$

Proof. The proof is obvious. In fact, since the solution of (2.4) is

$$I(t) = \begin{cases} (1-p_2)^{n-1}\left(I(0^+) - \dfrac{\mu}{1-(1-p_2)exp(-wT)}\right)\exp(-wt) + I_2^*(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad (n-1)T < t \leq (n+\widetilde{l}-1)T, \\ (1-p_2)^{n}\left(I(0^+) - \dfrac{\mu}{1-(1-p_2)exp(-wT)}\right)\exp(-wt) + I_2^*(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad (n+\widetilde{l}-1)T < t \leq nT, \end{cases}$$

the required convergence results immediately. □

Therefore, systems (2.3) and (2.4), respectively, have the susceptible pest-eradication periodic solution $(0, I_1^*(t))$ and $(0, I_2^*(t))$.

4. Extinction and Permanence

In this section, we first give sufficient conditions which assure the global asymptotic stability of the susceptible pest-eradication periodic solutions $(0, I_1^*(t))$ and $(0, I_2^*(t))$ of the above-mentioned models (2.3) and (2.4), respectively.

Theorem 4.1. *The susceptible pest-eradication periodic solution* $(0, I_1^*(t))$ *of* (2.3) *is globally asymptotically stable provided that the inequality*

$$\mu > \frac{wT}{\beta} \tag{4.1}$$

holds.

The proof of Theorem 4.1 is given in Appendix A.

Theorem 4.2. *The susceptible pest-eradication periodic solution* $(0, I_2^*(t))$ *of* (2.4) *is globally asymptotically stable provided that the inequality*

$$\mu > \frac{w\left(T - \ln\frac{1}{1-p_1}\right)(1-(1-p_2)(\exp(-wT)))}{\left(\frac{1}{K}+\beta\right)(1-p_2\exp(-w\widetilde{l}T)-(1-p_2)(\exp(-wT)))} \tag{4.2}$$

holds.

The proof of Theorem 4.2 is given in Appendix B.

Secondly, we will focus on analyzing the permanence of systems (2.3) and (2.4). Before stating our theorems, we give the following definition.

Definition 4.1. The system (2.3) (or (2.4)) is said to be permanent if there are constants m, $M > 0$ (independent of initial value) and a finite time T_0 such that for all solutions $S(t)$, $I(t)$ with all initial values $S(0^+) > 0$, $I(0^+) > 0$, $m \leq S(t) \leq M$, $m \leq I(t) \leq M$ hold for all $t \geq T_0$. Here, T_0 may depend on the initial values $S(0^+)$ and $I(0^+)$.

Theorem 4.3. *The system* (2.3) *is permanent provided*

$$\mu < \frac{wT}{\beta} \tag{4.3}$$

holds.

The proof of Theorem 4.3 is given in Appendix C.

Theorem 4.4. *The system* (2.4) *is permanent provided*

$$\mu < \frac{w\big(T - \ln\frac{1}{1-p_1}\big)(1-(1-p_2)(\exp(-wT)))}{(\frac{1}{K}+\beta)(1-p_2\exp(-w\widetilde{l}T)-(1-p_2)(\exp(-wT)))} \tag{4.4}$$

holds.

The proof of Theorem 4.4 is given in Appendix D.

From the above, we note that $\frac{\omega T}{\beta}$ and $\frac{w(T-\ln\frac{1}{1-p_1})(1-(1-p_2)(\exp(-wT)))}{(\frac{1}{K}+\beta)(1-p_2\exp(-w\widetilde{l}T)-(1-p_2)(\exp(-wT)))}$, respectively, are threshold parameters for the stability of the systems (2.3) and (2.4), as far as μ is concerned. Hence, we can evaluate the maximal period of the impulsive controls according to the parameters of systems (2.3) and (2.4), and the required percentage of pest removal by spraying chemical pesticides in system (2.4).

5. Discussion and Numerical Analysis

In this paper, we have investigated two pest management strategies which rely on impulsive and periodic controls. We showed that each system has a globally asymptotically stable susceptible pest-eradication periodic solution. Moreover, we gave sufficient conditions for the permanence of the systems and we pointed out the thresholds for the stability of the systems (2.3) and (2.4).

If we choose the single biological control strategy, with the intention of stabilizing the pest population at an acceptably low level, from Theorem 4.1, we have shown that the susceptible pest-eradication periodic solution $(0, I_1^*(t))$ is globally asymptotically stable if $\mu > \widetilde{\mu}_{\max} = \frac{\omega T}{\beta}$. In this section, we are ready to study the influence of impulsive perturbation μ on the system (2.3) to suggest a highly effective method of pest control.

We let $K = 10$, $\beta = 1$, $a = 1$, $l = 0.8$, $w = 0.8$, $T = 1$. From (4.1) and (4.3), we derive that when $\mu > \widetilde{\mu}_{\max} = 0.8$, the susceptible pest-eradication periodic solution is globally asymptotically stable, while for $\mu < \widetilde{\mu}_{\max} = 0.8$, the system (2.3) is permanent.

Figure 1 shows the bifurcation diagrams of system (2.3) with μ varying from 0.001 to 1. When $0.8 < \mu < 1$, all target pests turn into infected pests (see Fig. 4). When $0.53 < \mu < 0.8$, susceptible pests and infected pests ultimately co-exist in the

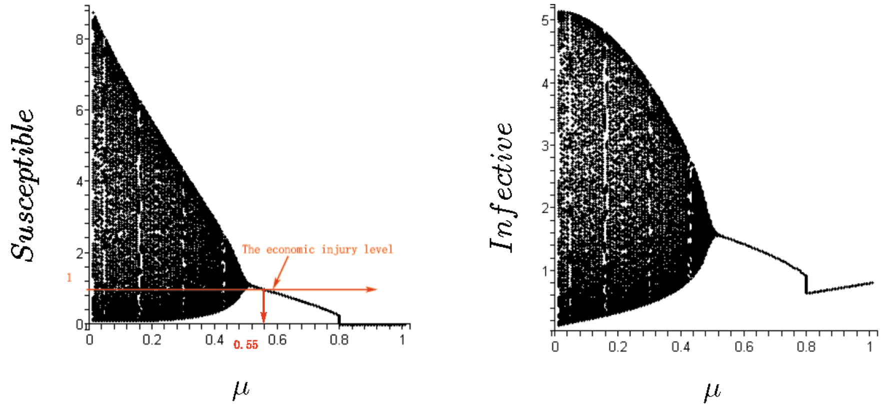

Fig. 1. Bifurcation diagram of system (2.3) for $K = 10$, $\beta = 1$, $a = 1$, $l = 0.8$, $w = 0.8$, $T = 1$ and $0.001 \leq \mu \leq 1$.

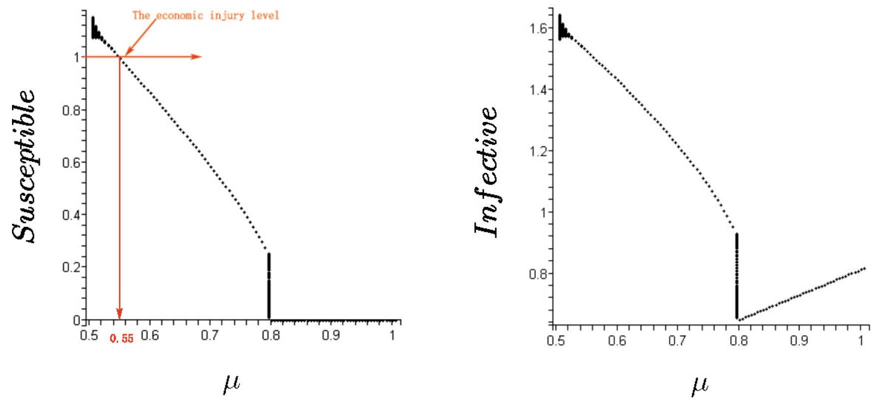

Fig. 2. Bifurcation diagram of system (2.3) for $K = 10$, $\beta = 1$, $a = 1$, $l = 0.8$, $w = 0.8$, $T = 1$ and $0.5 \leq \mu \leq 1$.

