## Chapter 3

# LINEAR MAPPINGS ON VECTOR SPACES

## § 3.1 LINEAR AND BILINEAR FORMS / FUNCTIONALS

Given a vector space $V$, several types of functions (or mappings) can be defined on it. They can be classified according to several criteria, but the most important regards *the nature of the values they take*. In other words, several kinds of mappings can be defined on a vector space $V$ (over a field $\mathbf{F}$ ) that may differ in what regards their ranges. Such a range may consist of vectors or of scalars in the field $\mathbf{F}$. Some important mappings taking scalar values are presented in this chapter. The first of them are introduced below.

### Linear Forms

***Definition 1.1.*** Let $V$ be a vector space over the field $\mathbf{F}$ (= $\mathbb{R}$ / = $\mathbb{C}$). A mapping $f : V \longrightarrow \mathbf{F}$ is said to be a *linear functional* (or *linear form*) if it satisfies both of the following properties (or axioms):

$(\mathbf{LF}_1)$ $(\forall\, x_1, x_2 \in V)\ f(x_1 + x_2) = f(x_1) + f(x_2)$ ;

$(\mathbf{LF}_2)$ $(\forall\, \lambda \in \mathbf{F})(\forall\, x \in V)\ f(\lambda\, x) = \lambda\, f(x)$ . ◊

Property $(\mathbf{LF}_1)$ is called the *additivity* of $f$, while $(\mathbf{LF}_2)$ may be termed the *homogeneity* of $f$ with respect to the multiplication by scalars. Certainly, the two operations should be differently understood in the two sides of each equation in the above definition : in $(\mathbf{LF}_1)$, the sum in the l.h.s. is the vector addition in $V$, while $+$ in the r.h.s. denotes the sum of scalars in $\mathbf{F}$ ; in $(\mathbf{LF}_2)$ the scalar $\lambda$ multiplies a vector in the l.h.s. (under $f$ ) while it multiplies the scalar $f(x)$ in its r.h.s.

***Remark 1.1.*** Properties $(\mathbf{LF}_1)$ and $(\mathbf{LF}_2)$ in *Def. 1.1* can be replaced by a single property, namely

$$\text{(LIN)}\quad \boxed{(\forall\, \lambda_1, \lambda_2 \in \mathbf{F})(\forall\, x_1, x_2 \in V)\ f(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 f(x_1) + \lambda_2 f(x_2).}$$

Indeed, $(\mathbf{LF}_1)$ & $(\mathbf{LF}_2)$ $\Rightarrow$ (LIN) since

$(\mathbf{LF}_1)$ (in *Def. 1.1* ) $\Rightarrow$

$$\lambda_1 x_1, \lambda_2 x_2 \in V \underset{(\mathrm{LF}_1)}{\Rightarrow} f(\lambda_1 x_1 + \lambda_2 x_2) = f(\lambda_1 x_1) + f(\lambda_2 x_2) \underset{(\mathrm{LF}_2)}{=}$$

$$= \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

Conversely, (LIN) $\Rightarrow$ $(\mathbf{LF}_1)$ for $\lambda_1 = \lambda_2 = 1$ (also using axiom $(\mathbf{L}_6)$ of a vector space see **§ 1.1** - *Def. 1.1* ) while (LIN) $\Rightarrow$ $(\mathbf{LF}_2)$ for $\lambda_1 = \lambda$, $\lambda_2 = 0$ and $x_1 = x$. Therefore, *Definition 1.1* may be replaced by an equivalent but simpler one :

***Definition 1.1'.*** Let $V$ be a vector space over the field $\mathbf{F}$ ( $= \mathbb{R}$ / $= \mathbb{C}$). A mapping $f : V \longrightarrow \mathbf{F}$ is said to be a *linear functional* (or *linear form*) if it satisfies

$$\text{(LIN)} \quad \boxed{(\forall\, \lambda_1, \lambda_2 \in \mathbf{F})(\forall\, x_1, x_2 \in V)\; f(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 f(x_1) + \lambda_2 f(x_2).}$$

◊

Property (LIN) is called the *linearity* of mapping $f$, and it just gives the terminology of *'linear forms'*. This property can be extended to linear combinations of several vectors under $f$, as stated in

**PROPOSITION 1.1.** *If $f$ is a linear form on the vector space $V$ then*

$$(\forall\, \lambda_1, \ldots, \lambda_m \in \mathbf{F})\,(\forall\, x_1, \ldots, x_m \in V)\; f\left(\sum_{i=1}^{m} \lambda_i\, x_i\right) = \sum_{i=1}^{m} \lambda_i\, f(x_i). \tag{1.1}$$

***Proof*** (by induction on $m$ ). For $m = 2$, $(\mathbf{P}_2)$ is just property (LIN). Let us assume that property (1.1) holds and denote it by $(\mathbf{P}_m)$. Then $(\mathbf{P}_{m+1})$ is implied by $(\mathbf{P}_m)$ and $(\mathbf{P}_2)$ as follows :

$$(\forall\, \lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{m+1} \in \mathbf{F})(\forall\, x_1, x_2, \ldots, x_m, x_{m+1} \in V)\; f\left(\sum_{i=1}^{m+1} \lambda_i x_i\right) =$$

$$= f\left(\sum_{i=1}^{m} \lambda_i\, x_i + \lambda_{m+1}\, x_{m+1}\right) \underset{(\mathbf{P}_2)}{=} f\left(\sum_{i=1}^{m} \lambda_i\, x_i\right) + f(\lambda_{m+1}\, x_{m+1}) \underset{(\mathbf{P}_m),\,(\mathbf{LF}_2)}{=}$$

$$= \sum_{i=1}^{m} \lambda_i\, f(x_i) + \lambda_{m+1} f(x_{m+1}) = \sum_{i=1}^{m+1} \lambda_i\, f(x_i) : \; (\mathbf{P}_{m+1}).$$

Hence, property $(\mathbf{P}_m) \iff$ Eq. (1.1) holds for any $m \in \mathbb{N}$. ■

This property (1.1) may be called the *extended linearity*. It can be written in a simpler way if we use the so-called 'matrix notations' introduced in **§ 2.1** - Eqs. (1.12) & (1.13) for linear combinations of several vectors with several scalars. Let us recall those notations :

$$\mathcal{X} = [x_1\; x_2 \ldots x_n] \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}; \tag{1.2}$$

With (1.2), a linear combination may be written as

$$\sum_{i=1}^{n} \lambda_i x_i = \Lambda^{\mathrm{T}} \cdot \mathcal{X}^{\mathrm{T}} = \mathcal{X} \cdot \Lambda = \sum_{i=1}^{n} x_i \lambda_i . \tag{1.3}$$

It follows from (1.1) with (1.3) that

$$f\left(\sum_{i=1}^{m} \lambda_i x_i\right) = f(\Lambda^{\mathrm{T}} \cdot \mathcal{X}^{\mathrm{T}}) = \Lambda^{\mathrm{T}} \cdot f(\mathcal{X}^{\mathrm{T}}) . \tag{1.4}$$

In (1.4), $f(\mathcal{X}^{\mathrm{T}})$ represents the column vector of the values

$$f(x_i),\ i = \overline{1, n} . \tag{1.5}$$

Using the alternative way to write a linear combination with the matrix notation (that is, the third side in the triple equality (1.3)), the property of extended linearity can be written as

$$f(\mathcal{X} \cdot \Lambda) = f(\mathcal{X}) \cdot \Lambda . \tag{1.6}$$

In this formula (1.6), the linear form's values of (1.5) appear as the component of a row vector :

$$f(\mathcal{X}) = [f(x_1)\ f(x_2) \ldots f(x_m)] . \tag{1.7}$$

In what follows, we will prefer the notational alternative (1.6).

This property (1.4) or (1.6) is involved in defining the coefficients of a linear form (LF) and its analytic expression in a certain basis $A$ of the space.

***Definition 1.2.*** Let $V$ be a vector space over the field $\mathbf{F}$, spanned by the basis $A = [a_1\ a_2\ \ldots\ a_n]$. The *coefficients* of the linear form $f : V \longrightarrow \mathbf{F}$ are the components of the (row) vector

$$f(A) = [f(a_1)\ f(a_2)\ \ldots\ f(a_n)] \underset{\text{not}}{=} [\alpha_1\ \alpha_2\ \ldots\ \alpha_n] = [\alpha] . \tag{1.8}$$

Hence, $f(a_i) =_{\text{not}} \alpha_i\ (i = \overline{1,n})$ and these values are written on a row. ♢

**PROPOSITION 1.2.** *If the vector space* $V$ *is spanned by the basis* $A = [a_1\ a_2\ \ldots\ a_n]$ *and the coefficients of the linear form* $f : V \longrightarrow \mathbf{F}$ *in this basis are* $f(A) =_{\text{not}} [\alpha_1\ \alpha_2\ \ldots\ \alpha_n] = [\alpha]$ *then the image* $f(x)$ *of a vector* $x = A X_A \in V$ is

$$\boxed{f(x) = [\alpha] X_A .} \tag{1.9}$$

***Proof.*** Formula (1.9) immediately follows from *Definition 1.2* of the coefficients of in a basis $A$ and from property of extended linearity, under its form (2.6), with $\mathcal{X} \to A$ and $\Lambda \to X_A$. ■

***Remarks 1.2.*** The analytic expression of $f(x)$ under its "matrix form" (1.9) is nothing else than the extended linearity of an LF applied to the linear expression of a vector in a basis $A$ :

$$x = \sum_{i=1}^{n} \xi_i a_i \underset{(1.4,8)}{\Rightarrow} f(x) = \sum_{i=1}^{n} \xi_i \alpha_i = \sum_{i=1}^{n} \alpha_i \xi_i. \tag{1.10}$$

Let us also see that the matrix notations like the ones in (1.2) – (1.4) allow for very quick and simple proofs.

