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Multigroup deterministic and stochastic *SEIRI* epidemic models with nonlinear incidence rates and distributed delays: A stability analysis

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In this paper, we investigate the dynamics of a multigroup disease propagation model with distributed delays and nonlinear incidence rates, which accounts for the relapse of recovered individuals. The main concern is the stability of the equilibria, sufficient conditions for global stability being obtained by applying Lyapunov-LaSalle invariance principle and using Lyapunov functionals, which are constructed using their single-group counterparts. The situation in which the deterministic model is subject to perturbations of white noise type is also investigated from a stability viewpoint.

KEYWORDS

delay differential equations, disease propagation, disease relapse, Lyapunov stability, multigroup model, nonlinear incidence

1 | INTRODUCTION

Deterministic disease propagation models, which provide a fruitful approach towards the study of various endemicity problems, are of very diverse natures and degrees of accuracy. In their simplest iterations, they deal with unstructured populations and do not account for the various sources of heterogeneity. For increased realism, however, one may structure the target population according to age, degree of infectivity, contact patterns, risk behavior, or to other criteria quantifying the risk of acquiring or transmitting the disease. This often leads to multicompartmental models, which describe the transition of an individual through various disease-related stages. To account for the effects of spatial heterogeneity, one may also consider multicompartmental structures in the framework of multigroup models, each compartment representing in this context a homogeneous population.

In most cases, use is made of at least 3 compartments, namely, the class of susceptible individuals (S), the class of infective individuals (I), and the class of recovered individuals (R). It may be the case, although, that after an initial inoculation of an individual with a relatively small number of pathogens, the level of infectivity is far too low for disease transmission, despite of rapid pathogen reproduction, and remains so for a certain amount of time. To keep track of this category of subjects, which are no longer susceptible, but also not yet infective, one needs to introduce the class of exposed (or latent) individuals (E) as well. This latency phenomenon may happen for diseases such as influenza, measles, or tuberculosis, for which subjects may become infected after an adequate contact with an infective individual without actually being infective during a certain initial period. One should not confuse, however, latency and incubation. While the former represents the time elapsed from the exposure to the acquisition of infectiousness, the latter designates the time elapsed from the exposure to the

onset of symptoms, and an individual may be infective prior to the onset of symptoms. Prolonged latency between exposure and infectiousness is a characteristic of tuberculosis, for instance, latent compartments being incorporated even in the earliest models of tuberculosis transmission.[1] Also, several models have included both fast and slow pathways from susceptible to actively infected, with a proportion of exposed susceptibles progressing immediately to active infection or several sequential latent compartments to simulate the increased risk of progression to active disease in the years immediately following initial infection.[2,3]

As noted in Lloyd,[4] an accurate description of the distribution of the latent and infectious periods is an important step towards the construction of an appropriate model. Assuming that the latency is exponentially distributed, the corresponding disease propagation model is represented by a system of nonlinear ODEs. This assumption, however, is equivalent to the fact that the chance of recovery within a given time interval is constant, regardless of the time since infection. This is sometimes unrealistic, as observed through the statistical studies of the transmission dynamics of measles in small communities (see, for instance, Lloyd[4] or Krylova and Earn[5]), the chance of recovery being initially small, but increasing over time and corresponding to a distribution of the infectious period, which is less dispersed than an exponential and more centered around the mean. If, however, a general distribution is assumed for the latent period, then one obtains a delayed integro-differential system.[6,7] The most commonly used distributions of the latent period are the Gamma distribution, used, for instance, to model the spread of avian influenza (H7N7) in chicken[8] and the log-normal distribution, used, for instance, to model the spread of Ebola.[9]

Since the host population can be divided geographically into communities, cities, and countries, or epidemiologically, to incorporate factors such as modes of transmission, contact patterns, and genetic susceptibility, a heterogeneous host population can subsequently be divided into several homogeneous groups, the within-group dynamics and the mixing patterns being then modeled separately. One of the earliest multigroup models was proposed by Lajmanovich and Yorke,[10] who obtained a complete characterization of an n-group *SIS* model with subpopulations of constant size from a stability viewpoint, proving the global stability of a unique endemic equilibrium using a quadratic global Lyapunov functional. In this regard, proving the global stability of the endemic equilibrium for multigroup models has always been a difficult undertaking, only recently a systematic graph-theoretic approach to the construction of Lyapunov functional being devised by M.Y. Li and his coworkers.[7,11,12] Under this approach, the Lyapunov functional for the multigroup model is constructed as a linear combination of Lyapunov functionals for each isolated group, assuming that they are stable in isolation, the coefficients being obtained by studying the properties of the directed graph associated to the transmission matrix, so that a certain summation lemma holds true. The presence of the delay caused by latency and the use of general (ie, nonfactorized) incidence functions are known then to cause further complications (in fact, the latter does so even for single-group models). See also Muroya and Kuniya,[13] Shu et al,[14] Huang et al,[15] Yuan et al,[16] Fall et al,[17] and the references therein.

For diseases such as herpes, tuberculosis, or malaria, recovered hosts may experience a relapse of the disease due to an incomplete or unsuccessful treatment, due to the emergence of drug resistance, or due to the reactivation of a latent infection and then reenter the infective group. For herpes, it is known that an infective individual may experience bouts of relapse all his life.[18] Sometimes, relapses may be more severe than the initial infection, as it happens for varicella zoster virus. For this virus, the initial infection, varicella, is usually benign, while the relapse (herpes zoster) may have life-altering neurological complications. Also, it is common for tuberculosis patients to relapse during or after treatment, although it should be noted that it is often difficult to distinguish between relapse (regrowth of the same strain of *Mycobacterium tuberculosis* that caused the previous tuberculosis episode) and reinfection by a different strain. In this regard, epidemic models with relapse have been investigated from a stability viewpoint in Moreira and Wang,[19] Vargas de León,[20] Georgescu and Zhang,[21] and Wang et al.[22]

Environmental noises may sometimes drastically limit the usefulness of deterministic epidemic models, being necessary to consider the effect of stochastic perturbations, as done, for instance, in Beretta et al,[23] Jiang et al,[24] and Yuan et al.[16] Particularly, stochastic perturbations around the positive endemic equilibrium of epidemic models were considered in Carletti,[25] Beretta et al,[23] and Ji et al.[26]

In this paper, we introduce and analyze from a stability viewpoint a multigroup deterministic SEIRI epidemic model with abstract nonlinear incidence and a general distributed time delay in which individuals may experience disease relapse, that is, the return of signs and symptoms of a disease after a remission.[20,27] We obtain the stability of the disease-free equilibrium and of the endemic equilibrium in terms of a threshold parameter that governs not only the stability of the equilibria but also the very existence of the endemic equilibrium, a certain inequality involving the nonlinear incidence function being also assumed. To this purpose, we use Lyapunov functionals constructed ad hoc on the basis of the ones used for the corresponding single-group model of Zhang et al.[28] Subsequently, we discuss the influence of stochastic perturbations of white noise type upon the stability of the system, also by Lyapunov's second method. Finally, the applicability of our results is further investigated by means of numerical simulations.

2 | THE MULTIGROUP DETERMINISTIC *SEIRI* EPIDEMIC MODEL AND ITS ANALYSIS

2.1 | The model and its relevance

Let us denote by $P(t)$ the probability that, if alive, an individual remains in the exposed class t time units after entering. By obvious biological considerations, we may suppose that P satisfies the following biologically motivated assumptions

(P) $P : [0, \infty) \to [0, 1]$ is nonincreasing and piecewise continuous, possibly with finitely many jumps, and satisfying $P(0+) = 1$, $\lim_{t\to\infty} P(t) = 0$ and $0 < \int_0^\infty P(u)du < \infty$.

Then $-P'(t)$ represents the removal rate from the exposed class to the infected class t units of time after being exposed.

Assuming that the force of infection is bilinear, $f(S, I) = \beta SI$, van den Driessche et al introduced and discussed in the paper[29] the following *SEIRI* disease propagation model with relapse

$$\begin{cases} \frac{dS}{dt} = \mu - \mu S(t) - \beta S(t)I(t) \\ \frac{dE}{dt} = \beta S(t)I(t) - \mu E(t) + \beta \int_0^t S(\xi)I(\xi)e^{-\mu(t-\xi)}d_tP(t-\xi)d\xi, \\ \frac{dI}{dt} = -\beta \int_0^t S(\xi)I(\xi)e^{-\mu(t-\xi)}d_tP(t-\xi)d\xi + \delta R(t) - (\mu+\gamma)I(t), \\ \frac{dR}{dt} = \gamma I(t) - (\mu+\delta)R(t). \end{cases} \tag{1}$$

In the above model 1, S, E, I, and R represent the (rescaled) sizes of the susceptible, exposed, infectious, and recovered populations, respectively, μ is the birth (and death) rate, β is the transmission coefficient, γ is the recover rate of infective individuals, and δ is the relapse rate of recovered individuals, which subsequently return to the infective compartment, the integrals being considered in the sense of Riemann-Stieltjes.

Model 1, using a general, unspecified form of P, encompasses several particular cases, with or without delay, which are important on their own.

In the paper,[30] Xu discussed a version of Equation 1, particularizing $P(t)$ as a step function, $P(t) = \begin{cases} 1, & t \in [0, \tau] \\ 0, & t > \tau \end{cases}$, and replacing the incidence rate considered in van den Driessche et al[29] with ratio-dependent incidence, in the form

$$\begin{cases} \frac{dS}{dt} = \mu - \mu S(t) - \frac{\beta S(t)I(t)}{S(t)+I(t)} \\ \frac{dE}{dt} = \frac{\beta S(t)I(t)}{S(t)+I(t)} - \frac{\beta S(t-\tau)I(t-\tau)}{S(t-\tau)+I(t-\tau)} - \mu E(t), \\ \frac{dI}{dt} = \frac{\beta S(t-\tau)I(t-\tau)}{S(t-\tau)+I(t-\tau)} + \delta R(t) - (\mu+\gamma)I(t), \\ \frac{dR}{dt} = \gamma I(t) - (\mu+\delta)R(t), \end{cases} \tag{2}$$

deriving conditions for the global stability of the endemic equilibrium and of the disease-free equilibrium in terms of a threshold parameter via appropriately constructed Lyapunov functionals.

Motivated these results, we have considered in Zhang et al[28] a general deterministic *SEIRI* model with an abstract nonlinear incidence rate, a distributed latent period, and the relapse of recovered individuals, in the form

$$\begin{cases} \frac{dS}{dt} = n(S(t)) - f(S(t), I(t)), \\ \frac{dE}{dt} = f(S(t), I(t)) - \mu E(t) - \int_0^h Q(\xi)e^{-\mu\xi} f\big(S(t-\xi), I(t-\xi)\big)d\xi, \\ \frac{dI}{dt} = \int_0^h Q(\xi)e^{-\mu\xi} f\big(S(t-\xi), I(t-\xi)\big)d\xi - (\mu+\gamma+\alpha)I(t) + \delta R(t), \\ \frac{dR}{dt} = \gamma I(t) - (\mu+\delta)R(t), \end{cases} \tag{3}$$

finding sufficient conditions for the global stability of equilibria, again by using Lyapunov functionals.

Generally, a multigroup model is formulated by dividing all individuals into n distinct groups, each representing a homogeneous subpopulation. For $1 \le k \le n$, the kth group can be further partitioned into 4 compartments, S_k, E_k, I_k, and R_k, which stand for susceptible, latent, infectious, and recovered populations in the kth group, respectively. Assuming that the incidence of infection can be factorized as the product of 2 transmission functions, the contact function $h_k(S_k)$ and the force of infection $g_k(I_k)$, Shu et al[14] introduced and discussed the following multigroup *SEIRI* disease propagation model with relapse and infinite delay

$$\begin{cases} \frac{dS_k}{dt} = n_k(S_k(t)) - \sum_{j=1}^{n} \beta_{kj} h_k(S_k(t)) \int_0^\infty f_j(\tau) g_j(I_j(t-\tau)) d\tau, \\ \frac{dE_k}{dt} = \sum_{j=1}^{n} \beta_{kj} h_k(S_k(t)) \int_0^\infty f_j(\tau) g_j(I_j(t-\tau)) d\tau - \left(d_k^E + \epsilon_k\right) E_k(t), \\ \frac{dI_k}{dt} = \epsilon_k E_k(t) - \left(d_k^I + r_k + \theta_k\right) I_k(t), \\ \frac{dR_k}{dt} = r_k I_k(t) - d_k^R R_k(t), \quad k = 1, 2, \cdots, n. \end{cases} \tag{4}$$

In the above, the kernel function $f_j(\tau)$ is used to express the variation of the infectivity according to the age of the infection τ. Within the kth group, n_k denotes the intrinsic growth rate of the susceptible compartment, β_{kj} is the coefficient of transmission between compartments S_k and I_j, and d_k^E, d_k^I, and d_k^R are respective natural death rates of the individuals in compartments E_k, I_k, and R_k. Also, in the same kth group, ϵ_k is the rate of progression from the compartment E_k to the compartment I_k, θ_k is the disease-induced death rate, and r_k is the recovery rate of infectious individuals. Along the idea in Guo et al,[11] global stability of the endemic equilibrium is established by exploiting a graph-theoretical approach to the task of constructing Lyapunov functionals.