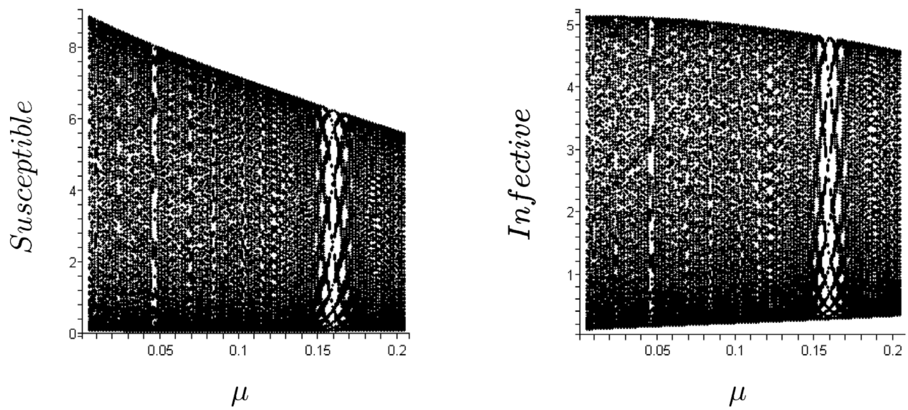

Fig. 3. Bifurcation diagram of system (2.3) for $K = 10$, $\beta = 1$, $a = 1$, $l = 0.8$, $w = 0.8$, $T = 1$ and $0 \leq \mu \leq 0.2$.

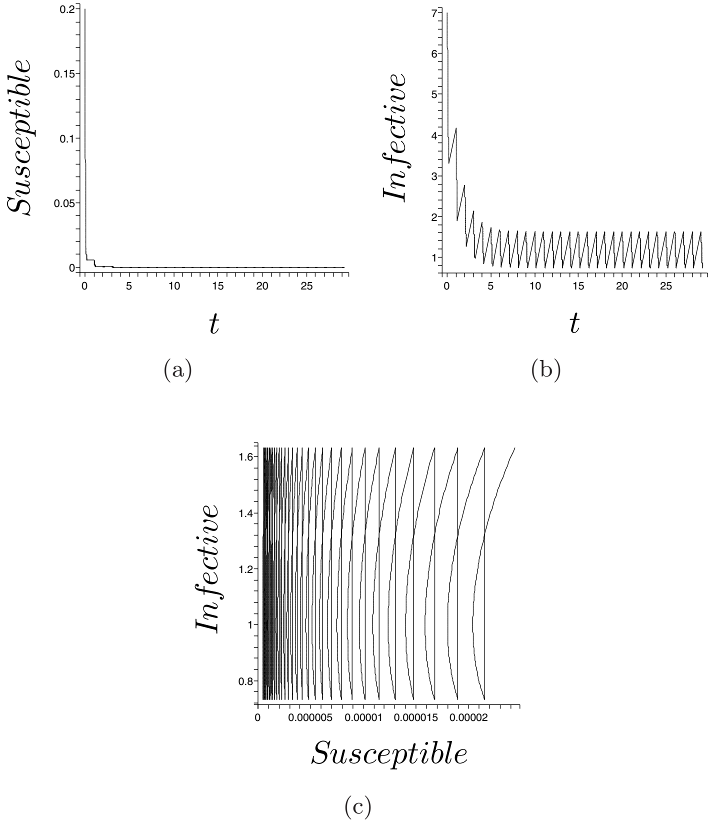

Fig. 4. ($\mu = 0.9$) Dynamical behavior of the system with impulsive control of epidemics for pest control: (a) time-series of the susceptible pest population; (b) time-series of the infective pest population; and (c) phase portraits of system (2.3).

form of a periodic solution (see Fig. 5). A stable periodic solution (see Fig. 5d) is captured when $\mu = 0.6$. With the further decrease of μ, we see that the dynamical behavior of system (2.3) is very complicated (see Fig. 3). In Fig. 6, we may find that there exists a strange attractor for $\mu = 0.35$.

Our aim is to keep susceptible pests at an acceptably low level (below the economic injury level (EIL) that indicates the pest densities (numbers of pests per unit area) at which artificial control measures are economically justified. In other words, at this level the cost of control is less than the loss the farmer, forester, or other resource producer would suffer if control action were not taken[34]) by releasing infected pests: not to infect all pests, only to control susceptible pests with a minimum use of the control variable (the amount of infective pests released).

For example, let $E_0 (= 1 < 1.25$, see Fig. 2) be the number of the susceptible pest population reaching the economic injury level (see Figs. 1 and 2). We only consider controlling the number of susceptible pests as infective pests cannot attack crops.

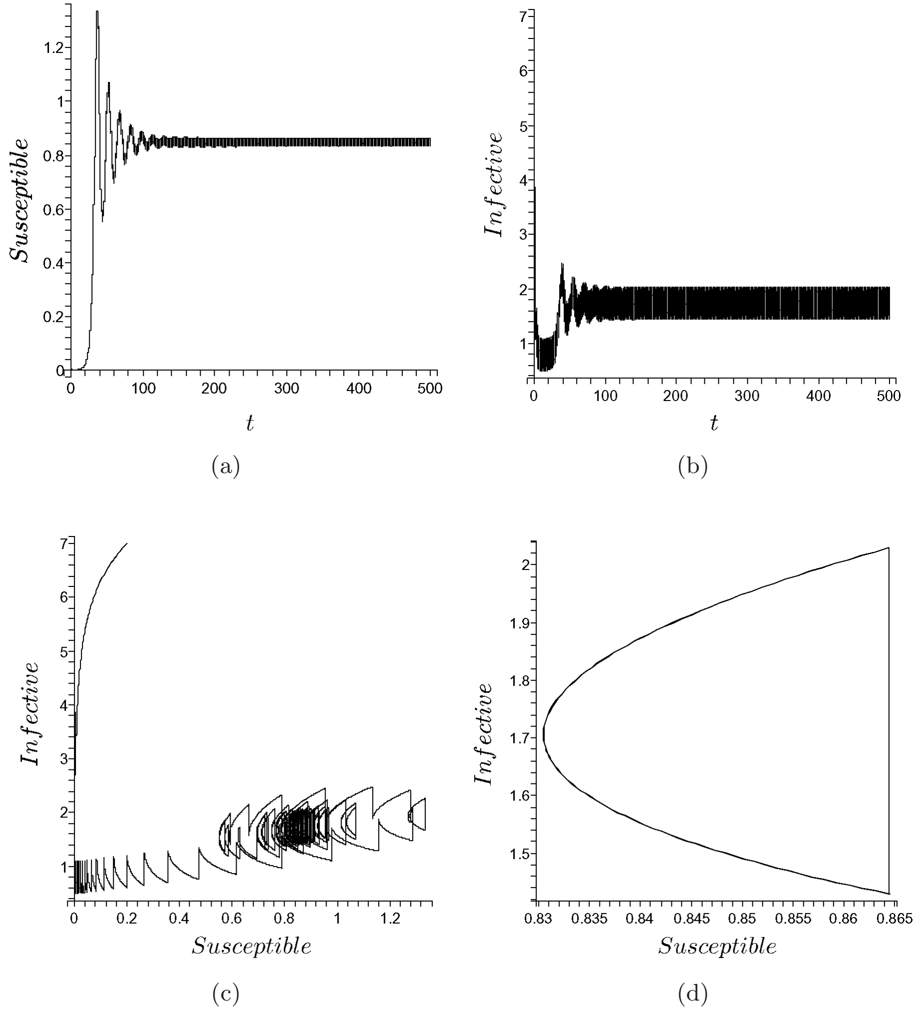

Fig. 5. ($\mu = 0.6$) Dynamical behavior of the system with impulsive control of epidemics for pest control: (a) time-series of the susceptible pest population; (b) time-series of the infective pest population; and (c) and (d) phase portraits of system (2.3).