**Example 1.1.** If $V$ is a vector space of dimension 4 over the field $\mathbb{R}$, $A = [a_1\ a_2\ a_3\ a_4]$ is a basis spanning $V$ and $f: V \longrightarrow \mathbb{R}$ is a linear form with its coefficients in $A$ $f(A) = [\alpha] = [-1\ \ 3\ \ 1\ \ 0]$ then the image through $f$ of (or the value taken by $f$ on ) the vector $x = a_1 - 2a_2 + 4a_3 + a_4$ is $f(x) = -3$, what results either by property (1.1) in the previous PROPOSITION applied to $x = a_1 - 2a_2 + 4a_3 + a_4$ with the given coefficients, or by formula (1.9) :

$$f(x) = [\alpha] X_A = [-1\ \ 3\ \ 1\ \ 0] \cdot \begin{bmatrix} 1 \\ -2 \\ 4 \\ 1 \end{bmatrix} = -1 - 6 + 4 = -3.$$

□

***Remarks 1.3.*** The explicit expression (1.10) of $f(x)$ in terms of the coordinates of a vector in a basis gives the reason for calling such a scalar mapping to be a *linear form* : it is just a linear function (or a homogeneous polynomial of order 1) in the coordinates $\xi_1, \xi_2, \ldots, \xi_n$ of $x$ in basis $A$ of $V$. In the particular (but very often met) case when $V = \mathbf{F}^n$ or $V = \mathbb{R}^n$ with the standard basis $E$ in each of these spaces, the analytic expression of $f(X)$ for $X = [x_1\ x_2\ \ldots\ x_n]^{\mathrm{T}} \in \mathbf{F}^n$ becomes

$$f(X) = \sum_{i=1}^{n} \varepsilon_i x_i \text{ with } [\varepsilon] = f(E) = [\varepsilon_1\ \varepsilon_2\ \ldots\ \varepsilon_n]. \tag{1.11}$$

We recall that $E$ was the notation used for the standard basis in $\mathbf{F}^n$ or $\mathbb{R}^n$, in which the coordinates of a vector coincide with their components that appear under the sum in (1.11), i.e. $X_E = X$.

A special subset of $V$ can be associated with any linear form defined on $V$:

***Definition 1.3.*** If $f: V \longrightarrow \mathbf{F}$ is a linear form, then its *kernel* is defined by

$$\boxed{\mathrm{Ker} f \underset{\text{def}}{=} \{x \in V : f(x) = 0 \in \mathbf{F}\}.} \tag{1.12}$$

In other words, the kernel is the subset of all vectors mapped by $f$ on the zero scalar of the field $\mathbf{F}$ (or on the real number $0$ when $\mathbf{F} = \mathbb{R}$). Using the subscript $_{-1}$ on $f$ to denote the counter-image of a scalar in the range of function $f$, we can define the kernel of the LF as

$$\boxed{\operatorname{Ker} f = f_{-1}(0) : 0 \in \mathbf{F}\ (\in \mathbb{R}).} \tag{1.13}$$

The kernel of any linear form is more that a simple subset of $V$, as stated in

**PROPOSITION 1.3.** *If* $f : V \longrightarrow \mathbf{F}$ *is a linear form, then its kernel is a subspace of* $V$:

$$\boxed{\operatorname{\mathbf{Ker}} f \subseteq_{\text{subsp}} V.} \tag{1.14}$$

***Proof.*** Let $x_1, x_2 \in \operatorname{Ker} f$ and let $\lambda_1, \lambda_2 \in \mathbf{F}$ be two arbitrary scalars. Then

$$f(\lambda_1 x_1 + \lambda_2 x_2) \underset{\text{(LIN)}}{=} \lambda_1 f(x_1) + \lambda_2 f(x_2) \underset{(1.12)}{=}$$
$$= \lambda_1 0 + \lambda_2 0 = 0 \underset{(1.12)}{\Rightarrow} \lambda_1 x_1 + \lambda_2 x_2 \in \operatorname{Ker} f. \tag{1.15}$$

The last membership in (1.15) + *Definition 1.1'* in the earlier section, **§ 2.2**, imply the inclusion of (1.14). ■

***Remarks 1.4.*** The condition on a vector $x = A X_A \in V = \mathcal{L}(A)$ to be in $\operatorname{Ker} f$ is equivalent, in view of (1.9) & (1.10), to the equation

$$[\alpha] X_A = 0 \iff \sum_{i=1}^{n} \alpha_i \xi_i = 0. \tag{1.16}$$

Except the trivial case when $x = \mathbf{0} \Rightarrow X_A = \mathbf{0} \in \mathbf{F}^n \Rightarrow f(x = \mathbf{0}) = 0$ in any basis $A$, it follows from (1.16) that the coordinates of a vector in the kernel should satisfy a linear equation with its coefficients = the components of $[\alpha] = f(A)$. Another property is involved by this PROPOSITION 1.3, and by *Remark 2.2* in **§ 2.2** (Eq. (3.7) at page 59): since any subspace contains the zero vector **0**, the kernel of any linear form also does it. If we denote as $\text{LINFORM}_V$ the set of the LF's defined on space $V$, then we may formally write this property as

$$\boxed{(\forall f \in \text{LINFORM}_V)\ \mathbf{0} \in \operatorname{Ker} f.} \tag{1.17}$$

**Example 1.2.** For the linear forms defined on a vector space like $\mathbf{F}^n$ or $\mathbb{R}^n$, equation (1.16) involves the components of the vector $X$ – candidate to the membership in $\operatorname{Ker} f$:

$$X \in \mathrm{Ker}\, f \iff \sum_{i=1}^{n} \varepsilon_i x_i = 0.$$

For instance, if $V = \mathbb{R}^n$ and $f(X) = x_1 - 2x_2 + x_3$,

$$X \in \mathrm{Ker} f \iff x_1 + x_3 = 2x_2.$$

By the way, many applications with linear forms (and other notions in the LINEAR ALGEBRA) are formulated on the most usual vector space, that is $V = \mathbb{R}^n$. In the case of LF's, instead of giving them by the coefficients in the standard basis $E$ it is simpler to give their analytic expression in terms of the components of the argument vector $X \in \mathbb{R}^n$, as we did it above. To conclude this example, let us see that the general form of a vector in $\mathrm{Ker} f$, after denoting $x_2 = \alpha$ & $x_3 = \beta$, is

$$X(\alpha,\beta) = \begin{bmatrix} 2\alpha - \beta \\ \alpha \\ \beta \end{bmatrix} \Rightarrow \mathrm{Ker} f = \mathcal{L}([b_1\ b_2]) \text{ with } b_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \ \&\ b_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Indeed, the image of this $X(\alpha,\beta)$ thru $f$ is

$$f[X(\alpha,\beta)] = 2\alpha - \beta - 2\alpha + \beta = 0. \qquad \square$$

As it follows from PROPOSITION 1.2, the coefficients and - implicitly - the analytic expression of a linear form depends on the considered basis. When the (initial) basis $A$ is changed to another basis $B$, the coefficients naturally change, too. The formula giving the "new" coefficients is given in

**PROPOSITION 1.4.** *If $f : V \longrightarrow \mathbf{F}$ is a linear form with its coefficients $f(A) =_{\text{not}} [\alpha_1\ \alpha_2\ \ldots\ \alpha_n] = [\alpha]$ in basis $A$ and if basis $A$ is changed to basis $B$ by the transformation*

$$B^{\mathrm{T}} = T A^{\mathrm{T}} \iff B = A T^{\mathrm{T}} \tag{1.18}$$

*then the coefficients of in basis B are given by*

$$\boxed{f(B) \underset{\text{not}}{=} [\beta_1\ \beta_2\ \ldots\ \beta_n] = [\beta] = [\alpha]\, T^{\mathrm{T}}.} \tag{1.19}$$

***Proof.*** Expression (1.19) of the coefficients $[\beta]$ in the new basis $B$ follows from (1.18) but not quite straightforward. Let us firstly remark that the property of extended linearity of a linear form in the "matrix notation" (1.6) may be written for several columns of scalars simultaneously. In other words, a linear form *f* with well (and uniquely) defined coefficients $[\alpha_1\ \alpha_2\ \ldots\ \alpha_n] =_{\text{not}} [\alpha]$ in basis $A$ satisfies property (1.1) $\Longleftrightarrow$ (1.6) for any set of $m$ or $n$ scalars, and if several such ordered $n$-tuples of scalars are considered, they may be arranged as column vectors forming a matrix that multiplies $\mathcal{X}$ (at right) under $f$ in the l.h.s. of (1.6), and $f(\mathcal{X})$ in the r.h.s. of that equation. If we consider the $n$-by-$k$ matrix $U = [\Lambda^1 \; \Lambda^2 \; \ldots \; \Lambda^k]$ and $f$ is a linear form with its coefficients in basis $A$ as in the statement and also rewritten above, $[\alpha_1 \; \alpha_2 \; \ldots \; \alpha_n] =_{\text{not}} [\alpha]$, then – if $\mathcal{X}$ is replaced by $A$ and Eq. (1.6) is ($k$ times) simultaneously written for the columns of $U$ – we get

$$f(A \cdot \Lambda) = f(A) \cdot U = [\alpha] \cdot U. \tag{1.20}$$

If we now take $T^{\mathrm{T}}$ instead of $U$ in (1.20) we obtain

$$[\beta] \underset{\text{not}}{=} f(B) = f(A \cdot T^{\mathrm{T}}) \underset{(1.20)}{=} f(A) \cdot T^{\mathrm{T}} = [\alpha]\, T^{\mathrm{T}}. \tag{1.21}$$

Hence formula (1.19) in the statement is thus proved. ■

As a technical matter, the C-dot ( $\cdot$ ) in equations (1.20) and (1.21) may be omitted. It stands for the matrix product, but it was not used in the statement - Eqs. (1.18) & (1.19).