In what follows, we shall consider first the following general deterministic multigroup *SEIRI* model with an abstract nonlinear incidence rate, a distributed latent period and relapse of recovered individuals

$$\begin{cases} \frac{dS_k}{dt} = n_k(S_k(t)) - \sum_{j=1}^{n} f_{kj}(S_k(t), I_j(t)), \\ \frac{dE_k}{dt} = \sum_{j=1}^{n} f_{kj}(S_k(t), I_j(t)) - \mu_k E_k(t) - \sum_{j=1}^{n} \int_0^h Q_k(\xi) e^{-\mu_k \xi} f_{kj}\left(S_k(t-\xi), I_j(t-\xi)\right) d\xi, \\ \frac{dI_k}{dt} = \sum_{j=1}^{n} \int_0^h Q_k(\xi) e^{-\mu_k \xi} f_{kj}\left(S_k(t-\xi), I_j(t-\xi)\right) d\xi - (\mu_k + \gamma_k + \alpha_k) I_k(t) + \delta_k R_k(t), \\ \frac{dR_k}{dt} = \gamma_k I_k(t) - (\mu_k + \delta_k) R_k(t), \quad k = 1, 2, \cdots, n. \end{cases} \tag{5}$$

This model is the multigroup version of the single-group one (Equation 3) discussed in our previous paper.[28] In the above, within the kth group, $n_k(S_k)$ is the recruitment rate of healthy individuals into the susceptible class, and μ_k and α_k are natural death rate and the disease-induced death rate, respectively. Also, γ_k is the recovery rate of the infective population, and δ_k is the relapse rate of recovered individuals, which then became infectious again. The incidence functions $f_{kj}(S_k, I_j)$ are given in an abstract, nonfactorized form, encompassing several classical situations, which shall be mentioned below. Here, h is the maximal length of the latent period of any group, that is, the maximal time in which infected individuals become infectious and $Q_k = -P_k'$, assuming implicitly that P_k, the probability that an individual remains in the kth exposed class t time units after entering, without taking death into account, is piecewise C^1.

Throughout this paper, we shall use the following assumptions (i) to (v) for $k, j = 1, 2, \ldots, n$.

(i) n_k is a continuous function on $\mathbb{R}^+$, and there exists a $S_k^0 > 0$ such that $n_k\left(S_k^0\right) = 0$ and $\left(n_k(S_k) - n_k\left(S_k^0\right)\right)(S_k - S_k^0) < 0$ for $S_k \neq S_k^0$.

(ii) f_{kj} is a locally Lipschitz continuous function on $\mathbb{R}^+ \times \mathbb{R}^+$ satisfying $f_{kj}(S_k, 0) = f_{kj}(0, I_j) = 0$.

(iii) f_{kj} is an increasing function of S_k for fixed I_j and an increasing function of I_j for fixed S_k. If the fixed variables are nonzero, then the monotonicity is strict.

(iv) For any fixed S_k, $\Phi_{kj}(S_k, I_j) \doteq \frac{f_{kj}(S_k, I_j)}{I_j}$ is a bounded and decreasing function of I_j.

(v) $\kappa_{kj}(S_k) \doteq \lim_{I_j \to 0^+} \Phi_{kj}(S_k, I_j)$ is a continuous function of S_k.

In particular, assumption (i) implies that each S_k^0, $1 \leq k \leq n$, is unique. Let us note that, by (iv) and (v), the following estimations for the nonlinear forces of infection $f_{kj}(S_k, I_j)$ hold

$$f_{kj}(S_k, I_j) \leq \kappa_{kj}(S_k) I_j \quad \text{for all } I_j > 0. \tag{6}$$

Also, κ_{kj} is increasing and has positive values for all $1 \leq k, j \leq n$. Let us now discuss the biological motivations behind assumptions (i) to (iv). Assumption (i) indicates that the growth rate of the susceptible hosts in the kth group is positive at lower densities, while negative at higher densities beyond the threshold S_k^0. This assumption contributes towards a limitation of the growth of the kth susceptible class, keeping its size under a maximal value S_k^0, and is satisfied by both the usual linear

and logistic growth rates. The meaning of (ii) is that if there are no susceptibles in one group or no infectives in the other group, then obviously, there is no disease transmission for the respective pathway. Also, (iii) describes the fact that increasing the size of the susceptible or of the infective class increases the occurrence of new infections if the size of the other class is kept constant. Assumption (iv) describes the fact that infection rate saturates as the size of the infective class grows larger, while (v) states the fact that a small number of infectives introduced in a totally susceptible population will produce a certain amount of new infections, which is larger if the initial population of susceptibles is larger. Also, if f_{kj}'s factorize in the form $f_{kj}(S_k, I_j) = g_k(S_k)h_j(I_j)$, then (iv) is satisfied provided that h_j is concave down.

Remark 1. *For example, for each group k, the set of growth functions $n_k(S)$ satisfying assumption (i) includes the linear growth*

$$n_k(S_k) = \Lambda - \mu_k S_k,$$

where Λ is a positive constant, and the combination of the linear growth and the logistic growth

$$n_k(S_k) = \alpha(\Lambda - \mu_k S_k) + \zeta\left(rS_k\left(1 - \frac{S_k}{K}\right)\right),$$

in which r is the intrinsic growth rate, K is the carrying capacity, α and ζ are positive constants.

Remark 2. *Examples of incidence functions $f_{kj}(S_k, I_j)$ that satisfy assumptions (ii) to (v) and are in common use include the bilinear incidence, the saturated incidence $\frac{\beta S_k I_j}{d_k + S_k + I_j}$, the modified saturated incidence $\frac{\beta S_k I_j}{1+\alpha_1 S_k + \alpha_2 I_j}$ (see Abta et al[31]), the standard incidence, and other common incidence functions such as $\beta \frac{S_k}{S_k + A_S} \frac{I_j}{I_j + A_I}$, in which A_S and A_I are positive constants, and $\beta S_k^p I_j^q (p \geqslant 0, 0 \leq q \leq 1)$.*

The initial conditions are given by

$$\begin{aligned} S_k(\theta) = \varphi_{1k}(\theta), \quad E_k(\theta) = \varphi_{2k}(\theta), \quad I_k(\theta) = \varphi_{3k}(\theta), \\ R_k(\theta) = \varphi_{4k}(\theta), \qquad \theta \in [-h, 0], \end{aligned}$$

where $\varphi_{1k}, \varphi_{2k}, \varphi_{3k}, \varphi_{4k} \in C([-h, 0], \mathbb{R}^+)$, the Banach space of continuous functions mapping the interval $[-h, 0]$ into $\mathbb{R}^+$, endowed with the sup-norm, such that $\varphi_{1k}(0) > 0, \varphi_{3k}(0) > 0$, $k = 1, 2, \ldots, n$. The existence and uniqueness of the solutions of the systems (Equation 5) then follow from standard results in the theory of delay differential equations (see, for instance, Kuang[32]).

2.2 | The well posedness of the system (Equation 5)

On the lines of lemmas 2.1 and 2.2 in Zhang et al[28] (see also lemma 2.1 of van den Driessche[29] and proposition 3.1 of Shu et al[14]), it is possible to prove that our system (Equation 5) is well posed from a biological viewpoint, in the sense that its solutions are posity-preserving and ultimately uniformly bounded, the ω-limit sets being contained in the following bounded region

$$\Gamma = \left\{ (S_k, E_k, I_k, R_k) | 0 \leq S_k \leq S_k^0, 0 \leq S_k + E_k + I_k + R_k \leq \frac{2\overline{n_k}}{\overline{d_k}}, 1 \leq k \leq n \right\},$$

where, for each group k,

$$\overline{n_k} = \sup_{0 \leq S_k \leq S_k^0} n_k(S_k) \quad \text{and} \quad \overline{d_k} = \min\left\{ \mu_k, \frac{\overline{n_k}}{S_k^0} \right\}.$$

We further observe that the first, third, and fourth equations of the system (Equation 5) do not refer to the exposed class E_k, which means that we could simplify the system (Equation 5) to the following lower-dimensional one

$$\begin{cases} \frac{dS_k}{dt} = n_k(S_k) - \sum_{j=1}^{n} f_{kj}(S_k(t), I_j(t)), \\ \frac{dI_k}{dt} = \sum_{j=1}^{n} \int_0^h Q_k(\xi) e^{-\mu_k \xi} f_{kj}\big(S_k(t-\xi), I_j(t-\xi)\big) d\xi - (\mu_k + \gamma_k + \alpha_k) I_k(t) + \delta_k R_k(t), \\ \frac{dR_k}{dt} = \gamma_k I_k(t) - (\mu_k + \delta_k) R_k(t). \end{cases} \tag{7}$$

Having established the biological well posedness of the multigroup model 5 (and, consequently, of its reduced version [Equation 7]), we are now ready to discuss the existence and stability of the equilibria for the reduced system (Equation 7). To this purpose, we shall use the graph-theoretical approach of Guo et al[11,12,33] and construct Lyapunov functionals defined ad hoc as linear combinations of Lyapunov functionals for each group in isolation. The stability results will be obtained in terms of a threshold parameter, the basic reproduction number, defined using the next generation method of van den Driessche and Watmough.[34] We start with a brief matrix theory interlude.

2.3 | A summation lemma

To discuss the stability of the endemic equilibrium, we shall need a summation lemma that will prove useful when evaluating the derivative of a Lyapunov functional along the solutions of Equation 5.

Let $U = (u_{kj})$, $V = (v_{kj})$ be $n \times n$ matrices. We shall write $U \leq V$ if $u_{kj} \leq v_{kj}$ for all $1 \leq j, k \leq n$ and $U < V$ if $U \leq V$ and $U \neq V$. If $O_n \leq U$, we shall say that U is nonnegative.

Given a nonnegative $n \times n$ matrix $A = (a_{kj})$, the directed graph $G(A)$ associated with A has vertices $1, 2, \dots, n$, with a directed arc (k, j) starting in vertex k and in vertex j if and only if $a_{kj} \neq 0$. The directed graph $G(A)$ is then said to be strongly connected if any 2 distinct vertices can be joined by an oriented path. Under these circumstances, the matrix A is irreducible if and only if the associated directed graph $G(A)$ is strongly connected. Equivalently, a $n \times n$ matrix A, $n \geqslant 2$, is irreducible if for any proper subset M of $\{1, 2, \dots, n\}$, there are $i \in M$ and $j \in \{1, 2, \dots, n\} \setminus M$ such that $a_{ij} \neq 0$.

For a nonnegative $n \times n$ matrix $A = (a_{kj})$, $n \geqslant 2$, let

$$L = \begin{pmatrix} \sum_{l \neq 1} a_{1l} & -a_{2l} & \dots & -a_{n1} \\ -a_{12} & \sum_{l \neq 1} a_{2l} & \dots & -a_{n2} \\ \vdots & \vdots & & \vdots \\ -a_{1n} & -a_{2n} & \dots & \sum_{l \neq n} a_{nl} \end{pmatrix}$$

be the Laplacian matrix of the directed graph $G(A)$ associated with A, and let C_{kj} be the cofactor of the (k, j) entry of L. Let also $c_i = C_{ii}$. The following result then holds as a consequence of Kirchoff's matrix tree theorem (see Guo et al,[33] appendix 1, for further details).

Lemma 1. *Let c_k, $1 \leq k \leq n$, defined as above. Then*

$$\sum_{k=1}^{n} \sum_{j=1}^{n} c_k a_{kj} G_k(x_k) = \sum_{k=1}^{n} \sum_{j=1}^{n} c_k a_{kj} G_j(x_j),$$

where $\{G_k(x_k)\}_{k=1}^{n}$ is an arbitrary family of functions.

We shall now observed that the existence and stability of the equilibria are governed by the values of the basic reproduction number, understood as a threshold parameter. As it is perhaps common, it will be determined that Equation 7 always has a disease-free equilibrium, while the endemic equilibrium exists only in certain conditions. We start with the seemingly easier tasks of discussing the stability of the disease-free equilibrium.