Recalling Fig. 5, we choose $\mu = 0.6$, then find that $S < E_0$ as $t \geq 75$. Obviously, our strategy to control target pests is successful. If we choose $\mu \leq 0.45$ (see Fig. 1), the system experiences chaotic behavior. Clearly, when $\mu = 0.35$, we know that the number of susceptible pests must exceed E_0 at some time (see Fig. 6). Recalling Fig. 1, if let $E_0 > 1.25$, we see that when $\mu < 0.53$, S may experience chaos. One can choose the release amount of infected pests μ to exceed 0.53, then the number of susceptible pests may be controlled below E_0.

If we choose our mixed impulsive control strategy, which uses a combination of biological and chemical tactics, for the purpose of suppressing the abundance of

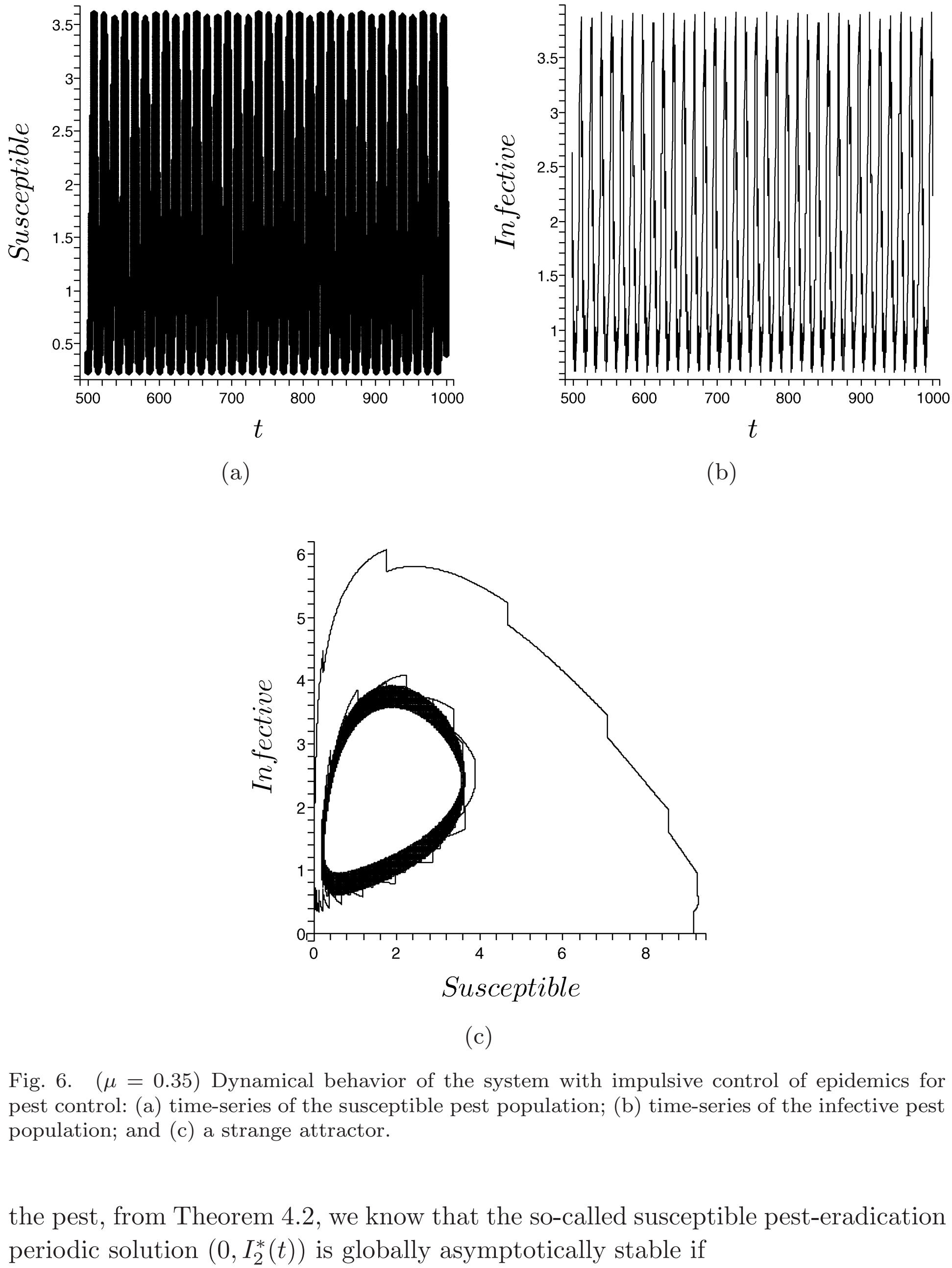

Fig. 6. ($\mu = 0.35$) Dynamical behavior of the system with impulsive control of epidemics for pest control: (a) time-series of the susceptible pest population; (b) time-series of the infective pest population; and (c) a strange attractor.

the pest, from Theorem 4.2, we know that the so-called susceptible pest-eradication periodic solution $(0, I_2^*(t))$ is globally asymptotically stable if

$$\mu > \mu_{\max} = \frac{w(T - \ln \frac{1}{1-p_1})(1 - (1 - p_2)(\exp(-wT)))}{(\frac{1}{K} + \beta)(1 - p_2 \exp(-w\widetilde{l}T) - (1 - p_2)(\exp(-wT)))}.$$

A typical susceptible pest-eradication periodic solution of system (2.4) is shown in Fig. 7, where we observe how the variable $I(t)$ oscillates in a stable cycle. In contrast,