**Example 1.3.** Let $f$ be a linear form whose coefficients in a basis $A$ (of the 4-dimensional vector space $V$ ) are $f(A) = [\alpha_1 \; \alpha_2 \; \alpha_3 \; \alpha_4] = [2 \;\; 3 \;\; -1 \;\; 4]$. If basis $A$ is changed to $B$ with the transformation matrix

$$T = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & -1 & 2 \\ 3 & 2 & 1 & -1 \end{bmatrix}, \tag{1.22}$$

then the new coefficients in basis $B$ will be, according to formula (1.19) and the numerical data just written,

$$[\beta] = [2 \;\; 3 \;\; -1 \;\; 4] \cdot \begin{bmatrix} 2 & 0 & 2 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 0 & -1 & 1 \\ 3 & 1 & 2 & -1 \end{bmatrix} = [13 \;\; 7 \;\; 16 \;\; 7]. \tag{1.23}$$

□

***Remark 1.5.*** Formula (1.19) expresses the dependence of the coefficients of a linear form on the basis selected in $V$. However, the value of an LF essentially depends on its argument $x$ (and also on its coefficients). Therefore, the value of a certain LF on a given argument remains the same irrespectively of the basis considered. Let us illustrate this remark using the previous example. If we take

$$x = 2\,a_1 + 3\,a_2 - a_3 - 2\,a_4, \tag{1.24}$$

formula (1.9) - (1.10) for the value of an LF gives

$$f(x) = [2 \;\; 3 \;\; -1 \;\; 4] \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \\ -2 \end{bmatrix} = 4 + 9 + 1 - 8 = 6. \tag{1.25}$$

The same value of (1.24) would have to be retrieved if we use the "new" basis $B$ instead of $A$. The coefficients of $f$ in $B$ have been just found - Eq. (1.23). We still need the coordinates of the vector $x$ in this basis. They can be found using the method presented in **§ 1.1** - Eq. (1.54) : the new coordinates $X_B = T^{-\mathrm{T}} \cdot X_A$ can be most conveniently determined by the Gaussian elimination, that is by transformations on the rows of the (augmented) matrix

$$[T^{\mathrm{T}} \,|\, X_A] = \left[\begin{array}{cccc|c} 2 & 0 & 2 & 3 & 2 \\ -1 & 1 & 1 & 2 & 3 \\ 0 & 0 & -1 & 1 & -1 \\ 3 & 1 & 2 & -1 & -2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 2 & 0 & 2 & 3 & 2 \\ -1 & 1 & 1 & 2 & 3 \\ 0 & 0 & -1 & 1 & -1 \\ 4 & 0 & 1 & -3 & -5 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cccc|c} 2 & 0 & 2 & 3 & 2 \\ -1 & 1 & 1 & 2 & 3 \\ 0 & 0 & -1 & 1 & -1 \\ 4 & 0 & 1 & -3 & -5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 2 & 0 & 0 & 5 & 0 \\ -1 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 4 & 0 & 0 & -2 & -6 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cccc|c} 2 & 0 & 0 & 5 & 0 \\ -1 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 4 & 0 & 0 & -2 & -6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 12 & 0 & 0 & 0 & -15 \\ 5 & 1 & 0 & 0 & -7 \\ -2 & 0 & 1 & 0 & 4 \\ -2 & 0 & 0 & 1 & 3 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cccc|c} 12 & 0 & 0 & 0 & -15 \\ 5 & 1 & 0 & 0 & -7 \\ -2 & 0 & 1 & 0 & 4 \\ -2 & 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -5/4 \\ 0 & 1 & 0 & 0 & -3/4 \\ -2 & 0 & 1 & 0 & 6/4 \\ 0 & 0 & 0 & 1 & 2/4 \end{array}\right]. \tag{1.27}$$

$$(1.27) \;\Rightarrow\quad X_B = \begin{bmatrix} -5/4 \\ -3/4 \\ 6/4 \\ 2/4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -5 \\ -3 \\ 6 \\ 2 \end{bmatrix}. \tag{1.28}$$

$$(1.23)\ \&\ (1.28) \Rightarrow [\beta]\cdot X_B = [13\ \ 7\ \ 16\ \ 7]\cdot\frac{1}{4}\begin{bmatrix}-5\\-3\\6\\2\end{bmatrix} =$$

$$= \frac{1}{4}(-65-21+96+14) = \frac{24}{4} = 6.$$

Hence, the value in (1.25) has been retrieved, working with basis $B$ as well.

□

This problem of changing bases and coefficients of an LF implies, in particular, a simple way for finding the coefficients of a linear form in a given basis of a space like $V = \mathbb{R}^n$ or $V = \mathbf{F}^n$.

If the LF is given by its coefficients *in the standard basis* $E$ of such a space, what is equivalent to its analytic expression in terms of the components of $X = [x_1\ \ x_2 \dots x_n]^{\mathrm{T}}$, like in Example 1.2, the coefficients of $f$ in basis $B = [b_1\ \ b_2 \dots b_n]$ can be found in two ways that lead to the same result. The transformation from the standard basis $E$ to another (given) basis $B$ involves a transformation matrix $T = B^{\mathrm{T}}$; see Eqs. (1.84) at page 23 in **§ 1.1.** Then, for each $b_i\ (1 \le i \le n)$, its components are its coordinates in the standard basis $E$ and it follows that

$$f(b_i) = [\varepsilon]\,b_i\ \ (1 \le i \le n) \ \Rightarrow\ f(B) = [\beta] = [\varepsilon]\,B. \tag{1.29}$$

But the same result follows from the coefficient transformation formula (1.19) with $[\alpha] = [\varepsilon]\ \&\ T^{\mathrm{T}} = B$, as it has just been recalled, from **§ 1.1.**

* * * * * *

**The Dual Space $V^\star$**

An interesting problem concerns the set of *all linear forms defined on the same vector space* $V$. We earlier considered it, in connection with the notion of kernel, and denoted it as $\mathrm{LINFORM}_V$. Two operations with / on the linear forms in this set can be naturally defined: the *sum* of two LF's and the *multiplication by a scalar* of an LF.

***Definition 1.4.*** If $f_1, f_2, f \in \mathrm{LINFORM}_V$ and $\lambda \in \mathbf{F}\ (\in \mathbb{R})$ then

$$(i) \quad (\forall\, x \in V)\ (f_1 + f_2)(x) \underset{\text{def}}{=} f_1(x) + f_2(x)\,; \tag{1.30}$$

$$(ii) \quad (\forall\, x \in V) \quad (\lambda f)(x) \underset{\text{def}}{=} \lambda f(x)\,. \tag{1.31}$$

By the way, let us recall that we had earlier met this definition on the set of all real functions $\mathcal{F}_I$ with $I \subseteq \mathbb{R}$ (see Example 1.5 in **§ 1.1**). The structure induced by these two (linear) operations on the set of real functions defined on an interval was that of a vector space, and the same property obviously holds for $\mathrm{LINFORM}_V$.

**THEOREM 1.1.** *The set* $\mathrm{LINFORM}_V$ *with the two operations defined in* $(i)$ *&* $(ii)$ *of Def. 1.4 is a vector space over the field* $\mathbf{F}\,(\mathbb{R})$.