2.4 | The stability of the disease-free equilibrium

It is easy to see that, as a direct byproduct of (i), the system (Equation 7) always has a disease-free equilibrium $\mathbf{E}^0$,

$$\mathbf{E}^0 = \left(S_1^0, 0, 0, \ldots, S_n^0, 0, 0\right).$$

With a view to the next generation matrix approach formulated in van den Driessche and Watmough,[34] let us define

$$\mathcal{M}^0 = \left(\frac{\left(\int_0^h Q_k(\xi) e^{-\mu_k \xi} d\xi\right) \kappa_{kj}\left(S_k^0\right)}{\mu_j + \gamma_j + \alpha_j - \frac{\delta_j \gamma_j}{\mu_j + \delta_j}} \right)_{\substack{1 \le k \le n \\ 1 \le j \le n}} = FV^{-1}, \quad \widetilde{\mathcal{M}}^0 = \left(\frac{\left(\int_0^h Q_k(\xi) e^{-\mu_k \xi} d\xi\right) \kappa_{kj}\left(S_k^0\right)}{\mu_k + \gamma_k + \alpha_k - \frac{\delta_k \gamma_k}{\mu_k + \delta_k}} \right)_{\substack{1 \le k \le n \\ 1 \le j \le n}} = V^{-1}F,$$

where

$$F = \left(\left(\int_0^h Q_k(\xi) e^{-\mu_k \xi} d\xi \right) \kappa_{kj}\left(S_k^0\right) \right)_{\substack{1 \le k \le n \\ 1 \le j \le n}}, \quad V = \operatorname{diag}\left(\mu_k + \gamma_k + \alpha_k - \frac{\delta_k \gamma_k}{\mu_k + \delta_k} \right)_{1 \le k \le n}.$$

Subsequently, following the approach of van den Driessche and Watmough,[34] we may define the basic reproduction number of Equation 7 as $\mathcal{R} \doteq \rho(\mathcal{M}^0)$, the spectral radius of the matrix $\mathcal{M}^0$. However, since $\rho(FV^{-1}) = \rho(V^{-1}F)$, it also follows that $\mathcal{R} = \rho(\widetilde{\mathcal{M}}^0)$. Note that $\mathcal{M}^0$ is irreducible if and only if $\widetilde{\mathcal{M}}_0$ is irreducible.

Let us also denote

$$\mathbf{S} = (S_1, S_2, \ldots, S_n), \quad \mathbf{S}^0 = (S_1^0, S_2^0, \ldots, S_n^0), \quad \mathbf{I} = (I_1, I_2, \ldots, I_n), \quad \mathbf{R} = (R_1, R_2, \ldots, R_n).$$

Theorem 1. *Assume that matrices $\mathcal{M}^0$ and $\widetilde{\mathcal{M}}(\mathbf{S})$ are irreducible.*

(a) If $\mathcal{R} \le 1$, then the disease-free equilibrium $\boldsymbol{E}^0$ is the unique equilibrium of Equation 7, and it is globally asymptotically stable in Γ.
(b) If $\mathcal{R} > 1$, then the disease-free equilibrium $\boldsymbol{E}^0$ is unstable.

Proof. *(a) Let us suppose that $\boldsymbol{E} = (\boldsymbol{S}, \boldsymbol{I}, \boldsymbol{R})$ is an equilibrium of Equation 7. Define*

$$\overline{\mathcal{M}}(\mathbf{S}) = \left(\overline{m}_{kj}\right)_{\substack{1 \le k \le n, \\ 1 \le j \le n}}, \quad \overline{m}_{kj} = \begin{cases} \frac{\left(\int_0^h Q_k(\xi) e^{-\mu_k \xi} d\xi\right) \kappa_{kj}(S_k)}{\mu_k + \gamma_k + \alpha_k - \frac{\delta_k \gamma_k}{\mu_k + \delta_k}} \frac{f_{kj}(S_k, I_j)}{I_j}, & I_j \ne 0 \\ \frac{\left(\int_0^h Q_k(\xi) e^{-\mu_k \xi} d\xi\right) \kappa_{kj}(S_k)}{\mu_k + \gamma_k + \alpha_k - \frac{\delta_k \gamma_k}{\mu_k + \delta_k}} \kappa_{kj}(S_k), & I_j = 0 \end{cases}.$$

Since κ_{kj} is increasing with positive values on Γ (in which $\boldsymbol{S} \le \boldsymbol{S}^0$ component-wise), it follows that $O_n \le \overline{\mathcal{M}}(\mathbf{S}) \le \widetilde{\mathcal{M}}^0$ on the feasible domain Γ. Also, if $\boldsymbol{S} \ne \boldsymbol{S}^0$, it follows that $O_n < \overline{\mathcal{M}}(\mathbf{S}) < \widetilde{\mathcal{M}}^0$, and from corollaries 2.1.5 and 2.1.10 of Berman and Plemmons,[35] one finds that $\rho(\overline{\mathcal{M}}(\mathbf{S})) < \rho(\widetilde{\mathcal{M}}^0)$. Therefore, if $\mathcal{R} \le 1$, then $\rho(\overline{\mathcal{M}}(\mathbf{S})) < 1$, and the equation $\overline{\mathcal{M}}(\mathbf{S})\mathbf{I}^T = \mathbf{I}^T$ has only the trivial solution. This leads to $\boldsymbol{R} = \boldsymbol{0}$ as well, and, combined with (i), to $\boldsymbol{S} = \boldsymbol{S}^0$. It follows that Equation 7 admits only the trivial equilibrium $\boldsymbol{E}^0$.

In the following, we shall consider the stability of $\boldsymbol{E}^0$ in the compact, positively invariant set Γ. Since $\widetilde{\mathcal{M}}^0$ is nonnegative and irreducible, it has a strictly positive left eigenvector $(\omega_1, \omega_2, \ldots, \omega_n)$ corresponding to the eigenvalue $\rho(\widetilde{\mathcal{M}}^0)$, that is,

$$(\omega_1, \omega_2, \ldots, \omega_n)\widetilde{\mathcal{M}}^0 = \rho(\widetilde{\mathcal{M}}^0)(\omega_1, \omega_2, \ldots, \omega_n).$$

We thereby construct the following candidate of a Lyapunov functional

$$V(t) = \sum_{k=1}^{n} \nu_k \left[\sum_{j=1}^{n} \int_0^h Q_k(\xi) e^{-\mu_k \xi} \left(\int_{t-\xi}^{t} f_{kj}(S_k(s), I_j(s)) ds \right) d\xi \right] + \sum_{k=1}^{n} \nu_k \left(I_k + \frac{\delta_k}{\mu_k + \delta_k} R_k \right),$$

in which

$$\nu_k = \frac{\omega_k(\mu_k + \delta_k)}{(\mu_k + \gamma_k + \alpha_k)(\mu_k + \delta_k) - \delta_k\gamma_k}.$$

The derivative of V along the solutions of Equation 7 reads then as

$$\begin{aligned}\frac{dV(t)}{dt} &= \sum_{k=1}^{n} \nu_k \left\{ \sum_{j=1}^{n} \int_0^h Q_k(\xi) e^{-\mu_k \xi} \left[f_{kj}(S_k(t), I_j(t)) - f_{kj}(S_k(t-\xi), I_j(t-\xi)) \right] d\xi \right\} \\ &\quad + \sum_{k=1}^{n} \nu_k \left(\frac{dI_k(t)}{dt} + \frac{\delta_k}{\mu_k + \delta_k} \frac{dR_k(t)}{dt} \right) \\ &= \sum_{k=1}^{n} \nu_k \sum_{j=1}^{n} \left\{ \int_0^h Q_k(\xi) e^{-\mu_k \xi} [f_{kj}(S_k(t), I_j(t)) - f_{kj}(S_k(t-\xi), I_j(t-\xi))] d\xi \right\} \\ &\quad + \sum_{k=1}^{n} \nu_k \left[\sum_{j=1}^{n} \int_0^h Q_k(\xi) e^{-\mu_k \xi} f_{kj}(S_k(t-\xi), I_j(t-\xi)) d\xi - (\mu_k + \gamma_k + \alpha_k) I_k + \frac{\delta_k \gamma_k}{\mu_k + \delta_k} I_k \right].\end{aligned}$$

By direct computations, one obtains that

$$\frac{dV(t)}{dt} = \sum_{k=1}^{n} \nu_k \sum_{j=1}^{n} \left[\left(\int_0^h Q_k(\xi) e^{-\mu_k \xi} d\xi \right) f_{kj}(S_k(t), I_j(t)) \right] - \sum_{k=1}^{n} \nu_k \left[(\mu_k + \gamma_k + \alpha_k) - \frac{\delta_k \gamma_k}{\mu_k + \delta_k} \right] I_k.$$

From the assumptions (iv) and (v) together with Equation 6, it is observed that

$$f_{kj}(S_k(t), I_j(t)) \le \kappa_{kj}(S_k) I_j(t) \le \kappa_{kj}\left(S_k^0\right) I_j(t).$$

Consequently, one then obtains that

$$\begin{aligned}\frac{dV(t)}{dt} &\le \sum_{k=1}^{n} \nu_k \left[\sum_{j=1}^{n} \left(\int_0^h Q_k(\xi) e^{-\mu_k \xi} d\xi \right) \kappa_{kj}\left(S_k^0\right) I_j - \left(\mu_k + \gamma_k + \alpha_k - \frac{\delta_k \gamma_k}{\mu_k + \delta_k} \right) I_k \right] \\ &\le \sum_{k=1}^{n} \nu_k \left(\mu_k + \gamma_k + \alpha_k - \frac{\delta_k \gamma_k}{\mu_k + \delta_k} \right) \left[\sum_{j=1}^{n} \frac{\left(\int_0^h Q_k(\xi) e^{-\mu_k \xi} d\xi \right) \kappa_{kj}\left(S_k^0\right)}{\left(\mu_k + \gamma_k + \alpha_k - \frac{\delta_k \gamma_k}{\mu_k + \delta_k} \right)} I_j - I_k \right] \\ &\le (\omega_1, \omega_2, \cdots, \omega_n)[\widetilde{\mathcal{M}}^0 \mathbf{I}^T - \mathbf{I}^T] = (\rho(\widetilde{\mathcal{M}}^0) - 1)(\omega_1, \omega_2, \cdots, \omega_n)\mathbf{I}^T \le 0.\end{aligned}$$

Also, $\{\boldsymbol{E}^0\}$ is the largest invariant set in

$$\left\{ (S_k, I_k, R_k) | \frac{dV(t)}{dt} = 0, k = 1, 2, \cdots, n \right\}.$$

Finally, by applying the Lyapunov-LaSalle invariance principle, we obtain that $\boldsymbol{E}^0$ is globally asymptotically stable. (b) If $\mathcal{R} > 1$ and $\boldsymbol{I}^T \neq 0$, we then have

$$(\omega_1, \omega_2, \cdots, \omega_n)[\widetilde{\mathcal{M}}^0 \mathbf{I}^T - \mathbf{I}^T] = (\rho(\widetilde{\mathcal{M}}^0) - 1)(\omega_1, \omega_2, \cdots, \omega_n)\mathbf{I}^T > 0.$$

By a continuity argument, one obtains that $\frac{dV(t)}{dt} > 0$ in a vicinity of the disease-free equilibrium $\boldsymbol{E}^0$, which implies that $\boldsymbol{E}^0$ is unstable. □

2.5 | The existence and stability of the endemic equilibrium

It has already been seen in Theorem 1 that if $\mathcal{R} \leq 1$, then the system (Equation 7) does not admit an endemic equilibrium. Let us now focus on the case $\mathcal{R} > 1$, in which, as previously mentioned, the disease-free equilibrium $\mathbf{E}^0$ is unstable.

Using Theorem 8.2.4 of Kuang[32] together with an argument similar to the one used in the proof of Proposition 3.3 of Li et al[36] (see also Liu et al,[37] section 5), one may prove that the system (Equation 7) is uniformly persistent.

As seen from corollary 2.8.8 in Bhatia and Szegö[38] or Theorem D.3 in Smith and Waltman,[39] the uniform persistence of the system (Equation 7) and the uniform boundedness of its solutions in Γ^0 ensure the existence of a rest point, that is, of an endemic equilibrium $\mathbf{E}^*$,

$$\mathbf{E}^* = \left(S_1^*, I_1^*, R_1^*, S_2^*, I_2^*, R_2^*, \cdots, S_n^*, I_n^*, R_n^*\right).$$

We may now focus on the study of its stability. To this purpose, we shall use again the Lyapunov-LaSalle invariance principle, after defining a suitable Lyapunov functional first.

Theorem 2. *If $\mathcal{R} > 1$ and*

(H) $D_{kj} \leq 0$, for all $S_k, I_k, I_j > 0$, where

$$D_{kj} = \left[\Phi_{kj}\left(S_k^*, I_j^*\right) f_{kk}\left(S_k, I_k^*\right) - \Phi_{kj}(S_k, I_j) f_{kk}\left(S_k^*, I_k^*\right)\right]\left[f_{kj}\left(S_k^*, I_j^*\right) f_{kk}\left(S_k, I_k^*\right) - f_{kj}(S_k, I_j) f_{kk}\left(S_k^*, I_k^*\right)\right],$$

then the endemic equilibrium $\boldsymbol{E}^*$ is globally asymptotically stable in Γ.