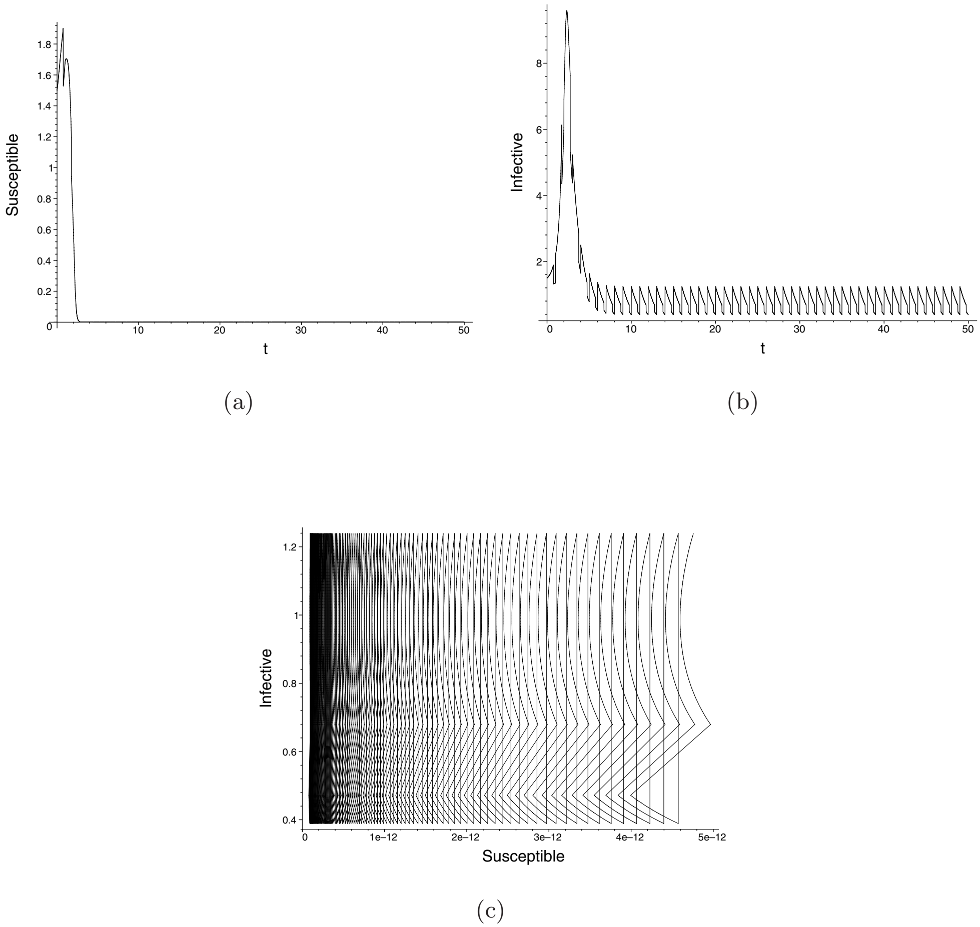

Fig. 7. ($\mu = 0.85$) Dynamical behavior of the system with impulsive control for pest management with $K = 100$, $a = 1$, $l = 0.8$, $w = 0.8$, $\widetilde{l} = 0.75$, $\beta = 1$, $p_1 = 0.2$, $p_2 = 0.3$, $T = 1$: (a) time-series of the susceptible pest population; (b) time-series of the infective pest population; and (c) phase portraits of system (2.4).

the susceptible pest $S(t)$ rapidly decreases to zero and $\mu_{\max} \approx 0.81$. In order to drive the susceptible pest population to extinction, we can determine the impulsive amount μ according to the effect of the chemical pesticides on the pest population and the cost of the releasing infective pests such that $\mu > \mu_{\max}$. With the further decrease of μ, numerical results show that susceptible pests and infective pests can co-exist on a stable limit cycle, which is a global attractor (see Fig. 8). A good pest control program should reduce susceptible pest population to levels acceptable to the public. It will be very interesting to consider the non-autonomous models with impulsive effects corresponding to models (2.3) and (2.4). These issues would be left for future consideration.

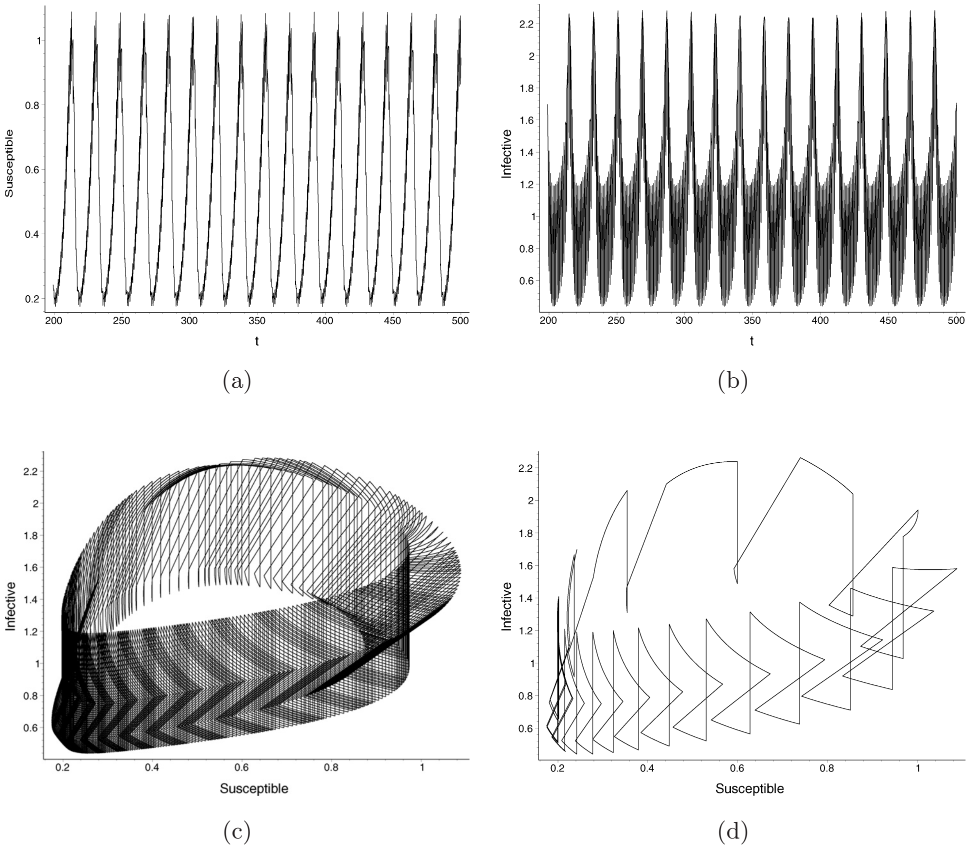

Fig. 8. ($\mu = 0.75$) Dynamical behavior of the system with impulsive control for pest management for $K = 100$, $a = 1$, $l = 0.8$, $w = 0.8$, $\widetilde{l} = 0.75$, $\beta = 1$, $p_1 = 0.2$, $p_2 = 0.3$, $T = 1$: (a) time-series of the susceptible pest population; (b) time-series of the infective pest population; and (c) phase portraits of system (2.4); (d) a global attractor.

Acknowledgments

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Appendix A: Proof of Theorem 4.1

Firstly, we prove the local stability by using small amplitude perturbation methods. Let us denote

$$S(t) = u(t), I(t) = v(t) + I_1^*(t),$$

where u, v are small amplitude perturbations. The system (2.3) can be expanded in a Taylor series. After neglecting higher-order terms, the linearized equations read as

$$\begin{cases} \left.\begin{aligned} \dot{u}_(t) &= u(t) - \beta u(t) I_1^*(t), \\ \dot{v}_(t) &= \beta I_1^*(t) u(t) - wv(t), \end{aligned}\right\} & t \neq nT, \\ \left.\begin{aligned} u(t^+) &= u(t), \\ v(t^+) &= v(t), \end{aligned}\right\} & t = nT. \end{cases} \tag{A.1}$$

Let $\Phi(t)$ be the fundamental matrix of (A.1). Then $\Phi(t)$ must satisfy

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} 1 - \beta I_1^*(t) & 0 \\ \beta I_1^*(t) & -w \end{pmatrix} \Phi(t)$$

and $\Phi(0) = I$. Hence the fundamental solution matrix is

$$\Phi(t) = \begin{pmatrix} \exp\left(\int_0^t (1 - \beta I_1^*(s)) ds\right) & 0 \\ * & \exp(-wt) \end{pmatrix}.$$

It follows from the linearization of the last two equations of (A.1) that

$$\begin{pmatrix} u(nT^+) \\ v(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u(nT) \\ v(nT) \end{pmatrix}.$$

Hence, if both eigenvalues of

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Phi(T)$$

have absolute values less than one, then the periodic solution $(0, I_1^*(t))$ is locally stable. Since the eigenvalues of M are

$$\lambda_1 = \exp(-wT) < 1, \lambda_2 = \exp\left(\int_0^T (1 - \beta I_1^*(s))ds\right),$$

$|\lambda_2| < 1$ if and only if (4.1) holds. According to the Floquet theory of impulsive differential equation (see the above), the periodic susceptible prey-eradication solution $(0, I_1^*(t))$ is locally stable.