***Proof.*** It suffices to show that $(i)$ & $(ii)$ $\Rightarrow$

$$\Rightarrow (\forall\, \lambda_1, \lambda_2 \in \mathbf{F})(\forall\, f_1, f_2 \in \mathrm{LINFORM}_V)\ \lambda_1 f_1 + \lambda_2 f_2 \in \mathrm{LINFORM}_V.$$

This implication follows from property (LIN) in *Definition 1.1'* :

$$(\forall\, \alpha, \beta \in \mathbf{F})(\forall\, x, y \in \mathbf{F})(\forall\, \lambda_1, \lambda_2 \in \mathbf{F})(\forall\, f_1, f_2 \in \mathrm{LINFORM}_V)$$

$$(\lambda_1 f_1 + \lambda_2 f_2)(\alpha x + \beta y) \underset{(1.31,30)}{=} \lambda_1 f_1(\alpha x + \beta y) + \lambda_2 f_2(\alpha x + \beta y) \underset{(\mathrm{LIN})}{=}$$

$$= \lambda_1 \alpha f_1(x) + \lambda_1 \beta f_2(y) + \lambda_2 \alpha f_2(x) + \lambda_2 \beta f_2(y) =$$

$$= \alpha\,[\lambda_1 f_1(x) + \lambda_2 f_2(x)] + \beta\,[\lambda_1 f_1(y) + \lambda_2 f_2(y)] =$$

$$= \alpha\,(\lambda_1 f_1 + \lambda_2 f_2)(x) + \beta\,(\lambda_1 f_1 + \lambda_2 f_2)(y). \tag{1.32}$$

This equation (1.32) – the leftmost side = the rightmost side) shows that

$$\lambda_1 f_1 + \lambda_2 f_2 \in \mathrm{LINFORM}_V$$

and the proof is thus over. ■

The vector space $\mathrm{LINFORM}_V$ is called the *dual space* (of the vector space $V$) and it is denoted, in many textbooks of LINEAR ALGEBRA, $V^\star$. Many notions defined in general vector spaces (see **§ 1.1** ) can be also defined in $V^\star$; for instance, the linear independence / dependence among the linear forms, bases in subspaces of $V^\star$, etc. The "zero vector" in $V^\star$ is the constant zero form and it is defined by $(\forall\, x \in V)\ \mathrm{O}(x) = 0$. Its coefficients in any basis $A$ of $V$ will be $[\mathrm{O}] = [0\ \ 0\ \ldots\ 0]$. It is also easy to see that the kernel of this trivial LF is the whole space : $\mathrm{Ker\,O} = V$. The negative of an LF $f$ is just $f$ times $(-1)$. If $f(A) = [\alpha]$ then $(-f)(A) = -[\alpha]$.

If a finite family of linear forms on $V$ is given, its linear independence / dependence can be effectively studied if the (row vectors of their) coefficieents in a basis $A$ are known. Their independence / dependence is equivalent to the corresponding relationship among their coefficient vectors $[\alpha]_j$ of $f_j$. This problem is illustrated by

**Example 1.4.** Let us consider three linear forms whose coefficients in a basis $A$ of $V$ (whose dimension is = 4) are the entries in the rows matrix $M$ given below :

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}(x) = \begin{bmatrix} 3 & 2 & -5 & 4 \\ 3 & -1 & 3 & -3 \\ 3 & 5 & -13 & 11 \end{bmatrix} \cdot X_A. \tag{1.33}$$

It is required to establish whether these linear forms are independent or not.

It can be seen that (for instance) $f_1$ and $f_2$ are linearly independent, but not all the three forms since $\text{rank}\, M = 2$. Writing down the equation involved in the definition of linear dependence / independence leads to a homogeneous system of matrix $M^{\mathrm{T}}$ which admits the nontrivial solution $[2\ \ -1\ \ -1]^{\mathrm{T}}$. Hence $f_3 = 2 f_1 - f_2$. □

If $\mathcal{LF}$ is a finite family of $m$ linear forms defined on $V$ with $\dim V = n$, it is clear that at most $n$ LF's in this family can be linearly independent. The number of the independent forms equals the rank of the matrix whose rows consist of their coefficients (in any basis of the space).

Let us close this section with a connection to the linear systems, studied in **§ 1.2** . It is obvious that the left hand side of each equation of a linear system (see Eq. (2.28) at page 43) is a linear form in the components of the unknown vector $X$; its coefficients are just the entries of the corresponding row of matrix $A$, in the matrix equation $AX = b$ that is equivalent to the system (2.28). If the system is homogeneous, that is its equivalent matrix equation is $AX = \mathbf{0}$, with $A \in \mathcal{M}_{m,n}(\mathbb{R})$, each equation is of the form

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = 0 \ \ (1 \le i \le m). \tag{1.34}$$

The solution of and equation of the form (1.34) is the kernel of the LF whose coefficients (usually in the standard basis $E$ ) are just the entries of the $i$-th row of matrix $A$. If we denote by $f_i(X)$ the left-hand side of Eq. (1.34), it follows from Eq. (2.30') in **§ 1.2** - page 44) that the solution set $S$ (in fact subspace) of such a homogeneous system is the common kernel of the $m$ linear forms :

$$S \underset{\text{def}}{=} \{X \in \mathbb{R}^n : AX = \mathbf{0}\} = \bigcap_{i=1}^{m} \operatorname{Ker} f_i. \tag{1.35}$$

## Bilinear Forms

The bilinear forms are (also, like the LFs) scalar mappings of two vector variables. More precisely, a bilinear form (BLF) is a function defined on two (possibly different) vector spaces over *the same* field **F** (or **K**, as it is denoted in some textbooks of LINEAR ALGEBRA, e.g. [E. Sernesi, 1993] ) and taking values in **F** / **K**. In fact, a BLF is defined on the Cartesian product of two vector spaces or of the same space by itself. The formal definition of a BLF is

***Definition 1.5.*** Let $U$ & $V$ be two vector spaces over the field $\mathbf{K}\,(=\mathbb{R}\,/\,=\mathbb{C})$. A mapping $f: U\times V \longrightarrow \mathbf{K}$ is said to be a *bilinear functional* (or *bilinear form*) if it satisfies both of the following properties (or axioms) :

$$(\mathbf{BLF}_1)\quad \begin{cases}(\forall\, \lambda_1,\lambda_2\in \mathbf{K})\,(\forall\, x_1,x_2\in U)\,(\forall\, y\in V)\\ \qquad f(\lambda_1 x_1+\lambda_2 x_2, y)=\lambda_1 f(x_1,y)+\lambda_2 f(x_2,y).\end{cases} \tag{1.36}$$

$$(\mathbf{BLF}_2)\quad \begin{cases}(\forall\, \mu_1,\mu_2\in \mathbf{K})\,(\forall\, x\in U)\,(\forall\, y_1,y_2\in V)\\ \qquad f(x,\mu_1 y_1+\mu_2 y_2)=\mu_1 f(x,y_1)+\mu_2 f(x,y_2).\end{cases} \tag{1.37}$$

◊

***Remark 1.6.*** Property $(\mathbf{BLF}_1)$ represents (for a fixed $y\in V$), the *linearity* of $f$ *in its first argument*, while $(\mathbf{BLF}_2)$ postulates the *linearity in the second argument* of $f$ (for a fixed $x\in V$). Hence, many of the properties of linear forms, earlier presented, hold for bilinear forms, too. Thus, the property (LIN) can be extended – for each of the arguments – to linear combinations of several vectors, as stated in

**PROPOSITION 1.5.** *If* $f: U\times V \longrightarrow \mathbf{K}$ *is a bilinear form, then*

$$(i)\quad \begin{cases}(\forall\, \lambda_1,\lambda_2,\ldots,\lambda_m\in \mathbf{K})\,(\forall\, x_1,x_2,\ldots,x_m\in U)\,(\forall\, y\in V)\\ \qquad f\left(\sum_{i=1}^{m}\lambda_i x_i, y\right)=\sum_{i=1}^{m}\lambda_i f(x_i,y).\end{cases} \tag{1.38}$$

$$(ii)\quad \begin{cases}(\forall\, \mu_1,\mu_2,\ldots,\mu_n\in \mathbf{K})\,(\forall\, x\in U)\,(\forall\, y_1,y_2,\ldots,y_n\in V)\\ \qquad f\left(x,\sum_{j=1}^{n}\mu_j y_j\right)=\sum_{j=1}^{n}\mu_j f(x,y_j).\end{cases} \tag{1.39}$$

***Proof.*** The proofs of $(i)$ & $(ii)$ can be effectively performed by induction on $m\,/\,n$ (respectively), but they may be omitted since these properties follow from PROPOSITION 1.1 in **§ 2.1** and from *Remark 2.1*. ■

The two properties (1.38) & (1.39) of extended linearity can be written in a matrix form, using appropriate notations - similar to those of (1.3) in the previous subsection (on LFs).

$$\mathcal{X} = [x_1\ x_2 \dots x_m] \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix}; \tag{1.40}$$

$$\mathcal{Y} = [y_1\ y_2 \dots y_n] \quad \text{and} \quad \mathrm{M} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}. \tag{1.41}$$

With these notations in (1.40) and (1.41), the linear combinations that occur in $(i)$ & $(ii)$ of the previous PROPOSITION may be (respectively) written as

$$\sum_{i=1}^{m} \lambda_i\, x_i = \Lambda^{\mathrm{T}} \cdot \mathcal{X}^{\mathrm{T}} = \mathcal{X} \cdot \Lambda = \sum_{i=1}^{n} x_i\, \lambda_i, \tag{1.42}$$

$$\sum_{j=1}^{n} \mu_j\, y_j = \mathrm{M}^{\mathrm{T}} \cdot \mathcal{Y}^{\mathrm{T}} = \mathcal{Y} \cdot \mathrm{M} = \sum_{j=1}^{n} y_j\, \mu_j. \tag{1.43}$$