Proof. *Let us now denote*

$$A = \left(f_{kj}\left(S_k^*, I_j^*\right)\left(\int_0^h Q_k(\xi) e^{-\mu_k \xi} d\xi\right)\right)_{\substack{1 \leq k \leq n \\ 1 \leq j \leq n}}, \tag{8}$$

and let $\overline{\nu}_k = C_{kk}$, the cofactor of the (k,k)-entry of the Laplacian matrix of the directed graph $G(A)$ associated with A, as described in Section 2.3, so that

$$\sum_{k=1}^{n}\sum_{j=1}^{n} \overline{\nu}_k a_{kj} G_k(x_k) = \sum_{k=1}^{n}\sum_{j=1}^{n} \overline{\nu}_k a_{kj} G_j(x_j), \tag{9}$$

for any arbitrary family of functions $\{G_k(x_k)\}_{k=1}^n$. The reason for this particular definition of $\overline{\nu}_k$ and the choice of the family of functions $\{G_k(x_k)\}_{k=1}^n$ will be made clear in the last few lines of this proof.

To prove the global stability of the endemic equilibrium $\boldsymbol{E}^*$, let us construct a candidate of a Lyapunov functional W by

$$W(t) = \sum_{k=1}^{n} \overline{\nu}_k W_k(t), \tag{10}$$

with

$$\begin{aligned}
W_k(t) &= W_{1k}(t) + W_{2k}t) + W_{3k}(t) + W_{4k}(t),\\
W_{1k}(t) &= \left(\int_0^h Q_k(\xi) e^{-\mu_k \xi} d\xi\right)\left(S_k(t) - S_k^* - \int_{S_k^*}^{S_k(t)} \frac{f_{kk}\left(S_k^*, I_k^*\right)}{f_{kk}(s, I_k^*)} ds\right),\\
W_{2k}(t) &= I_k(t) - I_k^* - I_k^* \ln \frac{I_k(t)}{I_k^*},\\
W_{3k}(t) &= \frac{\delta_k}{\mu_k + \delta_k}\left(R_k(t) - R_k^* - R_k^* \ln \frac{R_k(t)}{R_k^*}\right),\\
W_{4k}(t) &= \sum_{j=1}^{n} \int_0^h \left[\int_{t-\xi}^{t} Q_k(\xi) e^{-\mu_k \xi}\left(f_{kj}\left(S_k(s), I_j(s)\right) - f_{kj}\left(S_k^*, I_j^*\right) - f_{kj}\left(S_k^*, I_j^*\right) \ln \frac{f_{kj}\left(S_k(s), I_j(s)\right)}{f_{kj}\left(S_k^*, I_j^*\right)}\right) ds\right] d\xi.
\end{aligned}$$

In other words, the Lyapunov functional W is a linear combination of Lyapunov functionals W_k, of the type used in our previous paper[28] to discuss the stability of the corresponding single-group model considered therein. Note that each functional W_k refers (functionally) only to the kth group and it is radially unbounded. The fourth component of W_k, W_{4k} has as its exclusive role to deal with the delay terms. Without delay, the functional W_k would be exactly the sum of W_{1k} (which deals with the first class, the susceptibles), W_{2k} (which deals with the second class, the infectives), and W_{3k} (which deals with the third class, the recovered).

Computing the derivatives of W_{1k}, W_{2k}, W_{3k}, and W_{4k} with respect to t along the solutions of the system (Equation 7) yields

$$\frac{dW_{1k}(t)}{dt} = \left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)\left(1-\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}\right)\left(n_k(S_k(t))-\sum_{j=1}^n f_{kj}(S_k(t),I_j(t))\right),$$

$$\frac{dW_{2k}(t)}{dt} = \left(1-\frac{I_k^*}{I_k(t)}\right)\left(\sum_{j=1}^n\int_0^h Q_k(\xi)e^{-\mu_k\xi}f_{kj}\left(S_k(t-\xi),I_j(t-\xi)\right)d\xi+\delta_k R_k(t)-(\mu_k+\alpha_k+\gamma_k)I_k(t)\right),$$

$$\frac{dW_{3k}(t)}{dt} = \frac{\delta_k}{\mu_k+\delta_k}\left(1-\frac{R_k^*}{R_k(t)}\right)\left(\gamma_k I_k(t)-(\mu s_k+\delta_k)R_k(t)\right),$$

$$\frac{dW_{4k}(t)}{dt} = \sum_{j=1}^n\int_0^h Q_k(\xi)e^{-\mu_k\xi}\left[f_{kj}(S_k(t),I_j(t))-f_{kj}\left(S_k(t-\xi),I_j(t-\xi)\right)+f_{kj}\left(S_k^*,I_j^*\right)\ln\frac{f_{kj}\left(S_k(t-\xi),I_j(t-\xi)\right)}{f_{kj}(S_k(t),I_j(t))}\right]d\xi.$$

By summing up the above equalities, we obtain, after some algebraic manipulations, the expression of the derivative of the functional W_k, in the form

$$\begin{aligned}\frac{dW_k(t)}{dt} &= \left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)\left(1-\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}\right)n_k(S_k(t))+\sum_{j=1}^n\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}f_{kj}(S_k(t),I_j(t))\\ &+\sum_{j=1}^n f_{kj}\left(S_k^*,I_j^*\right)\int_0^h Q_k(\xi)e^{-\mu_k\xi}\left(-\frac{I_k^*f_{kj}\left(S_k(t-\xi),I_j(t-\xi)\right)}{I_k(t)f_{kj}\left(S_k^*,I_j^*\right)}+\ln\frac{f_{kj}\left(S_k(t-\xi),I_j(t-\xi)\right)}{f_{kj}(S_k(t),I_j(t))}\right)d\xi\\ &+\left\{-\left[(\mu_k+\alpha_k+\gamma_k)-\frac{\delta_k\gamma_k}{\mu_k+\delta_k}\right]I_k(t)-\frac{I_k^*}{I_k(t)}\delta_k R_k(t)+(\mu_k+\alpha_k+\gamma_k)I_k^*-\frac{\delta_k\gamma_k}{\mu_k+\delta_k}\frac{R_k^*}{R_k(t)}I_k(t)+\delta_k R_k^*\right\}.\end{aligned}\tag{11}$$

To apply Lyapunov-LaSalle invariance principle for the functional W, the sum of all W_k's, $1\le k\le n$, we need to further manipulate the right-hand side of Equation 11 to obtain as many terms with negative signs as possible. The first step is the use of the equilibrium conditions to rearrange the right-hand side of Equation 11. After that, we shall use monotonicity properties and a certain algebraic inequality.

Let us denote

$$T_k = -\left[(\mu_k+\alpha_k+\gamma_k)-\frac{\delta_k\gamma_k}{\mu_k+\delta_k}\right]I_k(t)-\frac{I_k^*}{I_k(t)}\delta_k R_k(t)+(\mu_k+\alpha_k+\gamma_k)I_k^*-\frac{\delta_k\gamma_k}{\mu_k+\delta_k}\frac{R_k^*}{R_k(t)}I_k(t)+\delta_k R_k^*.$$

For the endemic equilibrium $\mathbf{E}^*$ to exist, one notes that the following equalities need to be satisfied

$$n_k\left(S_k^*\right) = \sum_{j=1}^n f_{kj}\left(S_k^*,I_j^*\right),\tag{12}$$

$$\sum_{j=1}^n\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)f_{kj}\left(S_k^*,I_j^*\right) = (\mu_k+\gamma_k+\alpha_k)I_k^*-\delta_k R_k^*,\tag{13}$$

$$R_k^* = \frac{\gamma_k}{\mu_k+\delta_k}I_k^*.\tag{14}$$

Using the equilibrium relations 13 and 14, one notes that

$$T_k = \sum_{j=1}^{n}\left[-\frac{I_k(t)}{I_k^*}f_{kj}\left(S_k^*,I_j^*\right)\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi + f_{kj}\left(S_k^*,I_j^*\right)\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right] + \delta_k R_k^*(t)\left(-\frac{I_k^*}{I_k(t)}\frac{R_k(t)}{R_k^*} + 2 - \frac{R_k^*}{R_k(t)}\frac{I_k(t)}{I_k^*}\right)$$
$$= \left(1-\frac{I_k(t)}{I_k^*}\right)\sum_{j=1}^{n}\left[f_{kj}\left(S_k^*,I_j^*\right)\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right] - \delta_k R_k^*(t)\left(\sqrt{\frac{I_k^*}{I_k(t)}\frac{R_k(t)}{R_k^*}} - \sqrt{\frac{R_k^*}{R_k(t)}\frac{I_k(t)}{I_k^*}}\right)^2. \tag{15}$$

Now, the second term is clearly negative, while the first one will be paired with another one in the right-hand side of Equation 11. Substituting the expression of T_k given by Equation 15 back into Equation 11 gives

$$\frac{dW_k(t)}{dt} = \left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)\left(1-\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}\right)n_k(S_k(t)) + \sum_{j=1}^{n}\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}f_{kj}(S_k(t),I_j(t))$$
$$+\sum_{j=1}^{n}f_{kj}\left(S_k^*,I_j^*\right)\int_0^h Q_k(\xi)e^{-\mu_k\xi}\left(-\frac{I_k^*f_{kj}\left(S_k(t-\xi),I_j(t-\xi)\right)}{I_k(t)f_{kj}\left(S_k^*,I_j^*\right)} + \ln\frac{f_{kj}\left(S_k(t-\xi),I_j(t-\xi)\right)}{f_{kj}(S_k(t),I_j(t))}\right)d\xi \tag{16}$$
$$+\left(1-\frac{I_k(t)}{I_k^*}\right)\sum_{j=1}^{n}\left[f_{kj}\left(S_k^*,I_j^*\right)\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right] - \delta_k R_k^*(t)\left(\sqrt{\frac{I_k^*}{I_k(t)}\frac{R_k(t)}{R_k^*}} - \sqrt{\frac{R_k^*}{R_k(t)}\frac{I_k(t)}{I_k^*}}\right)^2.$$

To further rearrange the first term in the right-hand side of Equation 16 to obtain another term with negative sign out of it via sign conditions on n_k and f_{kk} (assumptions (i) and (iii)), we use again the equilibrium condition 12. It follows that

$$\frac{dW_k(t)}{dt} = \left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)\left(1-\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}\right)\left(n_k(S_k(t)) - n_k\left(S_k^*\right)\right)$$
$$+\sum_{j=1}^{n}f_{kj}\left(S_k^*,I_j^*\right)\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)\left[\frac{f_{kj}(S_k(t),I_j(t))}{f_{kk}(S_k(t),I_k^*)}\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kj}\left(S_k^*,I_j^*\right)} + 2 - \frac{I_k(t)}{I_k^*} - \frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}\right]$$
$$+\sum_{j=1}^{n}f_{kj}\left(S_k^*,I_j^*\right)\int_0^h Q_k(\xi)e^{-\mu_k\xi}\left(-\frac{I_k^*f_{kj}\left(S_k(t-\xi),I_j(t-\xi)\right)}{I_k(t)f_{kj}\left(S_k^*,I_j^*\right)} + \ln\frac{f_{kj}\left(S_k(t-\xi),I_j(t-\xi)\right)}{f_{kj}(S_k(t),I_j(t))}\right)d\xi$$
$$-\delta_k R_k^*(t)\left(\sqrt{\frac{I_k^*}{I_k(t)}\frac{R_k(t)}{R_k^*}} - \sqrt{\frac{R_k^*}{R_k(t)}\frac{I_k(t)}{I_k^*}}\right)^2.$$

Now, the terms with uncertain signs in the above inequality are the second and the third one. We set our sights on the third.