Next, we prove the global attractivity. Choose a $\epsilon > 0$ such that:

$$\zeta \doteq \exp\left(\int_0^T \left(1 - \frac{\beta}{1 + aM^l}(I_1^*(s) - \epsilon)\right) ds\right) < 1.$$

Noting that $I'(t) \geq -wI(t)$, from Lemmas 3.3 and 3.4, we have

$$I(t) > I_1^*(t) - \epsilon \tag{A.2}$$

for all t large enough. For simplification, we may assume that (A.2) holds for all $t \geq 0$. From (2.3) and (A.2), we get

$$S'(t) \leq S(t)\left(1 - \frac{\beta}{1 + aM^l}(I_1^*(t) - \epsilon)\right). \tag{A.3}$$

Integrating (A.3) on $(nT, (n+1)T]$, one obtains

$$S((n+1)T) \leq S(nT) \exp\left(\int_{nT}^{(n+1)T} \left(1 - \frac{\beta}{1 + aM^l}(I_1^*(t) - \epsilon)\right) dt\right) = S(nT)\zeta.$$

Thus $S(nT) \leq S(0^+)\zeta^n$ and $S(nT) \to 0$ as $n \to \infty$. Therefore, $S(t) \to 0$ as $t \to \infty$ since $0 < S(t) \leq S(nT)\exp(T)$ for $t \in (nT, (n+1)T]$.

In the following, we prove that $I(t) \to I_1^*(t)$ as $t \to \infty$. For $0 < \epsilon < \frac{w}{\beta}$, there must exist a $\widetilde{T} > 0$ such that $0 < S(t) < \epsilon$ as $t \geq \widetilde{T}$. Without loss of generality, we may assume $0 < S(t) < \epsilon$ as $t \geq 0$. Then, from (2.3), we have

$$-wI(t) \leq I'(t) < (-w + \beta\epsilon)I(t) \quad \text{for } t \geq 0.$$

From Lemmas 3.3 and 3.4, we obtain that $y_1(t) \leq I(t) \leq y_2(t)$ for $t \geq 0$ and consequently $y_1(t) \to I_1^*(t)$, $y_2(t) \to y_2^*(t)$ as $t \to \infty$, where $y_1(t)$ and $y_2(t)$ are the solutions of

$$\begin{cases} y_1'(t) = -wy_1(t), & t \neq nT, \\ \Delta y_1(t) = \mu, & t = nT, \\ y_1(0^+) = I_0 \end{cases}$$

and

$$\begin{cases} y_2'(t) = (-w + \beta\epsilon) y_2(t), & t \neq nT, \\ \Delta y_2(t) = \mu, & t = nT, \\ y_2(0^+) = I_0, \end{cases}$$

respectively, where

$$y_2^*(t) = \frac{\mu \exp((-w + \beta\epsilon)(t - nT))}{1 - \exp((-w + \beta\epsilon)T)}, \quad nT < t \leq (n+1)T.$$

Therefore, $I_1^*(t) - \varepsilon_1 < I(t) < y_2^*(t) + \varepsilon_1$, for t large enough. Let $\epsilon \to 0$, we get $y_2^*(t) \to I_1^*(t)$. Hence $I(t) \to I_1^*(t)$ as $t \to \infty$. This completes the proof. $\square$

Appendix B: Proof of Theorem 4.2

We firstly show the local stability by using small amplitude perturbation methods. Define

$$S(t) = u(t), \quad I(t) = v(t) + I_2^*(t),$$

where u, v are small amplitude perturbations. The system (2.4) can be expanded in a Taylor series. After neglecting higher-order terms, the linearized equations read as

$$\begin{cases} \left.\begin{aligned} \dot{u}(t) &= \big(1 - \big(\tfrac{1}{K} + \beta\big) I_2^*(t)\big) u(t), \\ \dot{v}(t) &= \beta I_2^*(t) u(t) - w v(t), \end{aligned}\right\} & t \neq (n + \widetilde{l} - 1)T, t \neq nT, \\ \left.\begin{aligned} \triangle u(t) &= -p_1 u(t), \\ \triangle v(t) &= -p_2 v(t), \end{aligned}\right\} & t = (n + \widetilde{l} - 1)T, \\ \left.\begin{aligned} \triangle u(t) &= 0, \\ \triangle v(t) &= 0, \end{aligned}\right\} & t = nT. \end{cases} \tag{B.1}$$

Let $\Phi(t)$ be the fundamental matrix of (B.1). Then $\Phi(t)$ must satisfy

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} 1 - \big(\frac{1}{K} + \beta\big) I_2^*(t) & 0 \\ \beta I_2^*(t) & -w \end{pmatrix} \Phi(t)$$

and $\Phi(o) = I$. Hence the fundamental solution matrix is

$$\Phi(t) = \begin{pmatrix} \exp\!\big(\int_0^t \big(1 - \big(\frac{1}{K} + \beta\big) I_2^*(s)\big) ds\big) & 0 \\ * & \exp(-wt) \end{pmatrix}.$$

The linearization of the third and fourth equation in (B.1) becomes

$$\begin{pmatrix} u((n + \widetilde{l} - 1)T^+) \\ v((n + \widetilde{l} - 1)T^+) \end{pmatrix} = \begin{pmatrix} 1 - p_1 & 0 \\ 0 & 1 - p_2 \end{pmatrix} \begin{pmatrix} u((n + \widetilde{l} - 1)T) \\ v((n + \widetilde{l} - 1)T) \end{pmatrix}.$$

The linearization of the fifth and sixth equation in (B.1) becomes

$$\begin{pmatrix} u(nT^+) \\ v(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u(nT) \\ v(nT) \end{pmatrix}.$$

Hence, if both eigenvalues of

$$M^* = \begin{pmatrix} 1-p_1 & 0 \\ 0 & 1-p_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Phi(T)$$

have absolute values less than one, then the periodic solution $(0, I_2^*(t))$ is locally stable. Since the eigenvalues of M^* are

$$\lambda_1 = (1-p_2)\exp(-wT) < 1, \lambda_2 = (1-p_1)\exp\left(\int_0^T \left(1 - \left(\frac{1}{K} + \beta\right) I_2^*(s)\right) ds\right),$$

$|\lambda_2| < 1$ if and only if (4.2) holds. According to Floquet theory of impulsive differential equations, the periodic susceptible prey-eradication solution $(0, I_2^*(t))$ is locally stable.

Next, we prove the global attractivity. Choose $\epsilon > 0$ such that:

$$\zeta \doteq (1-p_1)\exp\left(\int_0^T \left(1 - \left(\frac{1}{K} + \frac{\beta}{1+aM^l}\right)(I_2^*(s)-\epsilon)\right) ds\right) < 1.$$

Noting that $I'(t) \geq -wI(t)$, from Lemmas 3.3 and 3.5, we have

$$I(t) > I_2^*(t) - \epsilon \tag{B.2}$$

for all t large enough. For simplification, we may assume that (B.2) holds for all $t \geq 0$. From (2.4) and (B.2), we get

$$\begin{cases} S'(t) \leq S(t)\left(1 - \left(\dfrac{1}{K} + \dfrac{\beta}{1+aM^l}\right)(I_2^*(t)-\epsilon)\right), & t \neq (n+\widetilde{l}-1)T, \\ S(t^+) = (1-p_1)S(t), & t = (n+\widetilde{l}-1)T. \end{cases} \tag{B.3}$$

Integrating (B.3) on $((n+\widetilde{l}-1)T, (n+\widetilde{l})T]$, one obtains

$$\begin{aligned} S((n+\widetilde{l})T) &\leq S((n+\widetilde{l}-1)T)(1-p_1) \\ &\quad \times \exp\left(\int_{(n+\widetilde{l}-1)T}^{(n+\widetilde{l})T} \left(1 - \left(\frac{1}{K} + \frac{\beta}{1+aM^l}\right)(I_2^*(t)-\epsilon)\right) dt\right) = S((n+\widetilde{l}-1)T)\zeta. \end{aligned}$$

Thus $S((n+\widetilde{l})T) \leq S(\widetilde{l}T)\zeta^n$ and $S((n+\widetilde{l})T) \to 0$ as $n \to \infty$. Hence, $S(t) \to 0$ as $t \to \infty$ since $0 < S(t) \leq S((n+\widetilde{l}-1)T)(1-p_1)\exp(T)$ for $t \in ((n+\widetilde{l}-1)T, (n+\widetilde{l})T]$.