It follows from (1.38) with (1.42) that

$$f\left(\sum_{i=1}^{m} \lambda_i\, x_i, y\right) = f(\Lambda^{\mathrm{T}} \cdot \mathcal{X}^{\mathrm{T}}, y) = \Lambda^{\mathrm{T}} \cdot f(\mathcal{X}^{\mathrm{T}}, y). \tag{1.44}$$

Property $(ii)$ or (1.39) with notation (1.41) becomes

$$f(x, \mathcal{Y} \cdot \mathrm{M}) = f(x, \mathcal{Y}) \cdot \mathrm{M}. \tag{1.45}$$

Although the use of the "matrix notations" (that is rows or columns whose entries are vectors) is nothing new, let us write - explicitly - what mean two notations that occur in (1.44) and (1.45).

$$f(\mathcal{X}^{\mathrm{T}}, y) = \begin{bmatrix} f(x_1, y) \\ f(x_2, y) \\ \vdots \\ f(x_2, y) \end{bmatrix}; \tag{1.46}$$

$$f(x, \mathcal{Y}) = [f(x, y_1)\ f(x, y_2) \dots f(x, y_n)]. \tag{1.47}$$

Let us also mention that the symbol $\mathrm{M}$ that occurs in (1.41), (1.43), (1.45) is not the Latin letter capital $m$ but the Greek capital $\mu$.

The properties of extended linearity in each argument of a BLF can be combined, leading to the extended linearity in *both* arguments. The formula of (combined) extended linearity, in both arguments, can be obtained from Eq. (2.3) by replacing $y$ with the linear combination of the $n$ vectors $y_1, y_2, \dots, y_n \in V$ that occurs in (1.39). Alternatively, the same formula can be obtained from Eq. (1.39) by replacing the vector $x$ by the linear combination of the $m$ vectors that occurs in (1.38), that is $x_1, x_2, \dots, x_m \in U$. The explicit form of the property of simultaneous linearity (in both arguments) thus obtained (omitting the universal quantifiers + memberships) is

$$\boxed{f\left(\sum_{i=1}^{m} \lambda_i x_i, \sum_{j=1}^{n} \mu_j y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_i \mu_j f(x_i, y_j).} \tag{1.48}$$

This formula (1.48) takes a much simpler (and easier to remember) expression in terms of the matrix notations of (1.40) , . . . , (1.47) :

$$\boxed{f(\Lambda^{\mathrm{T}} \cdot \mathcal{X}^{\mathrm{T}}, \mathcal{Y} \cdot \mathbf{M}) = \Lambda^{\mathrm{T}} \cdot f(\mathcal{X}^{\mathrm{T}}, \mathcal{Y}) \cdot \mathbf{M}.} \tag{1.49}$$

The mid factor that occurs in (1.49) is an $m$-by-$n$ matrix whose entries are the values of the BLF on each possible couple of vectors from the two spaces, respectively.

$$f(\mathcal{X}^{\mathrm{T}}, \mathcal{Y}) = [f(x_i, y_j)]_{m,n} = \begin{bmatrix} f(x_1, y_1) & \dots & f(x_1, y_j) & \dots & f(x_1, y_n) \\ \vdots & & \vdots & & \vdots \\ f(x_i, y_1) & \dots & f(x_i, y_j) & \dots & f(x_i, y_n) \\ \vdots & & \vdots & & \vdots \\ f(x_m, y_1) & \dots & f(x_m, y_j) & \dots & f(x_m, y_n) \end{bmatrix}. \tag{1.50}$$

A matrix of the form (1.50) is involved in defining the coefficients of a BLF in a pair of bases.

***Definition 1.6.*** If $f : U \times V \longrightarrow \mathbf{K}$ is a bilinear form and $U$ is spanned by $A = [a_1 \ a_2 \dots a_m]$ while $V$ is spanned by $B = [b_1 \ b_2 \dots b_n]$ then the coefficients of $f$ in the pair of bases $(A, B)$ are the entries of the matrix

$$\boxed{F_{A,B} = f(A^{\mathrm{T}}, B) = [f(a_i, b_j)]_{m,n}.} \tag{1.51}$$

Clearly, the *m*-by-*n* matrix that occurs in (1.51) is obtained from (1.50) for $\mathcal{X} \to A$ and $\mathcal{Y} \to B$. ◊

**PROPOSITION 1.6.** (*The analytical expression of a BLF in a pair of bases*). *If* $\mathrm{f} : U \times V \longrightarrow \mathbf{K}$ *is a bilinear form with the matrix* $F_{A,B} = f(A^{\mathrm{T}}, B)$ *in the pair of bases* $(A, B)$ *then the value taken by* $f$ *on the pair of vectors* $(x \in U, y \in V)$ *with*

$$x = A X_A = X_A^{\mathrm{T}} A^{\mathrm{T}} \;\; \& \;\; y = B Y_B \tag{1.52}$$

*then*

$$\boxed{f(x, y) = X_A^{\mathrm{T}} \cdot F_{A,B} \cdot X_B .} \tag{1.53}$$

***Proof.*** Expression (1.53) immediately follows from property (1.49) with

$$\Lambda = X_A, \;\; \mathcal{X}^{\mathrm{T}} = A^{\mathrm{T}}, \;\; \mathcal{Y} = B, \;\; \mathrm{M} = Y_B \text{ and } F_{A,B} \text{ of (1.51)} .$$

Expressions (1.52) of $x$ and $y$ replace the two arguments of $f$ in the Eq. (1.52). ■

Let us remark the simplicity of a proof like this, when the “matrix formulations” are used. The expression (1.53) should be read as follows :

> If the coefficient matrix of a bilinear form in the pair of bases $(A, B)$ is known, the value $f(x, y)$ equals the scalar obtained by multiplying $F_{A,B} = f(A^{\mathrm{T}}, B)$ at left by the row of the coordinates of $x$ and, at right, by the column of the coordinates of $y$ (in the specific bases $(A, B)$ of the spaces $(U, V)$).

This analytical expression also offers a practical and easy-to-apply rule for calculating $f(x, y)$ as illustrated by

**Example 1.4.** Let $f : U \times V \longrightarrow \mathbb{R}$ be a bilinear form with its matrix

$$F_{A,B} = \begin{bmatrix} 0 & -1 & 3 & 2 \\ 1 & 2 & 1 & -2 \\ -1 & 0 & 3 & 1 \end{bmatrix}$$

in the pair of bases $A = [a_1 \; a_2 \; a_3]$ & $B = [b_1 \; b_2 \; b_3 \; b_4]$ of $U$ and $V$, respectively. It is required to find $f(-a_1 + 2a_2 - a_3, \; b_1 + 4b_2 - 5b_4)$.

Formula (1.53) with the given data leads to

$$f(x,y) = [-1\ \ 2\ \ -1]\cdot\begin{bmatrix} 0 & -1 & 3 & 2 \\ 1 & 2 & 1 & -2 \\ -1 & 0 & 3 & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 4 \\ 0 \\ -5 \end{bmatrix} = [-1\ \ 2\ \ -1]\cdot\begin{bmatrix} -14 \\ 19 \\ -6 \end{bmatrix} = 58.$$

□

### BLF's on the same vector space

A particular case regarding the (general) definition of a BLF is the one when the two vector spaces it is defined on *coincide*, that is $U = V$. Hence the mapping is of the form

$$f : V \times V \longrightarrow \mathbf{K} \quad \text{or} \quad f : V \times V \longrightarrow \mathbb{R}. \tag{1.54}$$

In this case, *Definition 1.5*, *Definition 1.6* and PROPOSITION 1.6 still hold, with slight modifications regarding the memberships of the vectors there involved:

***Definition 1.6'.*** Let $V$ be a vector space over the field $\mathbf{K}$ (= $\mathbb{R}$ / = $\mathbb{C}$). A mapping $f : U \times V \longrightarrow \mathbf{K}$ is said to be a *bilinear functional* (or *bilinear form*) if it satisfies both of the following properties (or axioms):

$$(\mathbf{BLF}_{1'}) \quad \begin{cases} (\forall\, \lambda_1, \lambda_2 \in \mathbf{K})\,(\forall\, x_1, x_2, y \in V) \\ \qquad f(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 f(x_1, y) + \lambda_2 f(x_2, y). \end{cases} \tag{1.36'}$$

$$(\mathbf{BLF}_{2'}) \quad \begin{cases} (\forall\, \mu_1, \mu_2 \in \mathbf{K})\,(\forall\, x, y_1, y_2 \in V) \\ \qquad f(x, \mu_1 y_1 + \mu_2 y_2) = \mu_1 f(x, y_1) + \mu_2 f(x, y_2). \end{cases} \tag{1.37'}$$

***Definition 1.6'.*** If $f : V \times V \longrightarrow \mathbf{K}$ is a bilinear form and $V$ is spanned by $A = [a_1\ a_2\ \dots\ a_m]$ then the coefficients of $f$ in the basis $A$ are the entries of the matrix

$$\boxed{F_A = f(A^{\mathrm{T}}, A) = [f(a_i, a_j)]_{n,n} \underset{\text{not}}{=} [\alpha].} \tag{1.38'}$$

**PROPOSITION 1.6'.** (*The analytical expression of a BLF in a basis*).
*If* $f : V \times V \longrightarrow \mathbf{K}$ *is a bilinear form with the matrix* $F_A = f(A^{\mathrm{T}}, A)$ *in basis* $A$ of $V$ *then the value taken by* $f$ *on the pair of vectors* $(x, y) \in V$ *with*

$$x = A X_A = X_A^{\mathrm{T}} A^{\mathrm{T}} \ \ \&\ \ y = A\, Y_A \tag{1.52'}$$

*then*

$$\boxed{f(x,y) = X_A^{\mathrm{T}} \cdot F_A \cdot Y_A \underset{\text{not}}{=} X_A^{\mathrm{T}} \cdot [\alpha] \cdot Y_A .} \qquad (1.53')$$

We do not rewrite the two extended linearities of PROPOSITION 2.1. Formulas (1.48) & (1.49) hold without any change, but the memberships in the statement have to be adapted (with $U = V$).