We intend to use to our advantage the inequality

$$1 - x + \ln x \le 0, \quad \textit{for all } x > 0,$$

with equality if and only if $x = 1$. Let us define

$$F : (0,\infty) \to (-\infty, 0], \quad F(x) = 1 - x + \ln x.$$

Since

$$-\frac{I_k^*f_{kj}\left(S_k(t-\xi),I_j(t-\xi)\right)}{I_k(t)f_{kj}\left(S_k^*,I_j^*\right)} + \ln\frac{f_{kj}\left(S_k(t-\xi),I_j(t-\xi)\right)}{f_{kj}(S_k(t),I_j(t))} = 1 - \frac{I_k^*f_{kj}\left(S_k(t-\xi),I_j(t-\xi)\right)}{I_k(t)f_{kj}\left(S_k^*,I_j^*\right)} + \ln\frac{I_k^*f_{kj}\left(S_k(t-\xi),I_j(t-\xi)\right)}{I_k(t)f_{kj}\left(S_k^*,I_j^*\right)}$$
$$+\ln\frac{I_k(t)f_{kj}\left(S_k^*,I_j^*\right)}{I_k^*f_{kj}(S_k(t),I_j(t))} - 1 = F\left(\frac{I_k^*f_{kj}\left(S_k(t-\xi),I_j(t-\xi)\right)}{I_k(t)f_{kj}\left(S_k^*,I_j^*\right)}\right) + \ln\frac{I_k(t)}{I_k^*} + \ln\frac{f_{kj}\left(S_k^*,I_j^*\right)}{f_{kj}(S_k(t),I_j(t))} - 1,$$

it follows that

$$\begin{aligned}\frac{dW_k(t)}{dt} &= \left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)\left(1-\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}\right)(n_k(S_k(t))-n_k\left(S_k^*\right))\\ &+\sum_{j=1}^n f_{kj}\left(S_k^*,I_j^*\right)\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)\\ &\cdot\left[\frac{f_{kj}(S_k(t),I_j(t))}{f_{kk}\left(S_k(t),I_k^*\right)}\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kj}\left(S_k^*,I_j^*\right)}+1-\frac{I_k(t)}{I_k^*}-\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}+\ln\frac{I_k(t)}{I_k^*}+\ln\frac{f_{kj}\left(S_k^*,I_j^*\right)}{f_{kj}(S_k(t),I_j(t))}\right]\\ &+\sum_{j=1}^n f_{kj}\left(S_k^*,I_j^*\right)\int_0^h Q_k(\xi)e^{-\mu_k\xi}F\left(\frac{I_k^*f_{kj}\left(S_k(t-\xi),I_j(t-\xi)\right)}{I_k(t)f_{kj}\left(S_k^*,I_j^*\right)}\right)d\xi\\ &-\delta_kR_k^*(t)\left(\sqrt{\frac{I_k^*}{I_k(t)}\frac{R_k(t)}{R_k^*}}-\sqrt{\frac{R_k^*}{R_k(t)}\frac{I_k(t)}{I_k^*}}\right)^2.\end{aligned}$$

It now only remains to establish the sign of the expression inside the square brackets, corresponding to the second term in the above inequality, since the other terms are already negative. Let us denote

$$P_k=\frac{f_{kj}(S_k(t),I_j(t))}{f_{kk}\left(S_k(t),I_k^*\right)}\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kj}\left(S_k^*,I_j^*\right)}+1-\frac{I_k(t)}{I_k^*}-\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}+\ln\frac{I_k(t)}{I_k^*}+\ln\frac{f_{kj}\left(S_k^*,I_j^*\right)}{f_{kj}(S_k(t),I_j(t))}.$$

Again, the main tool to establish the sign of P_k is to use the negative sign of F. In fact, we shall try to isolate inside of P_k as many values of F as necessary, for suitable arguments. It follows that

$$\begin{aligned}P_k &= \frac{f_{kj}(S_k(t),I_j(t))}{f_{kk}(S_k(t),I_k^*)}\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kj}\left(S_k^*,I_j^*\right)}+1-\frac{I_j(t)}{I_j^*}-\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}+\ln\frac{I_j(t)}{I_j^*}+\ln\frac{f_{kj}\left(S_k^*,I_j^*\right)}{f_{kj}(S_k(t),I_j(t))}\\ &+\left(-\frac{I_k(t)}{I_k^*}+\ln\frac{I_k(t)}{I_k^*}+\frac{I_j(t)}{I_j^*}-\ln\frac{I_j(t)}{I_j^*}\right)\\ &=F\left(\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}\right)+\frac{f_{kj}(S_k(t),I_j(t))}{f_{kk}(S_k(t),I_k^*)}\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kj}\left(S_k^*,I_j^*\right)}-\frac{I_j(t)}{I_j^*}+\ln\left(\frac{I_j(t)}{I_j^*}\frac{f_{kj}\left(S_k^*,I_j^*\right)}{f_{kj}(S_k(t),I_j(t))}\frac{f_{kk}(S_k(t),I_k^*)}{f_{kk}\left(S_k^*,I_k^*\right)}\right)\\ &+\left(-\frac{I_k(t)}{I_k^*}+\ln\frac{I_k(t)}{I_k^*}+\frac{I_j(t)}{I_j^*}-\ln\frac{I_j(t)}{I_j^*}\right)\\ &=F\left(\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}\right)+F\left(\frac{I_j(t)}{I_j^*}\frac{f_{kj}\left(S_k^*,I_j^*\right)}{f_{kj}(S_k(t),I_j(t))}\frac{f_{kk}(S_k(t),I_k^*)}{f_{kk}\left(S_k^*,I_k^*\right)}\right)+\frac{f_{kj}(S_k(t),I_j(t))}{f_{kk}(S_k(t),I_k^*)}\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kj}\left(S_k^*,I_j^*\right)}-\frac{I_j(t)}{I_j^*}-1\\ &+\frac{I_j(t)}{I_j^*}\frac{f_{kj}\left(S_k^*,I_j^*\right)}{f_{kj}(S_k(t),I_j(t))}\frac{f_{kk}(S_k(t),I_k^*)}{f_{kk}\left(S_k^*,I_k^*\right)}+\left(-\frac{I_k(t)}{I_k^*}+\ln\frac{I_k(t)}{I_k^*}+\frac{I_j(t)}{I_j^*}-\ln\frac{I_j(t)}{I_j^*}\right)\\ &=F\left(\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}\right)+F\left(\frac{I_j(t)}{I_j^*}\frac{f_{kj}\left(S_k^*,I_j^*\right)}{f_{kj}(S_k(t),I_j(t))}\frac{f_{kk}(S_k(t),I_k^*)}{f_{kk}\left(S_k^*,I_k^*\right)}\right)\\ &+\left(\frac{I_j(t)}{I_j^*}-\frac{f_{kj}(S_k(t),I_j(t))}{f_{kj}\left(S_k^*,I_j^*\right)}\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}\right)\left(\frac{f_{kj}\left(S_k^*,I_j^*\right)}{f_{kj}(S_k(t),I_j(t))}\frac{f_{kk}(S_k(t),I_k^*)}{f_{kk}\left(S_k^*,I_k^*\right)}-1\right)\\ &+\left(-\frac{I_k(t)}{I_k^*}+\ln\frac{I_k(t)}{I_k^*}+\frac{I_j(t)}{I_j^*}-\ln\frac{I_j(t)}{I_j^*}\right).\end{aligned}$$

We are now left with 2 terms that still resist the sign analysis. The first one (the product of parantheses) will be discussed by means of ***(H)****. Actually, this is the very (and only) reason why* ***(H)*** *is used. The second one (the one containing logarithms)*

will be analyzed by means of the summation lemma, Lemma 1, after all similar terms for each group are inserted back into the derivative of W (at this moment, P_k comes only from the derivative of W_k). The choice of A as given in Equation 8 is solely motivated by the need to estimate the corresponding terms in the derivative of W.

By the definition of Φ_{kj}, it follows that

$$\begin{aligned}
&\left(\frac{I_j(t)}{I_j^*}-\frac{f_{kj}(S_k(t),I_j(t))}{f_{kj}\left(S_k^*,I_j^*\right)}\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}\right)\left(\frac{f_{kj}\left(S_k^*,I_j^*\right)}{f_{kj}(S_k(t),I_j(t))}\frac{f_{kk}(S_k(t),I_k^*)}{f_{kk}\left(S_k^*,I_k^*\right)}-1\right)\\
&=\frac{I_j(t)}{I_j^*}\left(1-\frac{\Phi_{kj}(S_k(t),I_j(t))}{\Phi_{kj}\left(S_k^*,I_j^*\right)}\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}\right)\cdot\left(\frac{f_{kj}\left(S_k^*,I_j^*\right)}{f_{kj}(S_k(t),I_j(t))}\frac{f_{kk}(S_k(t),I_k^*)}{f_{kk}\left(S_k^*,I_k^*\right)}-1\right)\\
&=\frac{I_j(t)}{I_j^*}\frac{1}{\Phi_{kj}\left(S_k^*,I_j^*\right)f_{kk}(S_k(t),I_k^*)}\frac{1}{f_{kj}(S_k(t),I_j(t))f_{kk}\left(S_k^*,I_k^*\right)}\\
&\quad\cdot\left(\Phi_{kj}\left(S_k^*,I_j^*\right)f_{kk}(S_k(t),I_k^*)-\Phi_{kj}(S_k(t),I_j(t))f_{kk}\left(S_k^*,I_k^*\right)\right)\left(f_{kj}\left(S_k^*,I_j^*\right)f_{kk}(S_k(t),I_k^*)-f_{kj}(S_k(t),I_j(t))f_{kk}\left(S_k^*,I_k^*\right)\right)\\
&=\frac{I_j(t)}{I_j^*}\frac{1}{\Phi_{kj}\left(S_k^*,I_j^*\right)f_{kk}(S_k(t),I_k^*)}\frac{1}{f_{kj}(S_k(t),I_j(t))f_{kk}\left(S_k^*,I_k^*\right)}D_{kj}.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\frac{dW_k(t)}{dt}&=\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)\left(1-\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}\right)\left(n_k(S_k(t))-n_k\left(S_k^*\right)\right)\\
&\quad+\sum_{j=1}^n f_{kj}\left(S_k^*,I_j^*\right)\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)F\left(\frac{f_{kk}\left(S_k^*,I_k^*\right)}{f_{kk}(S_k(t),I_k^*)}\right)\\
&\quad+\sum_{j=1}^n f_{kj}\left(S_k^*,I_j^*\right)\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)F\left(\frac{I_j(t)}{I_j^*}\frac{f_{kj}\left(S_k^*,I_j^*\right)}{f_{kj}(S_k(t),I_j(t))}\frac{f_{kk}(S_k(t),I_k^*)}{f_{kk}\left(S_k^*,I_k^*\right)}\right)\\
&\quad+\sum_{j=1}^n f_{kj}\left(S_k^*,I_j^*\right)\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)D_{kj}\\
&\quad+\sum_{j=1}^n f_{kj}\left(S_k^*,I_j^*\right)\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)\left(-\frac{I_k(t)}{I_k^*}+\ln\frac{I_k(t)}{I_k^*}+\frac{I_j(t)}{I_j^*}-\ln\frac{I_j(t)}{I_j^*}\right)\\
&\quad+\sum_{j=1}^n f_{kj}\left(S_k^*,I_j^*\right)\int_0^h Q_k(\xi)e^{-\mu_k\xi}F\left(\frac{I_k^*f_{kj}\left(S_k(t-\xi),I_j(t-\xi)\right)}{I_k(t)f_{kj}\left(S_k^*,I_j^*\right)}\right)d\xi\\
&\quad-\delta_k R_k^*(t)\left(\sqrt{\frac{I_k^*}{I_k(t)}\frac{R_k(t)}{R_k^*}}-\sqrt{\frac{R_k^*}{R_k(t)}\frac{I_k(t)}{I_k^*}}\right)^2.
\end{aligned}$$

Now, all terms have been rearranged conveniently. Actually, all terms in the right-hand side of the above equality are known to be negative, except for a single one (the one containing logarithms). This yields

$$\frac{dW_k(t)}{dt}\le\sum_{j=1}^n f_{kj}\left(S_k^*,I_j^*\right)\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)\left(-\frac{I_k(t)}{I_k^*}+\ln\frac{I_k(t)}{I_k^*}+\frac{I_j(t)}{I_j^*}-\ln\frac{I_j(t)}{I_j^*}\right).$$

Consequently, since

$$\frac{dW(t)}{dt} = \sum_{k=1}^{n} \overline{\nu}_k \frac{dW_k(t)}{dt},$$

it follows that

$$\frac{dW(t)}{dt} \leq \sum_{k=1}^{n}\sum_{j=1}^{n} \overline{\nu}_k f_{kj}\left(S_k^*, I_j^*\right)\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right) G_k(I_k) - \sum_{k=1}^{n}\sum_{j=1}^{n} \overline{\nu}_k f_{kj}\left(S_k^*, I_j^*\right)\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right) G_j(I_j),$$

with

$$G_k(I_k) = -\frac{I_k}{I_k^*} + \ln\frac{I_k}{I_k^*}.$$

Using the notation given in Equation 8, this inequality may be restated as

$$\frac{dW(t)}{dt} \leq \sum_{k=1}^{n}\sum_{j=1}^{n} \overline{\nu}_k a_{kj} G_k(x_k) - \sum_{k=1}^{n}\sum_{j=1}^{n} \overline{\nu}_k a_{kj} G_j(x_j).$$

Now, because of Equation 9, the right-hand side of the above inequality is 0. Hence, $\frac{dW(t)}{dt} \leq 0$ *for all* $t \geqslant 0$*, and the equality holds only at the endemic equilibrium* E^**. By Lyapunov-LaSalle principle,* E^* *is globally asymptotically stable in* Γ*, which completes the proof.* □