In the following, we are ready to prove that $I(t) \to I_2^*(t)$ as $t \to \infty$. For $0 < \epsilon' < \frac{w}{\beta}$, there must exist a $\widetilde{T} > 0$ such that $0 < S(t) < \epsilon'$ as $t \geq \widetilde{T}$. Without loss of generality, we may assume that $0 < S(t) < \epsilon'$ as $t \geq 0$. Then, from (2.4), we have

$$-wI(t) \leq I'(t) < (-w + \beta\epsilon')I(t) \quad \text{for } t \geq 0.$$

From Lemmas 3.3 and 3.5, we obtain that $y_1(t) \le S(t) \le y_2(t)$ for $t \ge 0 y_1$ and consequently $(t) \to I_2^*(t)$, $y_2(t) \to y_2^*(t)$ as $t \to \infty$, where $y_1(t)$ and $y_2(t)$ are the solutions of

$$\begin{cases} y_1'(t) = -wy_1(t), & t \neq (n+\widetilde{l}-1)T, t \neq nT, \\ \triangle y_1(t) = -p_2 y_1(t), & t = (n+\widetilde{l}-1)T, \\ \triangle y_1(t) = \mu, & t = nT, \\ y_1(0^+) = I_0 \end{cases}$$

and

$$\begin{cases} y_2'(t) = (-w+\beta\epsilon')y_2(t), & t \neq (n+\widetilde{l}-1)T, t \neq nT, \\ \triangle y_2(t) = -p_2 y_2(t), & t = (n+\widetilde{l}-1)T, \\ \triangle y_2(t) = \mu, & t = nT, \\ y_2(0^+) = I_0, \end{cases}$$

respectively, where

$$y_2^*(t) = \begin{cases} \dfrac{\mu \exp((-w+\beta\epsilon')(t-(n-1)T))}{1-(1-p_2)\exp((-w+\beta\epsilon')T)}, & (n-1)T < t \le (n+\widetilde{l}-1)T, \\ \dfrac{\mu(1-p_2)\exp((-w+\beta\epsilon')(t-(n-1)T))}{1-(1-p_2)\exp((-w+\beta\epsilon')T)}, & (n+\widetilde{l}-1)T < t \le nT. \end{cases}$$

Therefore,

$$I_2^*(t) - \varepsilon_1 < I(t) < y_2^*(t) + \varepsilon_1, (\varepsilon_1 > 0)$$

for t large enough. Letting $\epsilon' \to 0$, we derive that $y_2^*(t) \to I_2^*(t)$ and hence $I(t) \to I_2^*(t)$ as $t \to \infty$. This completes the proof. □

Appendix C: Proof of Theorem 4.3

Suppose that $x(t)$ is a solution of (2.3) with $x(0) > 0$. From Lemma 3.2, there exists a positive constant $M(< K)$ such that $S(t) \le M$ and $I(t) \le M$ for t large enough. Without loss of generality, we may assume that $S(t) \le M$, $I(t) \le M$ for $t \ge 0$.

From (A.2), we know that $I(t) > I_1^*(t) - \epsilon$ for t large enough. Consequently, $I(t) \ge \frac{\mu \exp(-wT))}{1-\exp(-wT)} - \epsilon \doteq m_2$ for t large enough. Thus we only need to find $m_1 > 0$, such that $S(t) \ge m_1$ for t large enough. We have split the procedure of finding m_1 in the following two steps for convenience.

Step I. Let $m_3 > 0, \varepsilon > 0$ be small enough such that $\frac{\beta m_3}{1+am_3^l} < w$ and $\eta \doteq \exp\big((1-\frac{m_3}{K})T - \varepsilon\beta T + \frac{\beta\mu}{\frac{\beta m_3}{1+am_3^l}-w}\big) > 1$. We shall show that $S(t) < m_3$ cannot hold for all

$t \geq 0$. Otherwise,

$$\begin{cases} I'(t) \leq \left(-w + \dfrac{\beta S(t)}{1 + aS^l(t)} \right) I(t) \leq \left(\dfrac{\beta m_3}{1 + am_3^l} - w \right) I(t), & t \neq nT, \\ \Delta I(t) = \mu, & t = nT. \end{cases}$$

Then we obtain that $I(t) \leq y_3(t)$ and $y_3(t) \to y_3^*(t)$ as $t \to \infty$, where $y_3(t)$ is the solution of

$$\begin{cases} y_3'(t) = \left(\dfrac{\beta m_3}{1 + am_3^l} - w \right) y_3(t), & t \neq nT, \\ \Delta y_3(t) = \mu, & t = nT, \\ y_3(0^+) = I_0, \end{cases} \tag{C.1}$$

and

$$y_3^*(t) = \frac{\mu \exp\left(\left(\frac{\beta m_3}{1+am_3^l} - w \right)(t - (n-1)T) \right)}{1 - \exp\left(\left(\frac{\beta m_3}{1+am_3^l} - w \right) T \right)}, \quad (n-1)T < t \leq nT. \tag{C.2}$$

Therefore, there exists $\widehat{T} > 0$ such that

$$I(t) \leq y_3(t) < y_3^*(t) + \varepsilon$$

for $t > \widehat{T}$. From (2.3), we have

$$S'(t) \geq \left(1 - \frac{m_3}{K} - \beta(y_3^*(t) + \varepsilon) \right) S(t). \tag{C.3}$$

Let $N \in Z_+$ such that $(N-1)T \geq \widehat{T}$. Integrating (C.3) on $((n-1)T, nT], n \geq N$, we get

$$\begin{aligned} S(nT) &\geq S((n-1)T) \exp\left(\int_{(n-1)T}^{nT} \left(1 - \frac{m_3}{K} - \beta(y_3^*(t) + \varepsilon) \right) dt \right) \\ &= S((n-1)T)\eta. \end{aligned}$$

Then $S((n+k)T) \geq S(nT)\eta^k \to \infty$ as $k \to \infty$, which contradicts to the boundedness of $S(t)$. Thus, there exist a $t_1 > 0$ such that $S(t_1) \geq m_3$.