***Remark 2.1.*** If the spaces $U, V$ in *Definition 1.5*, *Definition 1.6* and PROPOSITION 1.6' are respectively equal to $\mathbb{R}^m$ **and** $\mathbb{R}^n$ and if the bases $A, B$ are replaced by the standard bases in these spaces, that is $A = E_m \wedge B = E_n$, then the coefficients of $f: \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}$ are the entries of the matrix

$$F_{E_m, E_n} \underset{\text{not}}{=} [\varepsilon] = [f(e_i, e_j')]_{m,n} \qquad (1.55)$$

and the analytical expression a BLF in the standard bases is

$$\boxed{f(X,Y) = X^{\mathrm{T}} \cdot F_{E_m, E_n} \cdot Y \underset{\text{not}}{=} X^{\mathrm{T}} [\varepsilon]\, Y.} \qquad (1.56)$$

Certainly, this remark still holds if $\mathbb{R}$ is replaced by a general field $\mathbf{F}$.

In the case when $U = V = \mathbf{F}^n$ or $U = V = \mathbb{R}^n$, the coefficient matrix in the standard basis is

$$F_{E_n} = f(E_n^{\mathrm{T}}, E_n) = [f(e_i, e_j)]_{n,n} \underset{\text{not}}{=} [\varepsilon]$$

and the analytical expression of $f: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ , that is of $f(X, Y)$, is

$$\boxed{f(X,Y) = X^{\mathrm{T}} \cdot F_{E_n} \cdot Y \underset{\text{not}}{=} X^{\mathrm{T}} [\varepsilon]\, Y.} \qquad (1.57)$$

Practically, the analytic expression in terms of the components of $X, Y \in \mathbb{R}^n$ is practically the same as in Eq. (1.56) but the matrix in the middle of the product is square in (1.57) while it was rectangular in (1.56).

**Example 1.5.** If $f: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is characterized by its coefficient matrix in the pair of standard bases $(E_2, E_4)$ by

$$[\varepsilon] = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 4 & -2 & -1 \end{bmatrix}$$

then

$$f([2\ \ 3]^{\mathrm{T}},[1\ \ 3\ \ -1\ \ 2]^{\mathrm{T}}) = [2\ \ 3]\cdot\begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 4 & -2 & -1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix} =$$

$$= [7\ \ 10\ \ 0\ \ -3]\cdot\begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix} = 31.$$

□

The next example involves a bilinear form expressed in the basis $E_3$ of $\mathbb{R}^3$.

**Example 1.6.** It is required to find the value $f(X,Y)$ of the bilinear form $f:\mathbb{R}^n\times\mathbb{R}^n \longrightarrow \mathbb{R}$, knowing that

$$F_{E_3} = f(E_3^{\mathrm{T}},E_3) \underset{\text{not}}{=} [\varepsilon] = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 3 & 0 \\ 2 & 4 & -1 \end{bmatrix},\ X = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}\ \&\ Y = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

Equation (1.57) with the data above gives

$$f(X,Y) =$$

$$= [2\ \ -2\ \ 0]\cdot\begin{bmatrix} -1 & 0 & 2 \\ 3 & 3 & 0 \\ 2 & 4 & -1 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = [-8\ \ -6\ \ 4]\cdot\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = -34.$$

□

A problem met in the previous subsection for the linear forms should be also approached for the bilinear forms :
If the bases $A,B$ in the spaces $U,V$ are changed to other ones, how change the coefficients of the BLF $f:U\times V \longrightarrow \mathbf{K}$ ? The answer is given by

**PROPOSITION 1.7.** (*Changing bases and coefficients of BLF 's*).
*If* $f:U\times V \longrightarrow \mathbf{K}$ *is a bilinear form with the matrix* $F_{A,B} = f(A^{\mathrm{T}},B)$ *in the pair of bases* $A,B$ *of the spaces* $U$ & $V$ *(respectively) and if*

$$A \rightarrow \bar{A}\ \text{ by }\ \bar{A}^{\mathrm{T}} = S\cdot A^{\mathrm{T}}\quad \textit{and}\quad B \rightarrow \bar{B}\ \textit{ by }\ \bar{B}^{\mathrm{T}} = T\cdot B^{\mathrm{T}} \tag{1.58}$$

*then the coefficient matrix of* $f$ *in the new bases* $\bar{A},\bar{B}$ *is given by*

$$\boxed{F_{\bar{A},\bar{B}} = f(\bar{A}^{\mathrm{T}},\bar{B}) = S\cdot F_{A,B}\cdot T^{\mathrm{T}}.} \tag{1.59}$$

***Proof.*** The proof of the formula (1.59) is essentially based on the one of PROPOSITION 1.4. In fact the properties of *extended linearity* in both arguments of a BLF, matriceally written in Eqs. (1.48) - (1.49), should be also extended to several vectors of scalars (with scalar components) multiplying, at left, a column of vectors $\mathcal{X}^{\mathrm{T}}$ (as row vectors) and a row of vectors $\mathcal{Y}$, at right, by several column vectors. These properties may be written, replacing $\mathcal{X}$ by $A$ and $\mathcal{Y}$ by $B$, as

$$f(L^{\mathrm{T}} \cdot A^{\mathrm{T}}, B) = L^{\mathrm{T}} \cdot f(A^{\mathrm{T}}, B) = L^{\mathrm{T}} \cdot F_{A,B} \quad \& \tag{1.60}$$

$$f(A^{\mathrm{T}}, B \cdot R) = f(A^{\mathrm{T}}, B) \cdot R = F_{A,B} \cdot R. \tag{1.61}$$

In (1.60), $L^{\mathrm{T}}$ is a matrix consisting of $k$ rows, each of them with $m$ entries provided $\dim U = m$, while $R$ is a matrix with $\ell$ columns with $n$ entries if $\dim V = n$. For instance,

$$L^{\mathrm{T}} = \begin{bmatrix} \Lambda_1^{\mathrm{T}} \\ \Lambda_2^{\mathrm{T}} \\ \vdots \\ \Lambda_k^{\mathrm{T}} \end{bmatrix} \;\&\; R = [\,\mathrm{M}^1\ \mathrm{M}^2 \ldots \mathrm{M}^\ell\,]. \tag{1.62}$$

Properties (1.60) and (1.61) can be taken together leading to

$$f(L^{\mathrm{T}} \cdot A^{\mathrm{T}}, B \cdot R) = L^{\mathrm{T}} \cdot f(A^{\mathrm{T}}, B \cdot R) = L^{\mathrm{T}} \cdot f(A^{\mathrm{T}}, B) \cdot R = L^{\mathrm{T}} \cdot F_{A,B} \cdot R. \tag{1.63}$$

The last step in this proof consists in replacing, in (1.63), $L^{\mathrm{T}} \rightarrow S$ & $R \rightarrow T^{\mathrm{T}}$. The resulting equation plus the connections from the old bases, that is $A \rightarrow \bar{A}$ and $B \rightarrow \bar{B}$ of Eqs. (1.58), results in the formula (1.59). ■

The particular case when the BLF $f$ is defined on the same space, that is $U = V$, should be again considered. In this case a single basis $A$ is involved, and it is changed to a "new" basis as in (1.58), but we denote the single transformation matrix by $T$:

$$A \rightarrow \bar{A} \text{ by } \bar{A}^{\mathrm{T}} = T \cdot A^{\mathrm{T}} \iff \bar{A} = A \cdot T^{\mathrm{T}}. \tag{1.64}$$

Taking now, instead of the two-spaces a single space, adapting the two-bases formula (1.59) with $S = T$, and also using the notations

$$F_A = f(A^{\mathrm{T}}, A) = [\alpha] \quad \text{and} \quad F_{\bar{A}} = f(\bar{A}^{\mathrm{T}}, \bar{A}) = [\bar{\alpha}],$$

PROPOSITION 1.7 becomes

**PROPOSITION 1.7'.** *If* $f : V \times V \longrightarrow \mathbf{K}$ *is a bilinear form with the coefficient matrix* $F_A = f(A^{\mathrm{T}}, A)$ *in the basis* $A$ *of space* $V$ *and if this basis is changed as in* (1.64) *then the coefficient matrix of* $f$ *in the new basis* $\bar{A}$ *is given by*

$$\boxed{F_{\bar{A}} = f(\bar{A}^{\mathrm{T}}, \bar{A}) = T \cdot F_A \cdot T^{\mathrm{T}}} \quad \text{or} \quad \boxed{[\bar{\alpha}] = T \cdot [\alpha] \cdot T^{\mathrm{T}}.} \tag{1.65}$$

Let us illustrate PROPOSITION 1.7 by taking two changes of the bases for the BLF and the two vectors considered in **Example 1.4** (at pages 63-64).