Remark 3. *Note that if the nonlinear incidence function* $f_{kj}(S_k, I_j)$ *factorizes as a product of functions depending upon a single variable, in the form*

$$f_{kj}(S_k, I_j) = g_k(S_k)h_j(I_j),$$

then (*H*) *reduces to*

$$\left(\frac{h_j(I_j^*)}{I_j^*} - \frac{h_j(I_j)}{I_j}\right)\left(h_j(I_j^*) - h_j(I_j)\right) \leq 0,$$

which holds true due to (iii) and (iv). Consequently, in this case (*H*) *does not need to be assumed separately. However, if* $f_{kj}(S_k, I_j)$ *does not factorize in the above form, then* (*H*) *does not follow from the other assumptions. In this regard, although it is somewhat symmetric (note that the second parenthesis in* (*H*) *is obtained from the first one by changing the* Φ*'s into f's), condition* (*H*) *involves both variables S and I in a nontrivial manner, as opposed to (iii) and (iv), which are single-variable conditions. Some examples of incidence functions that do not factorize are the saturated incidence and the modified saturated incidence mentioned in Remark 2.*

3 | THE MULTIGROUP STOCHASTIC *SEIRI* EPIDEMIC MODEL AND ITS ANALYSIS

3.1 | The model and its relevance

Apart from focusing on deterministic models, May also pointed out that environmental randomness plays an important role in the evolution of populations.[40] To further this line of thought, many papers modeled environmental randomness by means of using stochastic differential equations. In this regard, one of the interesting assumptions is that the environment noise affects mainly the infection-related parameters (see previous studies[41-43] and the references therein). Since in our model, we use a general incidence of infection, we thereby follow Carletti,[25] Beretta et al,[23] and Ji et al[26] and consider stochastic perturbations of white noise type, which are directly proportional to distances between S_k, I_k, R_k and the respective values of each component of the equilibrium $\hat{S}_k$, $\hat{I}_k$, $\hat{R}_k$. Under these assumptions, the system (Equation 7) is then reduced to the following form

$$\begin{cases} \frac{dS_k}{dt} = n_k(S_k) - \sum_{j=1}^{n} f_{kj}(S_k(t), I_j(t)) + \sigma_{1k}(S_k - \hat{S}_k)\frac{dB_{1k}}{dt}, \\ \frac{dI_k}{dt} = \sum_{j=1}^{n} \int_0^h Q_k(\xi) e^{-\mu_k \xi} f_{kj}\big(S_k(t-\xi), I_j(t-\xi)\big) d\xi - (\mu_k + \gamma_k + \alpha_k) I_k(t) + \delta_k R_k(t) + \sigma_{2k}(I_k - \hat{I}_k)\frac{dB_{2k}}{dt}, \\ \frac{dR_k}{dt} = \gamma_k I_k(t) - (\mu_k + \delta_k) R_k(t) + \sigma_{3k}(R_k - \hat{R}_k)\frac{dB_{3k}}{dt}, \end{cases} \tag{17}$$

in which B_{1k}, B_{2k}, and B_{3k}, $k = 1, 2, \dots, n$, are independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and σ_{ik}^2 represent the respective intensities of B_{ik}, $i = 1, 2, 3$, $k = 1, 2, \dots, n$. Here, the equilibrium $\hat{\mathbf{E}} = (\hat{S}_1, \hat{I}_1, \hat{R}_1, \dots, \hat{S}_n, \hat{I}_n, \hat{R}_n)$ may be either the disease-free equilibrium $\mathbf{E}^0$ or the endemic equilibrium $\mathbf{E}^*$.

Before starting to analyze the stochastic stability of the above-mentioned equilibria of Equation 17, we first introduce certain prerequisite notions and results. Let us consider the n-dimensional stochastic functional differential equation

$$d\mathcal{X} = \mathfrak{f}\big(\mathcal{X}_t, t\big)dt + \mathfrak{g}\big(\mathcal{X}_t, t\big)d\mathcal{B}(t) \tag{18}$$

with initial condition $\mathcal{X}(t_0) = \mathcal{X}_0 \in \mathbf{C}^+\big([-h, 0], \mathbb{R}^n\big)$, the space of continuous functions from $[-h, 0]$ to $\mathbb{R}^n$ with norm $\|\psi\| = \sup_{\theta \in [-h,0]} |\psi(\theta)|$.

Suppose that Equation 18 admits the trivial solution. Also, let $\mathbf{C}^{2,1}\big(\mathbb{R}^n \times [t_0, \infty); \mathbb{R}^+\big)$ be the family of all nonnegative functions $V(\mathcal{X}, t)$ defined on $\mathbb{R}^n \times [t_0, \infty)$, which are continuously differentiable, twice in $\mathcal{X}$ and once in t. Define the differential operator L associated with Equation 18 by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \mathfrak{f}_i(\mathcal{X}(t-\xi), t)\frac{\partial}{\partial \mathcal{X}_i} + \frac{1}{2}\sum_{i,j=1}^{n}\Big[\mathfrak{g}^T\big(\mathcal{X}(t-\xi), t\big)\mathfrak{g}\big(\mathcal{X}(t-\xi), t\big)\Big]\frac{\partial^2}{\partial \mathcal{X}_i \partial \mathcal{X}_j}.$$

Definition 1.

1. *The trivial solution of Equation 18 is said to be stochastically stable or stable in probability if for every pair of* $\epsilon \in (0, 1)$ *and* $r > 0$*, there exists a* $\widehat{\delta} = \widehat{\delta}(\epsilon, r, t_0)$ *such that*

$$\mathcal{P}\big\{|\mathcal{X}(t; t_0, \mathcal{X}_0)| < r \quad \text{for all} \quad t \geqslant t_0\big\} \geqslant 1 - \epsilon$$

   whenever $|\mathcal{X}_0| < \widehat{\delta}$*. Otherwise, the trivial solution of it is said to be stochastically unstable.*
2. The trivial solution of Equation 18 is said to be stochastically asymptotically stable if it is stochastically stable and, moreover, for every $\epsilon \in (0, 1)$, there exists a $\widehat{\delta} = \widehat{\delta}(\epsilon, t_0)$ such that

$$\mathcal{P}\big\{\lim_{t\to\infty} \mathcal{X}(t; t_0, \mathcal{X}_0) = 0\big\} \geqslant 1 - \epsilon$$

   whenever $|\mathcal{X}_0| < \widehat{\delta}$.

Definition 2. *A continuous nonnegative function* $V(\mathcal{X}, t)$ *is said to be decrescent if for some* $\nu \in \vartheta$

$$V(\mathcal{X}, t) \leq \nu(|\mathcal{X}|)$$

for all $(\mathcal{X}, t) \in \boldsymbol{C}^+\big([-h, 0], \mathbb{R}^n\big) \times [t_0, \infty)$*, where* ϑ *denotes the family of all continuous nondecreasing functions* ν*:* $\mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\nu(0) = 0$ *and* $\nu(s) > 0$ *if* $s > 0$*.*

Lemma 2. *If there exists a positive-definite decrescent function* $V(\mathcal{X}, t) \in \boldsymbol{C}^{2,1}\big(\mathbb{R}^n \times [t_0, \infty); \mathbb{R}^+\big)$ *such that* $LV(\mathcal{X}, t)$ *is negative definite, then the trivial solution of Equation 18 is stochastically asymptotically stable.*

3.2 | A stochastic stability analysis

In this section, we shall study the stochastic stability of $\hat{\mathbf{E}}$ by constructing an appropriate Lyapunov functional. By applying the variable change

$$x_k = S_k - \hat{S}_k, \quad y_k = I_k - \hat{I}_k, \quad z_k = R_k - \hat{R}_k,$$

the system (Equation 17) can be restated as

$$\begin{cases} \frac{dx_k}{dt} = n_k(x_k + \hat{S}_k) - \sum_{j=1}^{n} f_{kj}(x_k + \hat{S}_k, y_j + \hat{I}_j) + \sigma_{1k} x_k \frac{dB_{1k}}{dt}, \\ \frac{dy_k}{dt} = \sum_{j=1}^{n} \int_0^h Q_k(\xi) e^{-\mu_k \xi} f_{kj}\big(x_k(t-\xi) + \hat{S}_k, y_j(t-\xi) + \hat{I}_j\big) d\xi \\ \qquad -(\mu_k + \gamma_k + \alpha_k)(y_k + \hat{I}_k) + \delta_k(z_k(t) + \hat{R}_k) + \sigma_{2k} y_k \frac{dB_{2k}}{dt}, \\ \frac{dz_k}{dt} = \gamma y_k - (\mu + \delta) z_k + \sigma_{3k} z_k \frac{dB_{3k}}{dt}. \end{cases} \tag{19}$$

To obtain sufficient conditions for the stochastic stability of the null solution of the system (Equation 19), we shall consider its linearization

$$\begin{cases} \frac{dx_k}{dt} = n_k'(\hat{S}_k) x_k - \sum_{j=1}^{n} f_{kj_{\hat{S}_k}} x_k - \sum_{j=1}^{n} f_{kj_{\hat{I}_j}} y_j + \sigma_{1k} x_k \frac{dB_{1k}}{dt}, \\ \frac{dy_k}{dt} = \sum_{j=1}^{n} \int_0^h Q_k(\xi) e^{-\mu_k \xi} \big(f_{kj_{\hat{S}_k}} x_k(t-\xi) + f_{kj_{\hat{I}_j}} y_j(t-\xi)\big) d\xi - (\mu_k + \gamma_k + \alpha_k) y_k + \delta_k z_k + \sigma_{2k} y_k \frac{dB_{2k}}{dt}, \\ \frac{dz_k}{dt} = \gamma y_k - (\mu + \delta) z_k + \sigma_{3k} z_k \frac{dB_{3k}}{dt}, \end{cases} \tag{20}$$

in which we have used the notations $f_{kj_{\hat{S}_k}} = \frac{\partial f_{kj}}{\partial S_k}(\hat{S}_k, \hat{I}_j)$ and $f_{kj_{\hat{I}_j}} = \frac{\partial f_{kj}}{\partial I_j}(\hat{S}_k, \hat{I}_j)$. We then obtain the following result.

Theorem 3. *If n_k is differentiable and the following conditions hold*

(i) $\sigma_{1k}^2 < \sum_{j=1}^{n} f_{kj_{\hat{S}_k}} - 2n_k'(\hat{S}_k) - \sum_{j=1}^{n} f_{kj_{\hat{I}_j}}$;

(ii) $\sigma_{2k}^2 < 2(\mu_k + \alpha_k) + \gamma_k - \delta_k - \sum_{j=1}^{n} \left[(f_{kj_{\hat{S}_k}} + f_{kj_{\hat{I}_j}}) \big(\int_0^h Q_k(\xi) e^{-\mu_k \xi} d\xi \big)^2 \right] - 2 \sum_{j=1}^{n} f_{jk_{\hat{I}_k}}$;

(iii) $\sigma_{3k}^2 < 2\mu_k + \delta_k - \gamma_k$;

for $k, j = 1, 2, \ldots, n$, then the null solution of Equation 20 is stochastically asymptotically stable.

Proof. *Let us define*

$$U = U_1 + U_2 + U_3, \quad with \quad U_1 = \sum_{k=1}^{n} x_k^2, \quad U_2 = \sum_{k=1}^{n} y_k^2, \quad U_3 = \sum_{k=1}^{n} z_k^2.$$

It follows that

$$LU_1 = 2 \sum_{k=1}^{n} x_k \left[n_k'(\hat{S}_k) x_k - \sum_{j=1}^{n} f_{kj_{\hat{S}_k}} x_k - \sum_{j=1}^{n} f_{kj_{\hat{I}_j}} y_j \right] + \sum_{k=1}^{n} \sigma_{1k}^2 x_k^2;$$

$$LU_2 = 2 \sum_{k=1}^{n} y_k \left[\sum_{j=1}^{n} \int_0^h Q_k(\xi) e^{-\mu_k \xi} \big(f_{kj_{\hat{S}_k}} x_k(t-\xi) + f_{kj_{\hat{I}_j}} y_j(t-\xi)\big) d\xi - (\mu_k + \gamma_k + \alpha_k) y_k + \delta_k z_k \right] + \sum_{k=1}^{n} \sigma_{2k}^2 y_k^2;$$

$$LU_3 = 2 \sum_{k=1}^{n} z_k \left[\gamma_k y_k - (\mu_k + \delta_k) z_k \right] + \sum_{k=1}^{n} \sigma_{3k}^2 z_k^2.$$

We then have that

$$\begin{aligned} LU = & \sum_{k=1}^{n} x_k^2 \left[2n_k'(\hat{S}_k) - 2 \sum_{j=1}^{n} f_{kj_{\hat{S}_k}} + \sigma_{1k}^2 \right] + \sum_{k=1}^{n} y_k^2 \left[-2(\mu_k + \gamma_k + \alpha_k) + \sigma_{2k}^2 \right] + \sum_{k=1}^{n} z_k^2 \left[-2(\mu_k + \delta_k) + \sigma_{3k}^2 \right] \\ & - 2 \sum_{k=1}^{n} \sum_{j=1}^{n} f_{kj_{\hat{I}_j}} x_k y_j + 2 \sum_{k=1}^{n} \sum_{j=1}^{n} \left(\int_0^h Q_k(\xi) e^{-\mu_k \xi} x_k(t-\xi) d\xi \right) y_k f_{kj_{\hat{S}_k}} \\ & + 2 \sum_{k=1}^{n} \sum_{j=1}^{n} \left(\int_0^h Q_k(\xi) e^{-\mu_k \xi} y_j(t-\xi) d\xi \right) y_k f_{kj_{\hat{I}_j}} + 2 \sum_{k=1}^{n} (\delta_k + \gamma_k) y_k z_k. \end{aligned}$$