Step II. If $S(t) \geq m_3$ for all $t \geq t_1$, then our aim is obtained. Otherwise, $S(t) < m_3$ for some $t \geq t_1$. Set $t^* = \inf_{t > t_1}\{S(t) < m_3\}$. Then $S(t) \geq m_3$ for $t \in [t_1, t^*)$ and $t^* \in (n_1 T, (n_1 + 1)T]$, $n_1 \in Z_+$. It is easy to see that $S(t^*) = m_3$ since $S(t)$ is continuous. Choose n_2, $n_3 \in Z_+$ such that

$$\begin{gathered} n_2 T > \frac{1}{\frac{\beta m_3}{1+am_3^l} - w} \ln \frac{\varepsilon_1}{M + \frac{\mu}{1 - \exp\left(\left(\frac{\beta m_3}{1+am_3^l} - w \right) T \right)}}, \\ \exp(n_2 \eta_1 T)\eta^{n_3} > 1, \end{gathered}$$

where $\eta_1 = 1 - \frac{m_3}{K} - \beta M < 0$. Let $T' = n_2 T + n_3 T$. We claim that there exists a $t_2 \in (t^*, t^* + T']$ such that $S(t_2) > m_3$. Otherwise, we consider (C.1) with $y_3(t^{*+}) =$

$I(t^{*+})$ (only if $t^* = nT$ for some $n \in Z_+$) and we see that

$$y_3(t) = \left(y_3((n_1+1)T^+) - \frac{\mu}{1-\exp\left(\left(\frac{\beta m_3}{1+am_3^l} - w\right)T\right)}\right)$$
$$\exp\left(\left(\frac{\beta m_3}{1+am_3^l} - w\right)(t-(n_1+1)T\right) + y_3^*(t),$$

$nT < t \le (n+1)T$ where $n_1 + 1 \le n \le n_1 + n_2 + n_3$. Then $|y_3(t) - y_3^*(t)| < \left(M + \frac{\mu}{1-\exp\left(\left(\frac{\beta m_3}{1+am_3^l} - w\right)T\right)}\right)\exp\left(\left(\frac{\beta m_3}{1+am_3^l} - w\right)(t-(n_1+1)T)\right) < \varepsilon_1$ and $I(t) < y_3(t) \le y_3^*(t) + \varepsilon_1$ for $n_1T + (n_2-1)T \le t \le t^* + T'$, which implies that (C.3) holds for $t \in [(t^* + n_2T), (t^* + T')]$. As in Step I, we get

$$S(t^* + T') \ge S(t^* + n_2T)\eta^{n_3}. \tag{C.4}$$

From (C.3), we get

$$S'(t) \ge \left(1 - \frac{m_3}{K} - \beta M\right)S(t) \tag{C.5}$$

for $t \in [t^*, (t^* + n_2T)]$. Integrating (C.5) on $[t^*, (t^* + n_2T)]$, we obtain

$$S(t^* + n_2T) \ge m_3 \exp(n_2\eta_1 T).$$

Consequently, we have

$$S(t^* + T') \ge m_3 \exp(n_2\eta_1 T)\eta^{n_3} > m_3,$$

which is a contradiction. Let $\widetilde{t} = \inf_{t>t^*}\{S(t) > m_3\}$. Then for $t \in (t^*, \widetilde{t})$, $S(t) \le m_3$ and $S(\widetilde{t}) = m_3$. For $t \in (t^*, \widetilde{t})$, we get

$$S(t) \ge m_3 \exp((n_2 + n_3)\eta_1 T).$$

Let

$\overline{m}_1 = m_3 \exp((n_2 + n_3)\eta_1 T)$. With this notation we have $S(t) \ge \overline{m}_1$ for $t \in (t^*, \widetilde{t})$. For $t > \widetilde{t}$, the same arguments can be continued since $S(t) \ge m_3$. This completes the proof. □

Appendix D: Proof of Theorem 4.4

Suppose that $x(t)$ is a solution of (2.4) with $x(0) > 0$. From Lemma 3.2, there exists a positive constant $M(< K)$ such that $S(t) \le M$ and $I(t) \le M$ for t large enough. Without loss of generality, we may assume that $S(t) \le M$, $I(t) \le M$ for $t \ge 0$.

From (B.2), we know that $I(t) > I_2^*(t) - \epsilon$ for t large enough. Consequently $I(t) \ge \frac{\mu(1-p_2)\exp(-wT))}{1-\exp(-wT)} - \epsilon \doteq m_2$ for t large enough. Thus we only need to find $m_1 > 0$, such that $S(t) \ge m_1$ for t large enough. We have split the procedure of finding m_2 in the following two steps for convenience.

Step I. Let $m_3 > 0$, $\varepsilon > 0$ be small enough such that

$$\frac{\beta m_3}{1+am_3^l} < w$$

and

$$\eta \doteq (1-p_1)\exp\Bigg(\bigg(1-\frac{m_3}{K}\bigg)T-\varepsilon\bigg(\frac{1}{K}+\beta\bigg)T \\ -\frac{\big(\frac{1}{K}+\beta\big)\mu\big(1-p_2\exp\big(\big(\frac{\beta m_3}{1+am_3^l}-w\big)lT\big)-(1-p_2)\exp\big(\big(\frac{\beta m_3}{1+am_3^l}-w\big)T\big)\big)}{\big(w-\frac{\beta m_3}{1+am_3^l}\big)\big(1-(1-p_2)\exp\big(\big(\frac{\beta m_3}{1+am_3^l}-w\big)T\big)\big)}\Bigg) \\ >1,$$

We shall show that $S(t) < m_3$ cannot hold for all $t \geq 0$. Otherwise,

$$\begin{cases} I'(t) \leq \bigg(-w+\dfrac{\beta S(t)}{1+aS^l(t)}\bigg)I(t) & \\ \qquad \leq \bigg(\dfrac{\beta m_3}{1+am_3^l}-w\bigg)I(t), & t \neq (n+\widetilde{l}-1)T, t \neq nT, \\ \triangle I(t) = -p_2 I(t), & t = (n+\widetilde{l}-1)T, \\ \triangle I(t) = \mu, & t = nT. \end{cases}$$

Then we obtain that $I(t) \leq y_3(t)$ and $y_3(t) \to y_3^*(t)$ as $t \to \infty$, where $y_3(t)$ is the solution of

$$\begin{cases} y_3'(t) = \bigg(\dfrac{\beta m_3}{1+am_3^l}-w\bigg)y_3(t), & t \neq (n+\widetilde{l}-1)T, t \neq nT, \\ \triangle y_3(t) = -p_2 y_3(t), & t = (n+\widetilde{l}-1)T, \\ \triangle y_3(t) = \mu, & t = nT, \\ y_3(0^+) = x_{20} & \end{cases} \tag{D.1}$$

and

$$y_3^*(t) = \begin{cases} \dfrac{\mu\exp\big(\big(\frac{\beta m_3}{1+am_3^l}-w\big)(t-(n-1)T)\big)}{1-(1-p_2)\exp\big(\big(\frac{\beta m_3}{1+am_3^l}-w\big)T\big)}, \\ \qquad (n-1)T < t \leq (n+\widetilde{l}-1)T, \\ \dfrac{\mu(1-p_2)\exp\big(\big(\frac{\beta m_3}{1+am_3^l}-w\big)(t-(n-1)T)\big)}{1-(1-p_2)\exp\big(\big(\frac{\beta m_3}{1+am_3^l}-w\big)T\big)}, \\ \qquad (n+\widetilde{l}-1)T < t \leq nT. \end{cases} \tag{D.2}$$

Therefore, there exists $\widehat{T} > 0$ such that

$$I(t) \leq y_3(t) < y_3^*(t) + \varepsilon$$

for $t > \widehat{T}$. From (2.4), we have

$$\begin{cases} S'(t) \geq \bigg(1-\dfrac{m_3}{K}-\bigg(\dfrac{1}{K}+\beta\bigg)(y_3^*(t)+\varepsilon)\bigg)S(t), & t \neq (n+\widetilde{l}-1)T, \\ \triangle S(t) = -p_1 S(t), & t = (n+\widetilde{l}-1)T, \end{cases} \tag{D.3}$$

for $t \geq \widehat{T}$. Let $N \in Z_+$ such that $(N + \widetilde{l} - 1)T \geq \widehat{T}$. Integrating (D.3) on $((n + \widetilde{l} - 1)T,\, (n + \widetilde{l})T], n \geq N$, we get

$$\begin{aligned} S((n+\widetilde{l})T) &\geq S((n+\widetilde{l}-1)T)(1-p_1) \\ &\quad \times \exp\left(\int_{(n+\widetilde{l}-1)T}^{(n+\widetilde{l})T} \left(1 - \frac{m_3}{K} - \left(\frac{1}{K} + \beta\right)(y_3^*(t) + \varepsilon)\right) dt\right) \\ &= S((n+\widetilde{l}-1)T)\eta, \end{aligned}$$

then $S((N + n + \widetilde{l} - 1)T) \geq S((n + \widetilde{l})T)\eta^n \to \infty$ as $n \to \infty$, which contradicts to the boundedness of $S(t)$. Then there exist a $t_1 > 0$ such that $S(t_1) \geq m_3$.