**Example 1.7.** If the two bases $A = [a_1\ a_2\ a_3]$ & $B = [b_1\ b_2\ b_3\ b_4]$ are changed for two new bases, that is $A \rightarrow \bar{A}$ with $S$ and $B \rightarrow \bar{B}$ with $T$ where

$$S = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 2 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 \end{bmatrix} \tag{1.66}$$

the coefficient matrix in the new pair of bases is, according to formula (1.59),

$$F_{\bar{A},\bar{B}} \underset{(2.30)}{=} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 3 & 2 \\ 1 & 2 & 1 & -2 \\ -1 & 0 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 2 & 4 & -1 \\ 3 & 3 & 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 2 & 3 \\ 4 & 3 & -2 & 2 \\ 5 & 5 & 0 & 6 \end{bmatrix}. \tag{1.67}$$

We can use this "new" matrix for calculating $f(x, y)$ for the two vectors in the same example. Using the transformation method presented in § **1.1** – PROPOSITION 1.6, – the coordinates of $x$, $y$ in the new bases $\bar{A}$ & $\bar{B}$ can be found by Gaussian elimination on the augmented matrices $[S^{\mathrm{T}} | X_A]$ & $[T^{\mathrm{T}} | Y_B]$:

$$[S^{\mathrm{T}} | X_A] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 1 & 1 & 2 & 2 \\ -1 & 1 & -1 & -1 \end{array}\right] \sim \ldots \sim \left[\begin{array}{c|c} & -6 \\ I_3 & -2 \\ & 5 \end{array}\right] \Rightarrow X_{\bar{A}} = \begin{bmatrix} -6 \\ -2 \\ 5 \end{bmatrix}. \tag{1.68}$$

$$[T^{\mathrm{T}} \mid Y_B] = \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & -1 & 0 & 1 & 4 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -5 \end{array}\right] \sim \left[\begin{array}{c|c} & 11 \\ I_4 & 1 \\ & -2 \\ & 5 \end{array}\right] \Rightarrow Y_{\bar{B}} = \begin{bmatrix} 11 \\ 1 \\ -2 \\ 5 \end{bmatrix}. \tag{1.69}$$

With the new coordinates in (1.68) & (1.69), the value of the BLF is

$$f(x,y) = [-6\ \ -2\ \ 5] \cdot \begin{bmatrix} 3 & 3 & 2 & 3 \\ 4 & -3 & -2 & 2 \\ 5 & 5 & 0 & 6 \end{bmatrix} \cdot \begin{bmatrix} 11 \\ 1 \\ -2 \\ 5 \end{bmatrix} =$$

$$= [-1\ \ 13\ \ -8\ \ 8] \cdot \begin{bmatrix} 11 \\ 1 \\ -2 \\ 5 \end{bmatrix} = 58.$$

Therefore we have retrieved the same value as in **Example 1.4** , that was found using the initial bases. □

We close this section with giving the explicit analytical expression of a BLF. In other words, we give the explicit forms of expressions (1.53) & (1.53'), respectively.

$$\text{If } X_A^{\mathrm{T}} = [\xi_1\ \xi_2\ \dots\ \xi_m],\ Y_B = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix} \text{ and } F_{A,B} = f(A^{\mathrm{T}}, B) = [\varphi_{ij}]_{m,n}$$

then the explicit form of (1.53') is

$$\boxed{f(x,y) = X_A^{\mathrm{T}} [\varphi_{ij}]\, Y_B = \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi_{ij}\, \xi_i \eta_j .} \tag{1.70}$$

In the particular case when $U = V \Rightarrow B = A,\ F_A = f(A^{\mathrm{T}}, A) = [\alpha_{ij}]_{n,n}$ and the coordinates of the two vector arguments in basis $A$ are

$$X_A^{\mathrm{T}} = [\xi_1\ \xi_2\ \dots\ \xi_n],\ Y_A = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix}$$

then the explicit form of expression (1.17) is

$$\boxed{f(x,y) = X_A^{\mathrm{T}}\,[\alpha_{ij}]\,Y_A = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}\,\xi_i\,\eta_j\,.} \tag{1.71}$$

Therefore, in both cases (when $f$ is defined on two different spaces or on the same space), the analytical expression of $f(x,y)$ is a homogeneous function of order 2 in the coordinates $\xi_i,\ \eta_j\ (1 \le i \le m/n)(1 \le j \le n)$ of $x, y$ in the considered bases (basis).

The same property holds for the BLFs defined on the "standard" spaces $\mathbb{R}^m \times \mathbb{R}^n$, respectively $\mathbb{R}^n \times \mathbb{R}^n$. The difference consists in what regards expressions (1.70) & (1.71) if the vectors are expressed in the standard bases $E_m$ of $\mathbb{R}^m$, $E_n$ of $\mathbb{R}^n$. In fact, the coordinates in these bases are just the components $x_i, y_j$ of the two vectors $X, Y$. The corresponding expressions of $f(X,Y)$ defined on $\mathbb{R}^m \times \mathbb{R}^n$, respectively on $\mathbb{R}^n \times \mathbb{R}^n$, are

$$f(X,Y) = X^{\mathrm{T}}\,[\,\varepsilon_{ij}\,]\,Y = \sum_{i=1}^{m}\sum_{j=1}^{n} \varepsilon_{ij}\,x_i\,y_j\,; \tag{1.72}$$

$$f(X,Y) = X^{\mathrm{T}}\,[\varepsilon_{ij}]\,Y = \sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon_{ij}\,x_i\,y_j\,; \tag{1.73}$$

For instance, the explicit expression of the BLF presented in Example 1.6 is

$$f(X,Y) = X^{\mathrm{T}}\begin{bmatrix} -1 & 0 & 2 \\ 3 & 3 & 0 \\ 2 & 4 & -1 \end{bmatrix} Y =$$

$$= -x_1y_1 + 2x_1y_3 + 3x_2y_1 + 3x_2y_2 + 2x_3y_1 + 4x_3y_2 - x_3y_3\,. \tag{1.74}$$

Conversely, if a BLF is given by its analytical expression in the components $x_i, y_j$ of the two vectors $X$ & $Y$, then its matrix in the standard bases $(E_m, E_n)$ of $\mathbb{R}^m \times \mathbb{R}^n$, respectively $E_n$ of $\mathbb{R}^n$, can be immediately written. The subscripts of $x_i$'s in the terms $\varepsilon_{ij}\,x_i\,y_j$ correspond to the row index, similarly for the columns from $y_j$'s and the coefficient $\varepsilon_{ij}$ is just the entry in position $(i,j)$. If a term $\varepsilon_{ij}\,x_i\,y_j$ does not occur in the expression of $f(X,Y)$ then $\varepsilon_{ij} = 0$.

Let us close this section with the problem of finding the analytic expression of a BLF defined $\mathbb{R}^m \times \mathbb{R}^n$ or $\mathbb{R}^n \times \mathbb{R}^n$ by an expression of the form (1.74) in another pair of bases $(A, B)$ or in a basis $A$. The transformation matrix from $E_m \to A : \mathbb{R}^m = \mathcal{L}(A)$ is $S = A^{\mathrm{T}}$ while the similar transformation $E_n \to B : \mathbb{R}^n = \mathcal{L}(B)$ is performed with matrix $T = B^{\mathrm{T}}$. With these remarks, the transformation formula (1.59) becomes

$$\boxed{F_{A,B} = f(A^{\mathrm{T}}, B) = A^{\mathrm{T}} \cdot f(E_m^{\mathrm{T}}, E_n) \cdot B = A^{\mathrm{T}} \cdot [\varepsilon] \cdot B.} \qquad (1.75)$$

If the BLF is defined on the same space, that is on $\mathbb{R}^n \times \mathbb{R}^n$ and $E_n \to A : \mathbb{R}^n = \mathcal{L}(A)$ then the corresponding formula is easily obtained from (1.75) by taking $B = A$ & $E_m = E_n$:

$$\boxed{F_A = f(A^{\mathrm{T}}, A) \underset{\text{not}}{=} [\alpha] = A^{\mathrm{T}} \cdot f(E_n^{\mathrm{T}}, E_n) \cdot A = A^{\mathrm{T}} \cdot [\varepsilon] \cdot A.} \qquad (1.76)$$

## § 3.1 - A APPLICATIONS TO LINEAR & BILINEAR FORMS

### LINEAR FORMS

**1 – A.1** The linear expression of a linear form $f : V \longrightarrow \mathbb{R}$ in a basis $A$ of $V$ is $f(X_A) = 2\,\xi_1 - 3\,\xi_2 + 5\,\xi_3$. Find a new basis $A' = [a'_1\ a'_2\ a'_3]$ whose vectors are respectively proportional to the vector of basis $A = [a_1\ a_2\ a_3]$ so that the expression of $f$ in the new basis become $f(X_A') = \xi'_1 + \xi'_2 + \xi'_3$.