It is obvious that

$$2\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}x_k(t-\xi)d\xi\right)y_k f_{kj_{\hat{S}_k}} \le K_{1_{kj}}y_k^2 + \frac{1}{K_{1_{kj}}}f_{kj_{\hat{S}_k}}^2\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}x_k(t-\xi)d\xi\right)^2;$$
$$2\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}y_j(t-\xi)d\xi\right)y_k f_{kj_{\hat{j}_j}} \le K_{2_{kj}}y_k^2 + \frac{1}{K_{2_{kj}}}f_{kj_{\hat{j}_j}}^2\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}y_j(t-\xi)d\xi\right)^2,$$

in which $K_{1_{kj}}$ *and* $K_{2_{kj}}$, $k,j = 1, 2, \dots, n$, *are positive coefficients to be chosen later. We then obtain that*

$$\begin{aligned}LU \le &\sum_{k=1}^n x_k^2\left[2n_k'(\hat{S}_k) - 2\sum_{j=1}^n f_{kj_{\hat{S}_k}} + \sigma_{1k}^2\right] + \sum_{k=1}^n y_k^2\left[-2(\mu_k+\gamma_k+\alpha_k) + \sigma_{2k}^2 + \sum_{j=1}^n K_{1_{kj}} + \sum_{j=1}^n K_{2_{kj}} + \delta_k + \gamma_k\right]\\
&+ \sum_{k=1}^n z_k^2\left[-2(\mu_k+\delta_k) + \sigma_{3k}^2 + \delta_k + \gamma_k\right] - 2\sum_{k=1}^n\sum_{j=1}^n f_{kj_{\hat{j}_j}}x_k y_j + \sum_{k=1}^n\sum_{j=1}^n \frac{f_{kj_{\hat{S}_k}}^2}{K_{1_{kj}}}\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}x_k(t-\xi)d\xi\right)^2\\
&+ \sum_{k=1}^n\sum_{j=1}^n \frac{f_{kj_{\hat{j}_j}}^2}{K_{2_{kj}}}\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}y_j(t-\xi)d\xi\right)^2.\end{aligned}$$

It follows from the integral form of the Cauchy-Schwarz inequality that

$$\begin{aligned}LU \le &\sum_{k=1}^n x_k^2\left[2n_k'(\hat{S}_k) - 2\sum_{j=1}^n f_{kj_{\hat{S}_k}} + \sigma_{1k}^2\right] + \sum_{k=1}^n y_k^2\left[-2(\mu_k+\gamma_k+\alpha_k) + \sigma_{2k}^2 + \sum_{j=1}^n K_{1_{kj}} + \sum_{j=1}^n K_{2_{kj}} + \delta_k + \gamma_k\right]\\
&+ \sum_{k=1}^n z_k^2\left[-2(\mu_k+\delta_k) + \sigma_{3k}^2 + \delta_k + \gamma_k\right] - 2\sum_{k=1}^n\sum_{j=1}^n f_{kj_{\hat{j}_j}}x_k y_j\\
&+ \sum_{k=1}^n\sum_{j=1}^n \frac{f_{kj_{\hat{S}_k}}^2}{K_{1_{kj}}}\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}x_k^2(t-\xi)d\xi\right)\\
&+ \sum_{k=1}^n\sum_{j=1}^n \frac{f_{kj_{\hat{j}_j}}^2}{K_{2_{kj}}}\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}y_j^2(t-\xi)d\xi\right).\end{aligned}$$

Let

$$\begin{aligned}W =& U + U_4,\\
U_4 =& \sum_{k=1}^n\sum_{j=1}^n \frac{f_{kj_{\hat{S}_k}}^2}{K_{1_{kj}}}\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}\int_{t-\xi}^t x_k^2(s)dsd\xi\right)\\
&+ \sum_{k=1}^n\sum_{j=1}^n \frac{f_{kj_{\hat{j}_j}}^2}{K_{2_{kj}}}\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}\int_{t-\xi}^t y_j^2(s)dsd\xi\right).\end{aligned}$$

Then

$$\begin{aligned}LW \le &\sum_{k=1}^n x_k^2\left[2n_k'(\hat{S}_k) - 2\sum_{j=1}^n f_{kj_{\hat{S}_k}} + \sigma_{1k}^2\right] + \sum_{k=1}^n y_k^2\left[-2(\mu_k+\gamma_k+\alpha_k) + \sigma_{2k}^2 + \sum_{j=1}^n K_{1_{kj}} + \sum_{j=1}^n K_{2_{kj}} + \delta_k + \gamma_k\right]\\
&+ \sum_{k=1}^n z_k^2\left[-2(\mu_k+\delta_k) + \sigma_{3k}^2 + \delta_k + \gamma_k\right] - 2\sum_{k=1}^n\sum_{j=1}^n f_{kj_{\hat{j}_j}}x_k y_j + \sum_{k=1}^n\sum_{j=1}^n x_k^2\frac{f_{kj_{\hat{S}_k}}^2}{K_{1_{kj}}}\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)^2\\
&+ \sum_{k=1}^n\sum_{j=1}^n y_j^2\frac{f_{kj_{\hat{j}_j}}^2}{K_{2_{kj}}}\left(\int_0^h Q_k(\xi)e^{-\mu_k\xi}d\xi\right)^2.\end{aligned}$$

Therefore,

$$LW \le \sum_{k=1}^{n} x_k^2 \left[2n_k'(\hat{S}_k) - 2\sum_{j=1}^{n} f_{kj_{\hat{S}_k}} + \sum_{j=1}^{n} f_{kj_{\hat{I}_j}} + \sum_{j=1}^{n} \frac{f_{kj_{\hat{S}_k}}^2}{K_{1_{kj}}} \left(\int_0^h Q_k(\xi) e^{-\mu_k \xi} d\xi \right)^2 + \sigma_{1k}^2 \right]$$
$$+ \sum_{k=1}^{n} y_k^2 \left[-2(\mu_k + \gamma_k + \alpha_k) + \sigma_{2k}^2 + \sum_{j=1}^{n} K_{1_{kj}} + \sum_{j=1}^{n} K_{2_{kj}} + \delta_k + \gamma_k + \sum_{j=1}^{n} f_{jk_{\hat{I}_k}} + \sum_{j=1}^{n} \frac{f_{jk_{\hat{I}_k}}^2}{K_{2_{jk}}} \left(\int_0^h Q_k(\xi) e^{-\mu_j \xi} d\xi \right)^2 \right]$$
$$+ \sum_{k=1}^{n} z_k^2 \left[-2(\mu_k + \delta_k) + \sigma_{3k}^2 + \delta_k + \gamma_k \right].$$

Choose

$$K_{1_{kj}} = f_{kj_{\hat{S}_k}} \left(\int_0^h Q_k(\xi) e^{-\mu_k \xi} d\xi \right)^2, \quad K_{2_{kj}} = f_{kj_{\hat{I}_j}} \left(\int_0^h Q_k(\xi) e^{-\mu_k \xi} d\xi \right)^2, \quad k, j = 1, 2, \cdots, n.$$

Hence,

$$LW \le \sum_{k=1}^{n} x_k^2 \left[2n_k'(\hat{S}_k) - \sum_{j=1}^{n} f_{kj_{\hat{S}_k}} + \sum_{j=1}^{n} f_{kj_{\hat{I}_j}} + \sigma_{1k}^2 \right]$$
$$+ \sum_{k=1}^{n} y_k^2 \left\{ -2(\mu_k + \gamma_k + \alpha_k) + \sigma_{2k}^2 + \sum_{j=1}^{n} \left[(f_{kj_{\hat{S}_k}} + f_{kj_{\hat{I}_j}}) \left(\int_0^h Q_k(\xi) e^{-\mu_k \xi} d\xi \right)^2 \right] + \delta_k + \gamma_k + 2\sum_{j=1}^{n} f_{jk_{\hat{I}_k}} \right\}$$
$$+ \sum_{k=1}^{n} z_k^2 \left[-2(\mu_k + \delta_k) + \sigma_{3k}^2 + \delta_k + \gamma_k \right].$$

It now follows from conditions (i) to (iii) that LW is negative definite, which implies that $\hat{\mathbf{E}}$ *is stochastically asymptotically stable.*

□

4 | NUMERICAL SIMULATIONS AND CONCLUDING REMARKS

We first perform a numerical simulation of Equation 7 to illustrate our analytical results. For sexually transmitted diseases, it is natural to consider separate male and female groups, respectively, which leads to the particular choice of $n = 2$ for our future numerical considerations. We hereby consider the growth functions $n_k(S_k)$ and the incidence functions $f_{kj}(S_k, I_j)$, $k, j = 1, 2$ being particularized as

$$n_k(S_k) = 3 - S_k, \quad f_{kj}(S_k, I_j) = \beta_{kj} S_k I_j.$$

It is assumed that the maximal exposed period is 9 months (ie, $h = 0.75$ y). We take $\mu_k = 0.05$, $\delta_k = 0.45$, $\gamma_k = 0.5$, $\alpha_k = 0.2$, and $Q_k(\xi) = 2\exp(-2\xi)$. Furthermore, we choose that $\beta_{11} = 0.002$, $\beta_{12} = 0.012$, $\beta_{21} = 0.001$, and $\beta_{22} = 0.001$ and $\sigma_{1k} = 0.7$, $\sigma_{2k} = 0.3$, and $\sigma_{3k} = 0.2$, the concrete choices of parameter values being motivated by related investigations on sexually transmitted diseases pursued in Yuan and Wang,[44] Blower et al,[18] and van den Driessche et al.[29] Since $\mathcal{R} = 0.0766 < 1$, the disease-free equilibrium $\mathbf{E}^0 = (3, 0, 0, 3, 0, 0)$ is then globally asymptotically stable. Figure 1 offers a comparative (superimposed) view of the behavior of the susceptible population for the deterministic and stochastic models, respectively.

Letting $\beta_{11} = 0.05$, $\beta_{12} = 0.03$, $\beta_{21} = 0.03$, $\beta_{22} = 0.05$, and $\alpha_k = 0.05$ and keeping the other parameter values unchanged, we obtain that $\mathcal{R} = 1.2255 > 1$, which implies that the endemic equilibrium

$$\mathbf{E}^* \approx (2.4481, 2.8180, 2.8180, 2.4481, 2.8180, 2.8180)$$

of Equation 7 is globally asymptotically stable (note that (H) is satisfied if the incidence functions are all bilinear, since all D_{kj}'s are null). The convergence of S_k and I_k, $k = 1, 2$, to the respective components of $\mathbf{E}^*$ is illustrated in Figure 2 (for the deterministic model) and Figure 3 (for the stochastic model), respectively.