Step II. If $S(t_1) \geq m_3$ for all $t \geq t_1$, then our aim is obtained. Otherwise, $S(t) < m_3$ for some $t \geq t_1$. Set $t^* = \inf_{t > t_1}\{S(t) < m_3\}$. We should consider two possible cases for t^*.

Case I. $t^* = (n_1 + \widetilde{l} - 1)T$, $n_1 \in Z_+$. Then $S(t) \geq m_3$ for $t \in [t_1, t^*]$, and $(1 - p_1)m_3 \leq S(t^{*+}) = (1 - p_1)S(t^*) < m_3$. Choose n_2, $n_3 \in Z_+$ such that

$$n_2 T > \frac{1}{\frac{\beta m_3}{1+am_3^l} - w} \ln \frac{\varepsilon_1}{M + \frac{\mu}{1-(1-p_2)\exp\left(\left(\frac{\beta m_3}{1+am_3^l} - w\right)T\right)}},$$
$$(1 - p_1)^{n_2} \exp(n_2 \eta_1 T)\eta^{n_3} > 1,$$

where $\eta_1 = 1 - \frac{m_3}{K} - \left(\frac{1}{K} + \beta\right)M < 0$. Let $T' = n_2T + n_3T$. We claim that there exist a $t_2(t^*, t^* + T']$ such that $S(t_2) > m_3$. Otherwise, we consider (D.1) with $y_3(t^{*+}) = I(t^{*+})$, and get

$$y_3(t) = \begin{cases} (1-p_2)^{n-(n_1+1)}\left(y_3(n_1T^+) - \dfrac{\mu}{1-(1-p_2)\exp\left(\left(\frac{\beta m_3}{1+am_3^l} - w\right)T\right)}\right) \\ \exp\left(\left(\dfrac{\beta m_3}{1+am_3^l} - w\right)t\right) + y_3^*(t), \quad (n-1)T < t \leq (n+\widetilde{l}-1)T, \\ (1-p_2)^{n-n_1}\left(y_3(n_1T^+) - \dfrac{\mu}{1-(1-p_2)\exp\left(\left(\frac{\beta m_3}{1+am_3^l} - w\right)T\right)}\right) \\ \exp\left(\left(\dfrac{\beta m_3}{1+am_3^l} - w\right)t\right) + y_3^*(t), \qquad (n+\widetilde{l}-1)T < t \leq nT, \end{cases}$$

where $n_1 + 1 \leq n \leq n_1 + n_2 + n_3$. Then $|y_3(t) - y_3^*(t)| < \Big(M + \frac{\mu}{1-(1-p_2)\exp\left(\left(\frac{\beta m_3}{1+am_3^l} - w\right)T\right)}\Big) \exp\left(\left(\frac{\beta m_3}{1+am_3^l} - w\right)(t - (n_1+1)T)\right) < \varepsilon_1$ and $I(t) < y_3(t) \leq$

$y_3^*(t) + \varepsilon_1$ for $n_1T + (n_2 - 1)T \le t \le t^* + T'$, which implies that (D.3) holds for $t \in [(t^* + n_2T), (t^* + T')]$. As in Step I, we get

$$S(t^* + T') \ge S(t^* + n_2T)\eta^{n_3}. \tag{D.4}$$

From (2.4), we get

$$\begin{cases} S'(t) \ge \left(1 - \dfrac{m_3}{K} - \left(\dfrac{1}{K} + \beta\right) M\right) S(t), & t \ne (n + \widetilde{l} - 1)T, \\ \triangle S(t) = -p_1 S(t), & t = (n + \widetilde{l} - 1)T \end{cases} \tag{D.5}$$

for $t \in [t^*, (t^* + n_2T)]$. Integrating (D.5) on $[t^*, (t^* + n_2T)]$, we obtain

$$S(t^* + n_2T) \ge m_3(1 - p_1)^{n_2} \exp(n_2\eta_1 T).$$

Consequently, we get

$$S(t^* + T') \ge m_3(1 - p_1)^{n_2} \exp(n_2\eta_1 T)\eta^{n_3} > m_3,$$

which is a contradiction. Let $\widetilde{t} = \inf_{t>t^*}\{S(t) > m_3\}$. Then for $t \in (t^*, \widetilde{t})$, $S(t) \le m_3$ and $S(\widetilde{t}) = m_3$. For $t \in (t^*, \widetilde{t})$, we get

$$S(t) \ge m_3(1 - p_1)^{n_2+n_3} \exp((n_2 + n_3)\eta_1 T).$$

Let $\overline{m}_1 = m_3 \exp((n_2 + n_3)\eta_1 T)$. With this notation we have $S(t) \ge \overline{m}_1$ for $t \in (t^*, \widetilde{t})$. For $t > \widetilde{t}$, the same arguments can be continued since $S(t) \ge m_3$.

Case II. $t^* \ne (n + \widetilde{l} - 1)T$, $n \in Z_+$. Then $S(t) \ge m_3$ for $t \in [t_1, t^*]$ and $S(t^*) = m_3$. Suppose that $t^* \in ((\overline{n}_1 + \widetilde{l} - 1)T, (\overline{n}_1 + \widetilde{l})T), \overline{n}_1 \in Z_+$. There are two possible sub-cases for $t \in (t^*, (\overline{n}_1 + \widetilde{l})T)$.

Case II_1 For all $t \in (t^*, (\overline{n}_1 + \widetilde{l})T)$, $S(t) \le m_3$. Similarly to Case I we can prove that there exist a $t_2' \in [(\overline{n}_1 + \widetilde{l})T, (\overline{n}_1 + \widetilde{l})T + T']$ such that $S(t_2') > m_3$.

Let $\overline{t} = \inf_{t>t^*}\{S(t) > m_3\}$, then for $t \in (t^*, \overline{t})$, $S(t) \le m_3$ and $S(\overline{t}) = m_3$. For $t \in (t^*, \overline{t})$, we get

$$S(t) \ge m_3(1 - p_1)^{n_2+n_3} \exp((n_2 + n_3 + 1)\eta_1 T).$$

Let $m_1 = m_3(1 - p_1)^{n_2+n_3} \exp((n_2 + n_3 + 1)\eta_1 T) < \overline{m}_1(\eta_1 < 0)$. We then have $S(t) \ge m_1$ for $t \in (t^*, \overline{t})$. For $t > \widetilde{t}$, the same arguments can be continued since $S(\overline{t}) \ge m_3$.

Case II_2 There exists $t \in (t^*, (\overline{n}_1 + \widetilde{l})T)$ such that $S(t) > m_3$. Let $\widehat{t} = \inf_{t>t^*}\{S(t) > m_3\}$. Then for $t \in (t^*, \widehat{t})$, $S(t) \le m_3$ and $S(\widehat{t}) = m_3$. For $t \in (t^*, \widehat{t})$, (D.3) holds. Integrating (D.3) on $(t^*, \widehat{t})$, we obtain

$$S(t) \ge x_1(t^*) \exp(\eta_1(t - t^*)) \ge m_3 \exp(\eta_1 T) > m_1.$$

Since $S(\widehat{t}) \ge m_3$, for $t > \widehat{t}$ the same arguments can be continued, so we omit them. Hence, $S(t) \ge m_1$ for $t \ge t_1$. This completes the proof. $\square$