**1 – A.2** Determine the real parameter $\lambda$ so that the linear form $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$ with the coefficients $[\lambda\ \ 2\ \ {-1}\ \ \lambda]$ in the standard basis $E$ map the vector $X = [-3\ \ 1\ \ 2\ \ 1]^{\mathrm{T}}$ onto the value $-8$.

**1 – A.3** A basis and three vectors are considered in space $\mathbb{R}^3$ :

$$A : a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ a_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ a_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix};$$

$$X = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix},\ Y = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ Z = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}.$$

It is required to find the coordinates of the three vectors in basis $A$ and the coefficients of $f$ in this basis knowing that

$$f(X) = 3,\ f(Y) = 1,\ f(Z) = 5.$$

***Hint*** : The three vectors of coordinates can be simultaneously found by the method of transformations (Gaussian elimination) on the augmented matrix $[A \,|\, X\, Y\, Z]$. The coefficients $f(A) = [\alpha]$ of $f$ in basis $A$ can be determined by solving the matrix equation $[\alpha] \cdot [X_A\ Y_A\ Z_A] = [3\ \ 1\ \ 5]$. A previous transposition is recommended.

**1 – A.4** The coefficients of a linear form $f : V \longrightarrow \mathbb{R}$ in a basis $A$ and of $V$ are

$$f(A) = [\alpha] = [3 \ \ -1 \ \ 2 \ \ 0].$$

Find its coefficients in the basis $B$ obtained from $A$ through the change of matrix

$$T = \begin{bmatrix} 2 & 3 & 2 & 2 \\ 1 & 1 & 3 & 2 \\ -3 & 2 & -1 & -1 \\ 1 & -1 & 1 & 3 \end{bmatrix}.$$

***Hint***: The basis transformation means $B = A \cdot T^{\mathrm{T}} \vee B^{\mathrm{T}} = T \cdot A^{\mathrm{T}}$ and the new coefficients $f(B) = [\beta]$ can be found with formula (1.21) - p. 82 in **§ 2.1.**

**1 – A.5** Study the linear dependence / independence of the three linear forms defined on $\mathbb{R}^4$ whose coefficient rows are written together giving the matrices *A* and *B* below ; in the case when the LF's are dependent, find a dependence relation among them. In which space is this exercise stated ?

$$\text{a)} \quad A = \begin{bmatrix} 2 & 2 & 7 & -1 \\ 3 & -1 & 2 & 4 \\ 1 & 1 & 3 & 1 \end{bmatrix}; \qquad \text{b)} \quad A = \begin{bmatrix} 3 & 2 & -5 & 4 \\ 3 & -1 & 3 & -3 \\ 3 & 5 & -13 & 11 \end{bmatrix}.$$

**1 – A.6** Find the common kernel for each set of linear forms in the previous exercise.

**1 – A.7** The linear form $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$ is given by its analytical expression

$$f(X) = 3\,x_1 - 5\,x_2 + 4\,x_4. \tag{1.77}$$

Find its coefficients (and write its analytical expression) in the basis

$$A : a_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \ a_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \ a_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \ a_4 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}. \tag{1.78}$$

Find the value of $f(-a_1 + 3a_2 + 2a_3 + a_4)$ using both the standard basis and the basis $A$ in the statement.

***Hint***: The analytical expression (1.77) can be used after effectively finding the argument of $f$ as a vector in $\mathbb{R}^4$, using the vectors of (1.78). The same value can be found working in basis *A* but this needs to find the coordinates $X_A$ and the coefficients $[\alpha] = f(A)$ in the basis (1.78). The coordinates $X_A$ appear under $f$ in the statement, and the coefficients $[\alpha]$ can be easily found either by the appropriate formula in **§ 3.1** or by using (1.77).

**1 – A.8** The function $\mathrm{Tr} : \mathcal{M}_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is defined, for any square matrix $A \in \mathcal{M}_n(\mathbb{R}) \longrightarrow \mathbb{R}$ by

$$\mathrm{Tr}\, A = \sum_{i=1}^{n} a_{ii}.$$

$\mathrm{Tr}\, A$ is called the *trace* of matrix $A$. It is required to check that this function is a linear form on the space of square matrices of order $n$.

***Hint*** : A linear combination of two matrices, for instance $\lambda A + \mu B$, should be taken as the argument of the trace function.

**1 – A.9** Find the linear forms $f(X) = [\varepsilon]X$ such that

$$f([0 \;\; 1 \;\; -1]^{\mathrm{T}}) = 0 \quad \text{and} \quad f([-2 \;\; 1 \;\; 1]^{\mathrm{T}}) = 0.$$

### BILINEAR FORMS

**1 – A.9** Establish which of the following mappings are bilinear forms :

① $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R},\; f(x, y) = (x + a)(y + b)$ with $a, b \in \mathbb{R}$;

② $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R},\; f(x, y) = x^k y,\; k \in \mathbb{N}$;

③ $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R},\; f(X, Y) = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$;

④ $\begin{cases} f : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}, \\ f(X, Y) = \mathrm{Re}\,(x_1 + x_2 + \ldots + x_n) \cdot \mathrm{Im}\,(y_1 + y_2 + \ldots + y_n); \end{cases}$

⑤ $f : \mathcal{M}_n(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}) \longrightarrow \mathbb{R}$,

$$f(A, B) = \mathrm{Tr}\, A \cdot \mathrm{Tr}\, B\;;$$

⑥ $$\begin{cases} f : \mathrm{POL}_n(\mathbb{R}) \times \mathrm{POL}_n(\mathbb{R}) \longrightarrow \mathbb{R}, \\ f(p,q) = \int_0^1 \int_0^1 p(x)\, q(y)\, dx\, dy ; \end{cases}$$

⑦ $$f : \mathrm{POL}_{\le n}(\mathbb{R}) \times \mathrm{POL}_{\le n}(\mathbb{R}) \longrightarrow \mathbb{R},$$

$$f(p,q) = p(0)\, q'(0).$$

**1 – A.10** Given the bilinear form $f : \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}$,

$$f(X,Y) = 2x_1 y_1 + x_2 y_1 + + 3x_3 y_3 + x_4 y_1 + x_4 y_4,$$

write the coefficient matrix of $f$ in the standard basis $E$ of $\mathbb{R}^4$ and then in the basis

$$A : a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ a_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \ a_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \ a_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}.$$

**1 – A.11** A bilinear form $f : U \times V \longrightarrow \mathbb{R}$ is given by its matrix in a pair of bases $A$, $B$ :

$$F_{A,B} = \begin{bmatrix} 1 & 3 & 4 & 2 \\ -1 & 0 & 1 & -2 \\ 2 & 1 & 5 & 0 \end{bmatrix}.$$

It is required to find its matrix in the pair of bases $\bar{A}, \bar{B}$ obtained from the initial bases by the (respective) transformation matrices

$$S = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \text{ and } T = \begin{bmatrix} 2 & 0 & 0 & 2 \\ -1 & -2 & 3 & 0 \\ 4 & 1 & 0 & 2 \\ -1 & -1 & 1 & 1 \end{bmatrix}, \text{ respectively.}$$

It is also required to compute $f(x,y)$ where $x = 2a_1 - 5a_2 + 4a_3$ and $y = b_1 - 2b_2 - b_3 + 3b_4$, using the analytic expressions (or coefficients) of $f$ in both pairs of bases $(A,B)$ & $(\bar{A},\bar{B})$.

**1 – A.12** A bilinear form $f : V \times V \longrightarrow \mathbb{R}$ is given by its matrix in a basis $A$ of $V$, namely

$$[\alpha] = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 3 & 4 & -5 \end{bmatrix}.$$

It is required to determine the rank of $f$ $(= \text{rank}\,[\alpha])$, its value $f(x, y)$ for $x = 2a_1 - a_2 - a_3$ and $y = -a_1 - a_2$, and also the matrix of $f$ in a new basis $B$ obtained from $A$ by

$$B^{\mathrm{T}} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \cdot A^{\mathrm{T}}.$$

It is also asked to find the coordinates of the vectors $x, y$ in basis $A$ and to check whether the value $f(x, y)$ is retrieved when it is computed with the analytic information in the new basis, that is with $[\beta], X_B, Y_B$.

**1 – A.13** Write the matrix (in the standard basis $E_4$ of $\mathbb{R}^4$) of the BLF $f(X, Y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2 + 2x_3y_3 + 3x_4y_4$.

Then find the matrix $[\beta] = f(B^{\mathrm{T}}, B)$ of $f$ in the new basis

$$B : b_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ b_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ b_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix},\ b_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Finally, calculate the value $f(X, Y)$ for

$$X = [3\ \ 0\ \ 2\ \ -2]^{\mathrm{T}} \quad \text{and} \quad Y = [0\ \ 0\ \ -1\ \ -4]^{\mathrm{T}}$$

using both the analytic expression and the analytic information in the new basis, that is $[\beta], X_B, Y_B$.