In this paper, we investigate a multigroup disease propagation model with general incidence rates and distributed delay from a stability viewpoint. It is determined that the stability results that were obtained for the single-group model discussed in our previous paper[28] carry out to the multigroup model. In other words, from a stability viewpoint, the heterogeneity of the

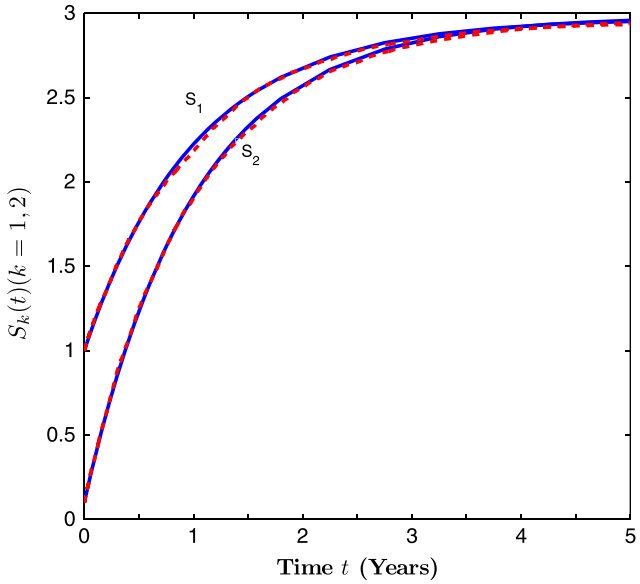

FIGURE 1 Time series graphs for the susceptible populations. The initial conditions: $\varphi_{11}(\theta) \equiv \varphi_{13}(\theta) \equiv 1, \varphi_{14}(\theta) \equiv 0$, $\varphi_{21}(\theta) \equiv \varphi_{23}(\theta) \equiv 0.1, \varphi_{24}(\theta) \equiv 0, \theta \in [-h, 0]$ (blue solid curve: the deterministic model; red dashed curve: the stochastic model) [Colour figure can be viewed at wileyonlinelibrary.com]

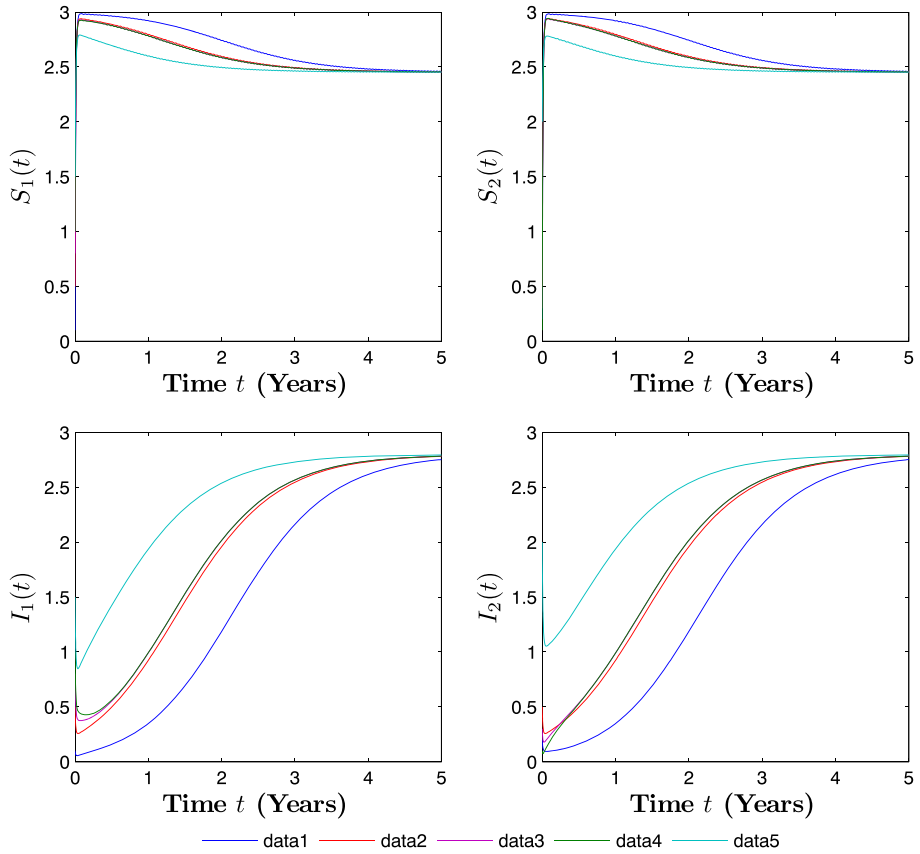

FIGURE 2 Time series graphs for the susceptible and infective populations of the deterministic model. The initial conditions: (data 1) $\varphi_{11}(\theta) \equiv \varphi_{13}(\theta) \equiv 0.1, \varphi_{14}(\theta) \equiv 0, \varphi_{21}(\theta) \equiv \varphi_{23}(\theta) \equiv 0.2, \varphi_{24}(\theta) \equiv 0$; (data 2) $\varphi_{11}(\theta) \equiv \varphi_{13}(\theta) \equiv 0.5, \varphi_{14}(\theta) \equiv 0$, $\varphi_{21}(\theta) \equiv \varphi_{23}(\theta) \equiv 0.5, \varphi_{24}(\theta) \equiv 0$; (data 3) $\varphi_{11}(\theta) \equiv \varphi_{13}(\theta) \equiv 0.8, \varphi_{14}(\theta) \equiv 0, \varphi_{21}(\theta) \equiv \varphi_{23}(\theta) \equiv 0.3, \varphi_{24}(\theta) \equiv 0$; (data 4) $\varphi_{11}(\theta) \equiv \varphi_{13}(\theta) \equiv 1, \varphi_{14}(\theta) \equiv 0, \varphi_{21}(\theta) \equiv \varphi_{23}(\theta) \equiv 0.1, \varphi_{24}(\theta) \equiv 0$; (data 5) $\varphi_{11}(\theta) \equiv \varphi_{13}(\theta) \equiv 1.5, \varphi_{14}(\theta) \equiv 0, \varphi_{21}(\theta) \equiv \varphi_{23}(\theta) \equiv 2, \varphi_{24}(\theta) \equiv 0$; $\theta \in [-h, 0]$ [Colour figure can be viewed at wileyonlinelibrary.com]

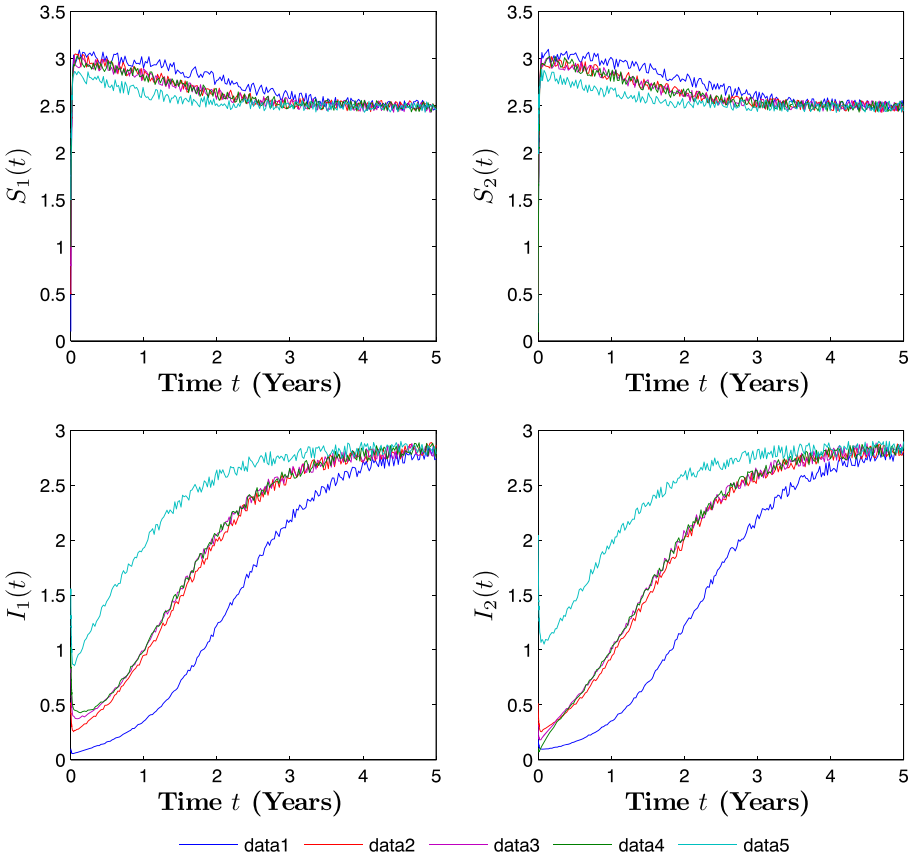

FIGURE 3 Time series graphs for the susceptible and infective populations of the stochastic model. The initial conditions: (data 1) $\varphi_{11}(\theta) \equiv \varphi_{13}(\theta) \equiv 0.1, \varphi_{14}(\theta) \equiv 0, \varphi_{21}(\theta) \equiv \varphi_{23}(\theta) \equiv 0.2, \varphi_{24}(\theta) \equiv 0$; (data 2) $\varphi_{11}(\theta) \equiv \varphi_{13}(\theta) \equiv 0.5, \varphi_{14}(\theta) \equiv 0, \varphi_{21}(\theta) \equiv \varphi_{23}(\theta) \equiv 0.5, \varphi_{24}(\theta) \equiv 0$; (data 3) $\varphi_{11}(\theta) \equiv \varphi_{13}(\theta) \equiv 0.8, \varphi_{14}(\theta) \equiv 0, \varphi_{21}(\theta) \equiv \varphi_{23}(\theta) \equiv 0.3, \varphi_{24}(\theta) \equiv 0$; (data 4) $\varphi_{11}(\theta) \equiv \varphi_{13}(\theta) \equiv 1, \varphi_{14}(\theta) \equiv 0, \varphi_{21}(\theta) \equiv \varphi_{23}(\theta) \equiv 0.1, \varphi_{24}(\theta) \equiv 0$; (data 5) $\varphi_{11}(\theta) \equiv \varphi_{13}(\theta) \equiv 1.5, \varphi_{14}(\theta) \equiv 0, \varphi_{21}(\theta) \equiv \varphi_{23}(\theta) \equiv 2, \varphi_{24}(\theta) \equiv 0$; $\theta \in [-h, 0]$ [Colour figure can be viewed at wileyonlinelibrary.com]

population, as expressed through the group structure, does not alter the qualitative behavior of a *SEIRI* model with general incidence rates and distributed delay. To this purpose, the graph-theoretic approach that was devised in Guo et al[11] to aid with the construction of Lyapunov functionals for multigroup models is used.

It is then seen that neither the relapse phenomenon nor the group structure induce sustained oscillations by themselves, the behavior of the multigroup model being completely characterized by a threshold parameter, the basic reproduction number, provided that a certain sign condition that involves the nonlinear incidence function is also met. In this regard, multigroup models provide a more accurate understanding of the ways in which a disease spreads through communities and global stability results and, in conjunction with an explicit expression for the basic reproduction number, allow for the preparation of successful measures to control the spread of the disease, designated to bring the basic reproduction number below unity. Our model is appropriate to discuss the treatment of tuberculosis and of certainly sexually transmitted diseases such as herpes and gonorrhea, which are prone to relapse and for which the partition of the population into specific groups can be done via the level of sexual activity (core and noncore groups).

Multigroup models have also become a useful tool to characterize disease dynamics in urban environments, which are naturally heterogeneous due to high population circulation. In this context, an explicit formula for the basic reproduction number allows for an estimation for the level of behavioral change required to tame and control an epidemic via a reduction of the contact patterns.

The fact that the nonlinear incidence functions are not necessarily assumed to factorize as products of functions depending only on the sizes of the susceptible and infective classes, respectively, allows for the treatment of more general saturation phenomena, modeled, for instance, by the saturated incidence and the modified saturated incidence nominated in Remark 2. The group structure makes the estimation of the derivative of the Lyapunov functional significantly more involved. To this purpose, a certain summation lemma derived from Kirchoff's matrix tree theorem is a key ingredient.

Finally, under certain assumptions on the magnitude of environmental perturbations, the stochastic stability of the equilibria is established, again via the use of Lyapunov functionals. It is to be noted that the choice of a Lyapunov functional may not be unique. For instance, to study the stability of the endemic equilibrium, our choice of a Lyapunov functional, given in Equation 10 and motivated by the ones used in Zhang et al[28] and Georgescu and Zhang,[21] contains the nonlinear incidence function only in a single term of the nondelayed part (the *S*-term, as defined by W_{1k}). From this viewpoint, our Lyapunov functional aligns with those used in Guo et al,[11,12,33] Li et al,[7] and Shu et al.[14] However, in the Lyapunov functional used in Georgescu et al[45] (albeit for a model without either delay or relapse), the nonlinear incidence function appears in both the *S*-term and the *I*-term (although the model treated in the paper[45] is a predator-prey model, it is functionally equivalent to a *SEI* model), which may lead to the possibility of using a different functional template for our multigroup model as well.

A related integro-differential multigroup *SIR* model with nonlinear incidence rates and distributed relapse has been analyzed in Wang et al.[22] Our model considers an additional exposed class for each group and a distributed latency, rather than a distributed relapse, as used in Wang et al,[22] while using a linear relapse term. Also, we postulate the existence of a maximal length of the exposed period, which leads to a transmission term consisting of an integral term over a finite interval, while in Wang et al,[22] the infective periods were assumed to be exponentially distributed in each group. In Shu et al,[14] the global stability of a multigroup *SEIR* model with distributed delays and nonlinear transmission terms has been considered, assuming that the incidence terms factorize and there is no relapse, while the multigroup *SEIRI* model discussed in Wang et al[46] features factorized incidence terms and linear growth terms for the susceptible classes, assuming also that the infective periods are exponentially distributed for each group. Multigroup *SEIR* and *SIR* models that are subject to white noise perturbations were analyzed in Yuan et al,[16] assuming that the incidence terms are bilinear and the growth terms for the susceptible classes are linear. In the nonperturbed case, our Theorem 2 extends their theorem 3.1, while in the stochastically perturbed case, our Theorem 3 is not directly comparable to their theorem 3.4, because of the use of a different quadratic functional, which leads to the use of different inequalities involving the intensities of Brownian motions as a priori assumptions. However, the quadratic functional used in the proof of Theorem 3 can be modified to have a similar form to the one used in the proof of Theorem 3.4 from Yuan et al[16] so that, after a suitable modification, Theorem 3 may encompass Theorem 3.4 from Yuan et al.[16]

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